Fast evaluation of asymptotic waveforms from gravitational perturbations

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Joint work with Alex Benedict (U. of New Mexico) and Stephen Lau (U. of New Mexico). CQG February 2013 (arXiv:1210.1565)



Introduction

Theoretical motivation of kernels

Implementation and results

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Why are waves so important in science?

- Wave equations arise in many areas of physics
 - Sound propagation (SONAR, echolocation of bats), seismic waves (earthquake warning system), electromagnetism (astronomy, cellphones)
- Propagate information over long distances to provide clues about...
 - ► Initial data, scatters, local sources, curvature and/or dimension of space
- Typically one records a time-dependent signal at some detector(s)







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Example: Scattering off an airplane



Figure courtesy of Andreas klockner

- Maxwell's equations
- Incoming plane wave
- Scattered electric field shown with arrows
- In the far-field is an enemy observer

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Example: Scattering off a black hole



- Einstein equation
- Gaussian perturbation
- Scattering and outgoing wave
- Far-field observations complicated by spacetime curvature

Obtaining the far-field signal

From data recorded in the extreme near-field we want:

- The far-field signal at arbitrarily large values of r
- Data \rightarrow simple operation \rightarrow far-field signal
- Should be equivalent to solving the PDE (free lunch?)

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In the black hole case...

- Observations are far from sources/scatters
 - Binary black hole systems are in other galaxies
- Perhaps many frequencies needed
 - Inspiral orbits will have a continuous frequency spectrum
- Accurate signal needed for scientific purposes
 - ► Searching data, parameter estimation, model building + verification
 - Systematic errors should be quantified and minimized

Existing techniques for the far-field

Geometric approaches

- Compactification of spatial infinity, hyperboloidal formulations, super-grids
- ► Intrusive to code, still solving the PDE, accuracy/efficiency

Extrapolation

- Record data at multiple values of r
- Fit for far-field $\approx \sum_{k=0}^{N} r^{-k} f_{(k)}(t-r)$ and then $r \to \infty$
- Multiple data recordings, ansatz for far-field, fit accuracy
- "Green's function" (this talk)
 - With a Green's function you know everything
 - Only for linear PDEs, hard to compute, could be hard to use

An almost Green's function

If we only care about outgoing solutions we can identify a function which, when convolved with data at a fixed r, *exactly* gives the far-field solution

- Similar in spirit to Green's function, for specific usage
- Built from outgoing solutions

Examples:

- Computational relativity: Gravitational multipoles for general relativity linearized about Minkowski (Abrahams and Evans matching)
- Computational electromagnetics: "Near-field to far-field transformation" for single frequency-domain signals on Minkowski

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Wave propagation on a curved spacetime with time-domain signals?

- Teleportation: From metric perturbations (i.e. signal) recorded at any radial location we seek to teleport this information to another location
 - Its cheap equivalent to solving the PDE but without solving it
 - "The safest way to travel" No accumulation of errors from numerical PDE simulation



 Asymptotic waveform evaluation: special case of teleportation, recover signal reaching future null infinity

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Goal of talk ...

- Theoretical construction of teleportation/AWE kernels
- Numerical approximation of these kernels
- How to handle backscattering due to spacetime curvature
 - ► The kernel, like the Green's function, has a branch cut

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1D wave equation 3D wave equation Gravitational perturbation equations Rational approximation techniques

Outline

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Theoretical motivation of kernels

Implementation and results

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A simple wave equation

1+1 wave equation...

$$(-\partial_t^2 + \partial_x^2)\psi = 0$$

 $\psi(0, x) = g(x), \quad \partial_t \psi(0, x) = 0$

d'Alembert's solution (1747)...

$$\psi(t,x) = G(x-t) + F(x+t)$$
$$= \frac{1}{2}g(x-t) + \frac{1}{2}g(x+t)$$

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From $\psi(t, x) = G(x - t) + F(x + t)$ we know

- "Outgoing" (right moving) piece is G(x t)
- G is constant $(\partial_t G + \partial_x G = 0)$ along null coordinate u = t x

To recover the asymptotic waveform at $x_{\infty} \gg x_0$...

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To recover the asymptotic waveform at $x_{\infty} \gg x_0$...

- 1. Place a "detector" at x_0 such that initial data is 0 to the right of x_0
- 2. Record solution $\psi(t, x_0)$ at x_0

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To recover the asymptotic waveform at $x_{\infty} \gg x_0$...

- 1. Place a "detector" at x_0 such that initial data is 0 to the right of x_0
- 2. Record solution $\psi(t, x_0)$ at x_0
- 3. Compute

Generalization of this relationship to other wave equations?

