A reduced basis representation for chirp and ringdown gravitational wave templates

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Motivation

- **Matched filtering:** gravitational waveform templates from a catalog are correlated with detector data
- Templates chosen so that few signals are "missed"
- *Minimal match* (MM) characterizes the event loss for a given catalog
- Many templates needed, scales like $(1 - MM)^{-(\text{Parameter dimension})/2}$

Desire for...

- Faster and cheaper searches, generate alerts for EM counterparts (LLOID [1]), use higher dimensional waveforms, reduced burden of overlap computation for parameter estimation

[1] K. Cannon et al., Toward Early-Warning Detection of Gravitational Waves from Compact Binary Coalescence, arXiv:1107.2665v1.

Reduced basis method

Highlights include...

- Generation of an accurate and compact reduced basis space of waveforms
- Significantly fewer templates (basis) for a given MM
- Selection of nearly optimal parameter points
- Non–linear "space of waveforms" can be represented as a small linear space with arbitrarily high accuracy

The reduced basis space and an algorithm to find it

Problem statement

- Given: P parameters $\vec{\mu} = \{\mu_1, \ldots, \mu_P\}$ and the space of all normalized waveforms $\mathcal{H}$
  - Chirp, ringdown, reduced models, analytic waveforms, etc.
- Each waveform is denoted $h_{\vec{\mu}} \in \mathcal{H}$

GOAL:

- Find an N dimensional linear space W_N to accurately represent $\mathcal{H}$
  - Ansatz: Basis vectors of W_N are waveforms chosen from $\mathcal{H}$
- W_N is called the *reduced basis space*

When should we look for a reduced basis space?

- Projection operator $P_N : \mathcal{H} \to W_N$ for an orthonormal basis e_i of W_N

$$P_N h \equiv \sum_{i=1}^{N} < e_i, h > e_i \qquad < g, f > \equiv \int_a^b \frac{g f^*}{S_n(f)} df$$

- **Remark:** $P_N h$ is the best approximation to h
- Kolmogorov N–width specifies error for an optimal W_N

$$d_N(\mathcal{H}) = \min_{W_N} \max_{\vec{\mu} \in \mathcal{H}} ||P_N h_{\vec{\mu}} - h_{\vec{\mu}}||$$

- If solutions have smooth dependence on parameters one expects

$$d_N(\mathcal{H}) \leq A e^{-bN}$$

A practical approach to finding W_N

Finding the best space is computationally challenging, so...

1. Sample $\mathcal{H}$ at a finite set of (*training space*) points $\mathcal{T}$
   - Now we seek to approximate $\mathcal{H}_{\mathcal{T}} = \{h_{\vec{\mu}} \in \mathcal{H} : \vec{\mu} \in \mathcal{T}\}$ by W_N
2. Build W_N by solving N easy problems (Greedy approach)
   - Suppose we have W_i. The algorithm optimally chooses W_{i+1} and continues to W_N
   - Sequence of hierarchical spaces are constructed $W_1 \subset W_2 \subset ... \subset W_N$

- W_N nearly optimal[2] compared to W_N^{Kol}. If N-width decays exponentially so does the approximation error for W_N.

[2] P. Binev et al., Convergence rates for greedy algorithms in reduced basis methods.

Greedy algorithm (setup)

- Choose a parameter and waveform space (continuous and discrete)
- Initialize reduced basis with choice of $\vec{\mu}_1$ and thus $W_1 = \mathrm{span}\left(\{h_1\}\right)$

To go from W_i to W_{i+1}...