1D wave equation **3D wave equation** Gravitational perturbation equations Rational approximation techniques

Wave equation on Minkowski spacetime

We wish to solve ...

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\psi = 0$$

Problem posed on spatially unbounded domain.

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Wave equation on Minkowski spacetime

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Problem posed on spatially unbounded domain.

We actually solve...

- For computational reasons the problem is solved on a spatially finite domain
- Detector records signal on a sphere defined by $r = r_b$

How do we recover the solution which escapes to (null) infinity??

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Some facts...

- Outgoing solutions are no longer G(r-t)
 - Instead a power series in 1/r.
- Sharp Huygen's principle: source/data at X₀ = (t₀, x₀, y₀, z₀) only influences X = (t, x, y, z) if |X₀ − X| = 0

3D Gaussian Wave (Frans Pretorius' webpage)

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A roadmap to the asymptotic (far-field) signal

- 1. What are the outgoing solutions?
- 2. What is the asymptotic solution?
- 3. Derive an equation for $\psi_{\infty}(t)$ using data recorded on the sphere $\psi(t, r_0)$

Preview: Wave equations on flat and curved geometry follow similar approach. However, we may carry these steps analytically for Minkowski, while relying more heavily on numerical results for the gravitational perturbations of Schwarzschild.

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Implementation Impleme

Set of spherical harmonics $\{Y_{\ell m}\}$ form complete orthonormal basis on unit sphere. Expand solution as

$$\psi = \frac{1}{r} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Psi_{\ell m}(t,r) Y_{\ell m}(\theta,\phi)$$

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\psi = 0 \quad \rightarrow \qquad \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{r^2}\right]\Psi_{\ell m} = 0$$

Outgoing solutions...

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Outgoing solutions...

$$\Psi_{\ell m}(t,r) = \sum_{k=0}^{\ell} \frac{1}{r^k} c_{\ell k} f^{(\ell-k)}(t-r), \quad c_{\ell k} = \frac{1}{2^k k!} \frac{(\ell+k)!}{(\ell-k)!},$$

Starting point for derivation

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A scalar Abrahams-Evans procedure

- ► AE considered gravitational perturbation multipoles linearized about flat spacetime (1988). Outgoing multipoles of tensorial wave equation.
- Consider the procedure applied to our scalar wave equation

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A *scalar* Abrahams-Evans procedure

- The sphere defined by $r = r_b$ is our recording surface.
 - ► *r_b* large enough such that initial data compactly supported inside
- Consider an $\ell = 2$, any *m* procedure

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- 1. Outgoing solution is $\Psi(t,r) = f''(t-r) + \frac{3}{r}f'(t-r) + \frac{3}{r^2}f(t-r)$

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- 2. Record data $\Psi(t, r_b)$ until T such that $\Psi(t > T, r_b) \approx 0$

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A *scalar* Abrahams-Evans procedure

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- 2. Record data $\Psi(t, r_b)$ until T such that $\Psi(t > T, r_b) \approx 0$
- 3. f(t-r) is found by solving

$$y'' + \frac{3}{r_b}y' + \frac{3}{r_b^2}y = \Psi(t, r_b), \qquad y(0) = 0 = y'(0)$$

f''(t-r), the signal reaching $r = \infty$, is found from f(t-r) = y

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A need for reformulated Abrahams-Evans procedure

- ► This will *not* generalize to wave equations on curved backgrounds
 - Will return to difficulties later
- Reformulate their procedure so that generalizations are possible.

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Laplace transformed wave equation

Laplace transformed solution

$$\hat{\Psi}_{\ell m}(s,r) = \int_0^\infty \Psi_{\ell m}(t,r) \mathrm{e}^{-st} dt$$

solves the transformed wave equation

$$\left[s^2 - \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{r^2}\right]\hat{\Psi}_{\ell m} = \frac{\partial\hat{\Psi}_{\ell m}}{\partial t}(0,r) + s\hat{\Psi}_{\ell m}(0,r)$$

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Outgoing solutions

$$\left[s^2 - \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{r^2}\right]\hat{\Psi}_{\ell} = 0$$

- Ordinary differential equation
- A modified Bessel equation solutions well studies

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• General outgoing solution:
$$\widehat{\Psi}_{\ell}(s,r) = a(s)e^{-sr}W_{\ell}(sr)$$

- ► *a*(*s*) encodes the initial data
- ▶ *e^{-sr}* is correct exponential dependence for outgoing