Greedy algorithm

- The *greedy error* $\varepsilon_i \equiv \max_{\vec{\mu}\in\mathcal{T}} ||h_{\vec{\mu}} - P_i h_{\vec{\mu}}||$

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$i \to i + 1$

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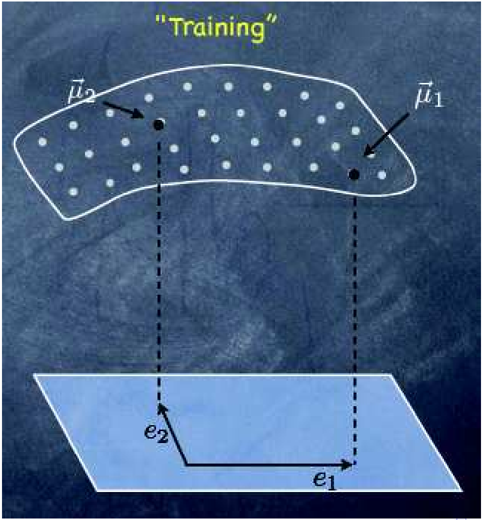
- W_N approximates $\mathcal{H}_{\mathcal{T}}$ with an error of better than Tol
- Outputs a collection of points $\{\vec{\mu}_i\}_{i=1}^N$ and corresponding waveforms $\{h_i\}_{i=1}^N$ such that $W_N = \mathrm{span}\left(\{h_i\}_{i=1}^N\right)$

Greedy construction



Remarks

- Straightforward to implement
- Constructing W_N is $O(N)$
- Resulting reduced basis is an "application–specific spectral basis"
- Simplest matched filtering search between the signal s and every member of the training space

$$< s, P_N h_j > = \sum_{i=1}^{N} \langle s, e_i \rangle \langle e_i, h_j \rangle$$

- Since $\langle e_i, h_j \rangle$ has been precomputed offline, filtering now involves significantly fewer integrals

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Chirp Waveforms

- 2PN stationary-phase-approximation chirp waveforms
- 2 dimensional parameter space described by the compact objects' masses, m_1 and m_2
- Recall total mass $M = m_1 + m_2$ and symmetric mass ratio $\eta = m_1 m_2 / M^2$
- Dense training space where points are uniformly spaced in m_1 and m_2

Results for [1-3]$M_\odot$ with Initial LIGO

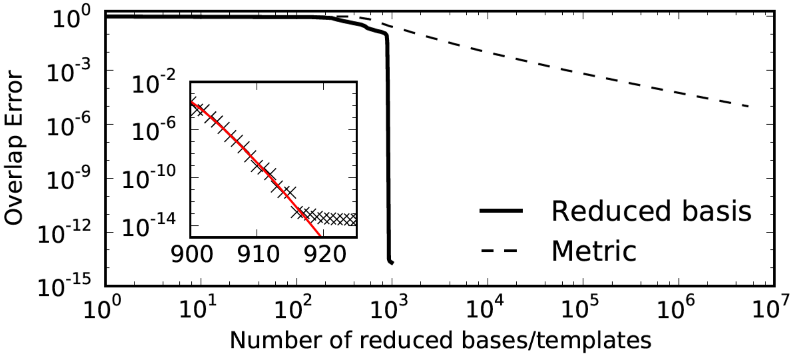


Figure: Expected exponential convergence of algorithm. Same generic feature for all mass ranges considered.

Distribution of selected parameters

[1-3]$M_\odot$ with Initial LIGO (ChirpM $= \eta^{3/5} M$)

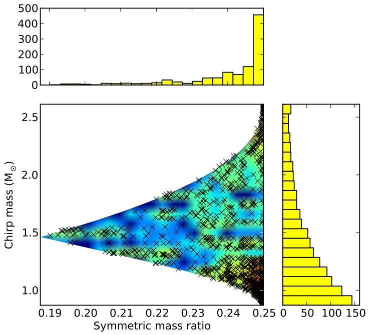


Figure: Reduced basis with $MM = 1 - 10^{-12}$

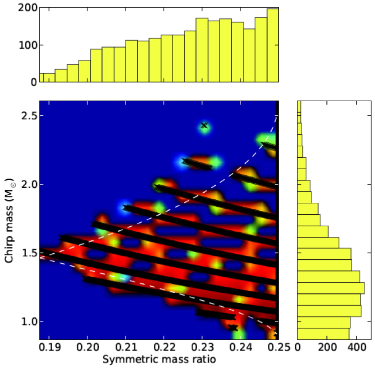


Figure: Metric placement with $MM = .97$

Results

Table: Number of reduced basis ($N_{\rm RB}$) and metric placed ($N_{\rm metric}$) waveforms for different minimal matches $\rm MM$.