•
$$W_{\ell}(sr) = (sr)^{-\ell} \sum_{k=0}^{\ell} c_{\ell k}(sr)^{k}$$

- Coefficients $c_{\ell k}$ known (e.g. *Classical Electrodynamics* by Jackson)
- Example $W_2(sr) = (sr)^{-2} [(sr)^2 + 3sr + 3]$
- Recall $\Psi_{\ell=2}(t,r) = f''(t-r) + \frac{3}{r}f'(t-r) + \frac{3}{r^2}f(t-r)$

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Asymptotic waveform evaluation/teleportation kernel

Starting with $\widehat{\Psi}_{\ell}(s,r) = a(s)e^{-sr}W_{\ell}(sr)$

$$\begin{split} \mathrm{e}^{s(r_2-r_1)} \widehat{\Psi}_{\ell}(s,r_2) &= \left[\frac{W_{\ell}(sr_2)}{W_{\ell}(sr_1)} - 1\right] \widehat{\Psi}_{\ell}(s,r_1) + \widehat{\Psi}_{\ell}(s,r_1) \\ &\equiv \widehat{\Phi}_{\ell}(s,r_1,r_2) \widehat{\Psi}_{\ell}(s,r_1) + \widehat{\Psi}_{\ell}(s,r_1) \end{split}$$

Kernel defined with "minus 1" so $\widehat{\Phi}_\ell(s, r_1, r_2) o 0$ as $s o \infty$

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Kernel defined with "minus 1" so $\widehat{\Phi}_\ell(s, r_1, r_2) o 0$ as $s o \infty$

Case $r_2 \ll \infty$: $\widehat{\Phi}_{\ell}(s, r_1, r_2)$ teleports a signal of frequency s from r_1 to r_2

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Kernel defined with "minus 1" so $\widehat{\Phi}_\ell(s,r_1,r_2) o 0$ as $s o \infty$

Case $r_2 \ll \infty$: $\widehat{\Phi}_{\ell}(s, r_1, r_2)$ teleports a signal of frequency s from r_1 to r_2 Case $r_2 \approx \infty$: $\widehat{\Phi}_{\ell}(s, r_1, r_2)$ recovers the asymptotic signal for each s

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Case $\ell = 0$

Specializing to

$$(\partial_t^2 - \partial_r^2)\Psi = 0$$
 $W_0(sr_2) = W_0(sr_1) = 1$

This

$$\mathrm{e}^{s(r_2-r_1)}\widehat{\Psi}_0(s,r_2)=\left[rac{W_0(sr_2)}{W_0(sr_1)}-1
ight]\widehat{\Psi}_0(s,r_1)+\widehat{\Psi}_0(s,r_1)$$

becomes

$$\mathrm{e}^{s(r_2-r_1)}\widehat{\Psi}(s,r_2)=\widehat{\Psi}(s,r_1) \quad o \quad \Psi(t+(r_2-r_1),r_2)=\Psi(t,r_1)$$

Agrees with previous result obtained via d'Alembert's solution. What have we gained?

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Recall the product $\widehat{\Phi}_\ell(s,r_1,r_2)\widehat{\Psi}_\ell(s,r_1)$ becomes a time-domain convolution

$$\int_0^t \Phi_\ell(t-t',r_1,r_2)\Psi_\ell(t',r_1)dt'$$

where

$$\Phi_{\ell}(t, r_1, r_2) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \widehat{\Phi}_{\ell}(s, r_1, r_2) \mathrm{e}^{st} ds$$

Allowing the signal/waveform at r_2 to be written as
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Allowing the signal/waveform at r_2 to be written as

$$\Psi_{\ell}(t + (r_2 - r_1), r_2) = \int_0^t \Phi_{\ell}(t - t', r_1, r_2) \Psi(t', r_1) dt' + \Psi_{\ell}(t, r_1)$$

This *does* generalize to curved geometries *if* we can invert $\widehat{\Phi}_{\ell}(s, r_1, r_2)$

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Sum-of-poles representation

One can show

$$\widehat{\Phi}_{\ell}(s, r_1, r_2) = \frac{W_{\ell}(sr_2) - W_{\ell}(sr_1)}{W_{\ell}(sr_1)} = \sum_{k=1}^{\ell} \frac{\gamma_{\ell,k}}{s - b_{\ell,k}/r_1}$$

where $b_{\ell,k}$ are zeros of $z^{\ell}W_{\ell}(z)$. And so

$$\Psi_{\ell}(t+(r_{2}-r_{1}),r_{2})=\sum_{k=1}^{\ell}\gamma_{\ell,k}\int_{0}^{t}\mathrm{e}^{\frac{b_{\ell,k}}{r_{1}}(t-t')}\Psi_{\ell}(t',r_{1})dt'+\Psi_{\ell}(t,r_{1})$$

Exact expression for $\Psi_{\ell}(t + (r_2 - r_1), r_2)$ provided we know $b_{\ell,k}$ and $\gamma_{\ell,k}$.

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$$\Phi_{\ell}(t, r_1, r_2) = \sum_{k=1}^{\ell} \gamma_{\ell,k} \exp\left(\frac{b_{\ell,k}}{r_1}t\right)$$

Scaled zeros
$$\frac{b_{\ell,j}}{(\ell+1/2)}$$
 of $z^{\ell}W_{\ell}(z)$.
+ $z^{1}W_{1}(z) = z\left(1+\frac{1}{z}\right)$
 $\diamond z^{2}W_{2}(z) = z^{2}\left(1+\frac{3}{z}+\frac{3}{z^{2}}\right)$
 $\diamond z^{3}W_{3}(z) = z^{3}\left(1+\frac{6}{z}+\frac{15}{z^{2}}+\frac{15}{z^{3}}\right)$
 $\ast z^{4}W_{4}(z) = z^{4}\left(1+\frac{10}{z}+\frac{45}{z^{2}}+\frac{105}{z^{3}}+\frac{105}{z^{4}}\right)$



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Notice zeros in the left half-plane, finite number and no branch cuts.

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A summary for 3D wave equation (Minkowski)

- 1. Record (ℓ, m) multipoles $\Psi_{\ell m}(t, r_1)$ at a fixed r_1
- 2. Convolve each multipole $\Psi_{\ell m}(t, r_1)$ with time-domain kernel to recover

$$\Psi_{\ell m}(t+(r_2-r_1),r_2) = \sum_{k=1}^{\ell} \gamma_{\ell,k} \int_0^t e^{\frac{b_{\ell,k}}{r_1}(t-t')} \Psi_{\ell m}(t',r_1) dt' + \Psi_{\ell m}(t,r_1)$$

- ▶ $\Psi(t + (r_2 r_1), r_2)$ is the exact solution observed at $r_2 \leq \infty$
- Automatically includes all features of wave propagation
 - Picks out correct piece from ℓ terms with 1/r fall-off

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Perturbation equations (I)

Recall Einstein's equation $G_{\mu
u}(g_{lphaeta})=T_{\mu
u}$

- ► Assume a *background* solution $\hat{g}_{\alpha\beta}dx^{\alpha}dx^{\beta} = -fdt^2 + f^{-1}dr^2 + r^2d\Omega^2$, f = 1 - 2M/r.
- Assumption: small metric perturbations, $g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$.
- Decompose perturbation $h_{\alpha\beta}$ into (tensorial) multipoles
- Key insight: Introduce a "master function" $\Psi_{\ell m}(h_{\alpha\beta}^{\ell m})$
- $[\Psi, \Psi', \Psi'', \dot{\Psi}] \iff [h_{lphaeta}]$ which carry (ℓ, m) multipole labels

 $(-\partial_t^2 + \partial_x^2 - V)\Psi =$ possible source terms

with the tortoise coordinate $x = r + 2M \log(\frac{1}{2}r/M - 1)$

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Perturbation equations (II)

$$(-\partial_t^2 + \partial_x^2 - V)\Psi =$$
possible source terms

$$V^{\text{axial}}(r) = \frac{f(r)}{r^2} \left[\ell(\ell+1) - \frac{6M}{r} \right]$$
$$V^{\text{polar}}(r) = \frac{2f(r)}{(nr+3M)^2} \left[n^2 \left(1 + n + \frac{3M}{r} \right) + \frac{9M^2}{r^2} \left(n + \frac{M}{r} \right) \right]$$

with $n = \frac{1}{2}(\ell - 1)(\ell + 2)$

Focus on simpler case V^{axial} known as Regge-Wheeler equation.

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Relevant quantities

- From Ψ one can calculate...
 - Gravitational wave signal

$$h_{+}^{\ell m}+ih_{x}^{\ell m}=\frac{1}{2r}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\Psi^{\mathrm{Polar}}+i\Psi^{\mathrm{Axial}}\right]_{-2}Y^{\ell m}$$

Energy carried away by waves

$$\dot{E}_{\ell m} = rac{1}{64\pi} rac{(\ell+2)!}{(\ell-2)!} (|\dot{\Psi}_{\ell m}|^2), \qquad \dot{L}_{\ell m} = rac{\mathrm{i}m}{64\pi} rac{(\ell+2)!}{(\ell-2)!} (ar{\Psi}_{\ell m} \dot{\Psi}_{\ell m})$$

For circular orbits self-force quantities are possible too

$$\dot{E}_{p} = -\frac{1}{2u^{t}}u^{\alpha}u^{\beta}\frac{\partial h_{\alpha\beta}}{\partial t}, \qquad \dot{L}_{p} = \frac{1}{2u^{t}}u^{\alpha}u^{\beta}\frac{\partial h_{\alpha\beta}}{\partial \phi},$$

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The asymptotic waveform Ψ

- **Problem:** We need the gravitational waveform at large distances (black holes are in other galaxies)
- **Goal:** From a signal recorded at $r_b \approx 30 M$, recover the signal at (say) $r \approx 10^{15} M$
- **Preview:** Schematically similar to previous 3+1 wave equation



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Difficulties

Conceptual

- Backscattered wave seems to have "incoming" and "outgoing" pieces. How to separate these?
- ► Weak Huygen's principle: source/data at X₀ = (t₀, x₀, y₀, z₀) influences ALL points in future light cone

[Fig from Adam Pound, Capra 15]

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Mathematical

- No closed-form expressions for the outgoing solution
- Expect kernels to have a branch cut (similar to the Green's function)

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As for the Minkowski case (same steps)...

✓ Laplace transformed equation [say M = 1/2]

$$\left(-s^2+\partial_x^2-\frac{f(r)}{r^2}\left[\ell(\ell+1)-\frac{3}{r}\right]\right)\hat{\Psi}=0$$

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$$\left(-s^2+\partial_x^2-\frac{f(r)}{r^2}\left[\ell(\ell+1)-\frac{3}{r}\right]\right)\hat{\Psi}=0$$

 \checkmark Outgoing solution has the form

$$\widehat{\Psi}_\ell(s,r) = \mathsf{a}(s) e^{-sx} W_\ell(x,s), \qquad W_\ell(x,s) \mathop{\sim}\limits_{x o \infty} 1$$

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As for the Minkowski case (same steps)...

 \checkmark Laplace transformed equation [say M=1/2]

$$\left(-s^2+\partial_x^2-\frac{f(r)}{r^2}\left[\ell(\ell+1)-\frac{3}{r}\right]\right)\hat{\Psi}=0$$

 \checkmark Outgoing solution has the form

$$\widehat{\Psi}_\ell(s,r) = \mathsf{a}(s) e^{-sx} W_\ell(x,s), \qquad W_\ell(x,s) \mathop{\sim}\limits_{x o \infty} 1$$

Enacting teleportation/asymptotic-waveform evaluation

$$\mathrm{e}^{s(x_2-x_1)}\widehat{\Psi}_\ell(s,x_2) = \left[rac{W_\ell(x_2,s)}{W_\ell(x_1,s)} - 1
ight]\widehat{\Psi}_\ell(s,x_1) + \widehat{\Psi}_\ell(s,x_1)$$

...finding $W_{\ell}(x,s)$ and time-domain relationship requires something new

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Difficulties in computing " $\frac{W_{\ell}(x_2,s)}{W_{\ell}(x_1,s)} - 1$ " (1 slide only!)

- Need to evaluate $W_\ell(x,s)$ along the along path of inversion $s\in\mathrm{i}\mathbb{R}$
- Neither asymptotic expansion nor numerical integration are accurate enough.
- Solution: define an auxiliary variable $\hat{\Omega}_{\ell}$ for the logarithmic derivative of W which i) satisfies a Riccati equation; ii) can be accurately computed; iii) and

$$\frac{W_{\ell}(x_2,s)}{W_{\ell}(x_1,s)} = \exp\left[\int_{r_1}^{r_2} \frac{\widehat{\Omega}_{\ell}(s,\eta)}{\eta} d\eta\right]$$

Numerics: For each $\eta \in [r_1, r_2]$ we find $\widehat{\Omega}$ then perform above integral

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$\ell = 2, r = 30M \rightarrow 10^{15}M$ kernel evaluated along s = iy

Numerically computed
$$\widehat{\Phi}_2(s) = \frac{W_2(10^{15}M,s)}{W_2(30M,s)} - 1$$



- For 3D Minkowski equation we had a closed-form expression.
- Inverse Laplace transform known.
- How to invert $\widehat{\Phi}_2(s)$?

Fast asymptotic waveforms from gravitational perturbations

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Motivation

$$\widehat{\Phi}_2(s) pprox rac{\mathsf{degree } d-1 \text{ polynomial}}{\mathsf{degree } d \text{ polynomial}} = \sum_{i=1}^d rac{\gamma_i}{s - \beta_i}$$

- Rational approximation won't be accurate for all $s \in \mathbb{C}$
 - In fact VERY bad in the left half-plane where poles/branch cuts are located
- ► If rational approximation is accurate for s ∈ iℝ we can analytically perform the inversion!
- Theorem: Alpert, Greengard and Hagstrom (2002) showed this approximation is accurate and convergent for a wide class of kernels
- Our kernel is not obviously part of this class, but we can try and empirically check the accuracy

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Black box rational approximation

Look for the least squares solution

$$\min_{P,Q} \int_{a}^{b} \left| \frac{P(s)}{Q(s)} - \widehat{\Phi}_{2}(s) \right|^{2} ds$$

degree P = d - 1, degree Q = d

- ▶ Input: guess for degree d, polynomial Q and evaluations of $\widehat{\Phi}_2(s)$
- Output: the best d pole locations and strengths
- Typical pointwise relative error $10^{-12} \implies$ accuracy of teleported signal

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Output is a table of numbers

Pole #	Gamma strengths			
1	-1.7576263057e-08 +	0i		
2	-6.4180514293e-08 +	0i		
	17	More	Entires	He

Beta locations -5.4146529341e-01 + 0i -4.1310954989e-01 + 0i

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Output is a table of numbers



Equivalently



$$egin{aligned} \widehat{\Phi}_2(s) &\approx \sum_{i=1}^{19} rac{\gamma_i}{s-eta_i} \ \Phi_2(t) &pprox \sum_{i=1}^{19} \gamma_i \exp\left(eta_i t
ight) \end{aligned}$$

- Black crosses: Approximation to direct propagation
- Red circles: Approximation to branch cut

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Error control: can check that true kernel and sum-of-poles representation accurate on $s \in i\mathbb{R}$

Same information: table of numbers, pole location/strengths (frequency-domain), exponential strength/damping rate (time-domain)

Usage: if you have a time-series of RWZ data $\Psi(t, x_1)$ for some ℓ recorded at x_1 , teleport it to x_2

$$\Psi(t + (x_2 - x_1), x_2) = \sum_{k=1}^{\ell} \gamma_{\ell,k} \int_0^t e^{\beta_{\ell,k}(t-t')} \Psi(t', x_1) dt' + \Psi(t, x_1)$$

 \mathscr{I}^+ waveform: send $x_2 \to \infty$

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Key features of asymptotic waveform evaluation

- A free lunch if you didn't pay
 - Building a kernel table is costly
 - Processing data with a table is free $(\approx 1s)$
- A post-processing step on existing data (non-intrusive to code)
- Only 1 time-series at a fixed r needed (extrapolation requires many)
- ► If Ψ not a solution to the RWZ equation (estimated another way) one can still use kernels
 - would capture physics of wave propagation to the far-field

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Implementation with tables Results: Price tails Results: EMRIs

Outline

Introduction

Theoretical motivation of kernels

Implementation and results

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Implementation with tables Results: Price tails Results: EMRIs

Kernel tables

- Kernels computed using MPI and quad precision
- Once a table has been generated very easy to use
- All kernels are (or will be) available online

www.dam.brown.edu/people/sfield/KernelsRWZ www.math.unm.edu/~lau/KernelsRWZ

Experiment setup

 \blacktriangleright Consider the $\ell=2$ Regge-Wheeler equation and Gaussian initial data

$$\Psi = e^{-[2-x/(2M)]^2}, \quad \partial_x \Psi = \frac{4M-x}{2M^2}e^{-[2-x/(2M)]^2}, \quad -\partial_t \Psi = \Phi(0, r(x)).$$

• Ψ recorded at 30*M* and evaluated at 2*M* imes 10¹⁵

Expected late-time Price tails t^{-7} (fixed small x and $t \to \infty$) and t^{-4} (fixed large x and $t \to \infty$)

Test that method captures known physics unique to large x

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Recorded $\Psi(t, 20M)$ Evaluated $\Psi(t + 2M \times 10^{15}, 20M + 2M \times 10^{15})$



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Late-time decay of $\Psi(t + 2M \times 10^{15}, 20M + 2M \times 10^{15})$



The rate p for $\Psi_{\infty}(t) \propto t^{p}$ has been computed using logarithmic difference quotients based on $\partial_{\ln t} \ln |\Psi_{\infty}(t)|$

less than 5 seconds to generate (relative accuracy better than 12 digits)

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Setup

- ▶ Astrophysical sources where a compact object, $m_p < 30 M_{\odot}$, orbits a massive black hole $M > 10^5 M_{\odot}$. Require $\mu = m_p/M \ll 1$
- Distributional forcing terms due to the perturbing object

$$(-\partial_t^2 + \partial_x^2 - V(x))\Psi = G(x,t)\delta(x - x_p(t)) + F(x,t)\delta'(x - x_p(t))$$

- Expressions for source term known
- Small black hole m_p follows geodesic motion providing $x_p(t)$

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Setup

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- Expressions for source term known
- Small black hole m_p follows geodesic motion providing $x_p(t)$

Code

- RWZ equations solved with a multi-domain discontinuous Galerkin method, very similar to pseudo-spectral methods.
- Delta function located between domains and *exactly* handled with prescribed jump conditions

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Circular Orbits: $r_p = 10M$ (outer boundary at 30*M*)

$(\ell, m) = (2, 2)$ perturbations. Scale Ψ by $m_p << 1$

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Comparison with frequency domain results

Luminosities computed from $\Psi(t, r_b = 30M)$

т	Alg.	\dot{E}_{2m}^{∞}		Ĺ _{2m}	
1	FR	1.93160935116	$ imes 10^{-7}$	6.10828509933	$ imes 10^{-6}$
	AWE	1.93160935114	$ imes 10^{-7}$	6.10828509953	$ imes 10^{-6}$
2	FR	5.36879547910	$ imes 10^{-5}$	1.69776220056	$ imes 10^{-3}$
	AWE	5.36879547910	$ imes 10^{-5}$	1.69776220057	$ imes 10^{-3}$

Mode-by-mode $\ell = 2$ luminosities. For a particle of mass m_p these values should be scaled by m_p^2 . The table compares our asymptotic-waveform evaluation (AWE) results with frequency domain (FR) results. Thanks to Seth Hopper for generating these previously unpublished FR luminosity values.

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Junk radiation at \mathscr{I}^+

Trivial initial data cause "junk" radiation and thus energy luminosity errors



 $|\dot{E}_{22}^{\infty}(t) - \dot{E}_{22,\text{FR}}^{\infty}|/\dot{E}_{22,\text{FR}}^{\infty}$ (red line, r_{∞}) $|\dot{E}_{22}^{b}(t) - \dot{E}_{22}^{b}(6500M)|/\dot{E}_{22}^{b}(6500M)$ (black line, r_{b}), (black line, r_{b})

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Eccentric Orbit

- $\begin{pmatrix} \ell, m \end{pmatrix} = (2, 2) \\ M 1$
- M = 1
- eccentricity = 0.76412402
- semi-latus rectum = 8.75456059



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Outer boundary at 60M

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Luminosities computed from $\Psi(t, r_b = 60M)$

т	Alg.	\dot{E}_{2m}^{∞}		Ĺ _{2m}	
0	FR	1.27486196317	$ imes 10^{-8}$	0	
	AWE	1.27486196187	$ imes 10^{-8}$	0	
1	FR	1.15338054092	$ imes 10^{-6}$	1.44066000650	$\times 10^{-5}$
	AWE	1.15338054091	$ imes 10^{-6}$	1.44066000619	$ imes 10^{-5}$
2	FR	1.55967717209	$\times 10^{-4}$	2.07778922470	$\times 10^{-3}$
	AWE	1.55967717211	$ imes 10^{-4}$	2.07778922439	$ imes 10^{-3}$

Mode-by-mode $\ell = 2$ luminosities. FR results refer to Table III of Hopper and Evans (2011) and are quoted to a relative error of 10^{-12} .

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Final remarks

- Reformulated Abrahams-Evans method such that generalizations possible
- RWZ case similar to ordinary wave equation but requires more numerics
 - Rational approximation allows for a time-domain expression
- Computing a kernel table is hard, using it is easy
- Extremely efficient for time-domain solvers without sacrificing accuracy

Possible future work?

- \blacktriangleright Kernels/tables will be made available up to high ℓ
- Generalization to other wave equations, spheroidal wave equation, perturbations of Kerr
- Full numerical GR data

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Rational approximation of kernels

Given an exact kernel $\Phi_{\ell}(t, r_1, r_2)$ and its rational approximation $\phi_{\ell}(t, r_1, r_2)$ we have the following long-time error bound

$$\begin{split} \|\Phi_{\ell}(\cdot,r_{1},r_{2})*\Psi_{\ell}(\cdot,r_{1})-\phi_{\ell}(\cdot,r_{1},r_{2})*\Psi_{\ell}(\cdot,r_{1})\|_{L_{2}(0,\infty)} \\ &\leq \sup_{s\in\mathrm{i}\mathbb{R}}\frac{|\widehat{\phi}(s,r_{1},r_{2})-\widehat{\Phi}(s,r_{1},r_{2})|}{|\widehat{\Phi}(s,r_{1},r_{2})|}\|\Phi_{\ell}(\cdot,r_{1},r_{2})*\Psi_{\ell}(\cdot,r_{1})\|_{L_{2}(0,\infty)}, \end{split}$$

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Typical pointwise relative error

$$\sup_{s \in i\mathbb{R}} \frac{|\widehat{\phi}(s, r_1, r_2) - \widehat{\Phi}(s, r_1, r_2)|}{|\widehat{\Phi}(s, r_1, r_2)|} \leq 10^{-12}$$

achieved (computed on a dense s-grid).

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Computing the kernel " $\frac{W_{\ell}(x_2,s)}{W_{\ell}(x_1,s)} - 1$ "

Need to evaluate $W_\ell(x,s)$ along the along path of inversion $s\in \mathrm{i}\mathbb{R}$

Neither asymptotic expansion nor numerical integration were found to be suitable.

Instead

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Computing the kernel " $\frac{W_{\ell}(x_2,s)}{W_{\ell}(x_1,s)} - 1$ "

Need to evaluate $W_\ell(x,s)$ along the along path of inversion $s\in \mathrm{i}\mathbb{R}$

Neither asymptotic expansion nor numerical integration were found to be suitable.

Instead define $\hat{\Omega} = (sx) \frac{\partial_x W(x,s)}{W(x,s)}$ to give

$$\frac{W_{\ell}(x_2,s)}{W_{\ell}(x_1,s)} = \exp\left[\int_{r_1}^{r_2} \frac{\widehat{\Omega}_{\ell}(s,\eta)}{\eta} d\eta\right]$$

The auxiliary variable $\hat{\Omega}$ solves a non-linear ODE, accurate to high ℓ Numerically solve for $\hat{\Omega}$ then compute kernel

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$\ell = 2$ auxiliary kernel evaluated along s = iy



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ole #	Gamma strengths	Beta locations
1	-1.7576263057e-08 + 0i	-5.4146529341e-01 + 0i
2	-6.4180514293e-08 + 0i	-4.1310954989e-01 + 0i
3	-6.2732971050e-06 + 0i	-3.1911338482e-01 + 0i
4	-6.9363117988e-05 + 0i	-2.4711219871e-01 + 0i
5	-5.7180637750e-04 + 0i	-1.9108163722e-01 + 0i
6	-2.7884247577e-03 + 0i	-1.4749601558e-01 + 0i
7	-5.8836792033e-03 + 0i	-1.1366299945e-01 + 0i
8	-3.6549136132e-03 + 0i	-8.6476935381e-02 + 0i
9	-1.0498746767e-03 + 0i	-6.4512065175e-02 + 0i
10	-2.4204781878e-04 + 0i	-4.7332374442e-02 + 0i
11	-5.5724464176e-05 + 0i	-3.4115775484e-02 + 0i
12	-1.2157296793e-05 + 0i	-2.4048935704e-02 + 0i
13	-2.6651813247e-06 + 0i	-1.6468632919e-02 + 0i
14	-4.8661708981e-07 + 0i	-1.0845690423e-02 + 0i
15	-8.6183677612e-08 + 0i	-6.7552918597e-03 + 0i
16	-9.3735071189e-09 + 0i	-3.8525630196e-03 + 0i
17	-8.7881787023e-10 + 0i	-1.8481215040e-03 + 0i
18	-9.1164536027e-02 -5.3953709155e-02i	-9.4779490815e-02 +5.9927979877e-02i
19	-9.1164536027e-02 +5.3953709155e-02i	9.4779490815e-02 -5.9927979877e-02i
	$\widehat{\Phi}_2(s) pprox \sum_{i=1}^{19} rac{\gamma_i}{s-eta_i} o$	$\Phi_2(t) \approx \sum_{i=1}^{19} \gamma_i \exp\left(\beta_i t\right)$

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Signal teleportation

Experiment setup

- Finite boundary to finite boundary location (RW potential)
- "bump" function with support $-10M < r_* < 3M$
- Record $\Psi(t, 480M)$ as a time-series
- Record $\Psi(t, r_1)$ as a time-series at some location $r_1 < r_2$
- Find $\Psi(t, r_2)$ by convolving $\Psi(t, r_1)$ with a teleportation kernel

Clean test of method error

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Teleportation from $r \rightarrow 480M$

Plot shows difference $|\Psi(t, 480M) - \Psi(t, r_1 \rightarrow 480M)|$



This is for a high $\ell = 64$ solution.