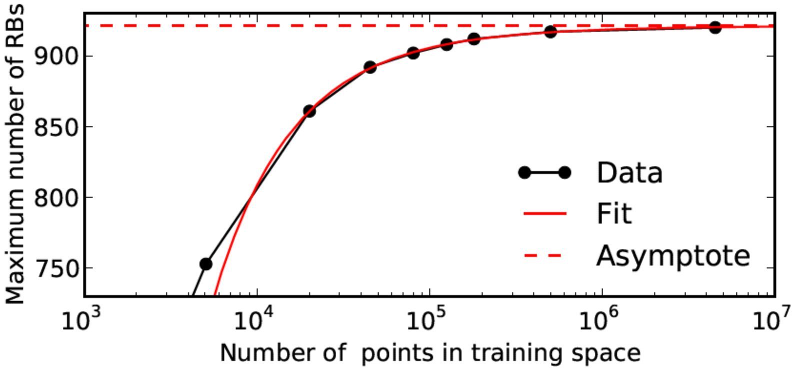
Detector	$1-\rm MM$	BBH		BNS	
		$N_{\rm RB}$	$N_{\rm metric}$	$N_{\rm RB}$	$N_{\rm metric}$

Results

Table: Number of reduced basis (N_{RB}) and metric placed (N_{metric}) waveforms for different minimal matches MM.

Detector	$1-\mathrm{MM}$	BBH		BNS	
		N_{RB}	N_{metric}	N_{RB}	N_{metric}
AdvLIGO	10^{-2}	$1,058$	$19,336$	$5,395$	$72,790$
	10^{-5}	$1,687$	1.1×10^{7}	$8,958$	3.2×10^{7}
	2.5×10^{-13}	$1,700$	8.0×10^{13}	$8,976$	1.4×10^{14}
AdvVirgo	10^{-2}	$1,395$	$42,496$	$7,482$	$156,127$
	10^{-5}	$1,690$	3.2×10^{7}	$8,960$	2.6×10^{7}
	2.5×10^{-13}	$1,703$	1.4×10^{14}	$8,977$	2.9×10^{14}

Results for [1-3]$M_\odot$ with Initial LIGO



Asymptotic result $\dim(W_N) = 921$. Confirms expectation that we are converging to finite dimensional space for fixed error tolerance.

Ringdown Waveforms

- Superposition of quasi-normal (i.e. exponentially damped) modes
- Dimension of parameter space is...
  - 2 for dominate $\ell = m = 2$ one-mode waveform: central frequency f_{22} and quality factor Q_{22}
  - 5 for two-mode waveform: central frequencies f_{22} and f_{33}, quality factors Q_{22} and Q_{33}, and relative amplitude $\mathcal{A}$
  - 3 for two-mode waveform constrained by general relativity as $f_{33}(f_{22}, Q_{22})$ and $Q_{33}(f_{22}, Q_{22})$
- Training space given by metric placement algorithm and $MM = .99$

Single mode: convergence and initialization

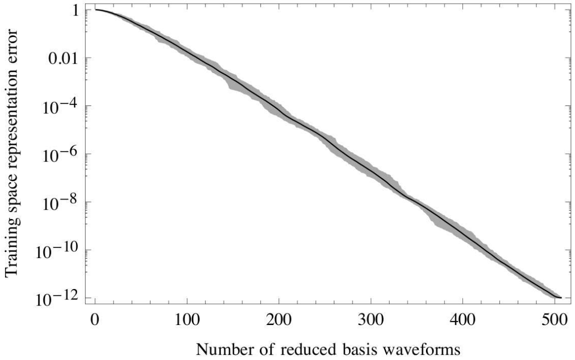


Figure: Greedy error as a function of the number of reduced basis waveforms for *all* possible initializations. The dark line shows the average and the shaded area the maximum dispersion around it. No fine tuning is needed!

Constrained two-mode: Selected parameters

Waveform given by $h = \mathcal{C}\left[(1-\mathcal{A})h_{220} + \mathcal{A}h_{330}\right]$

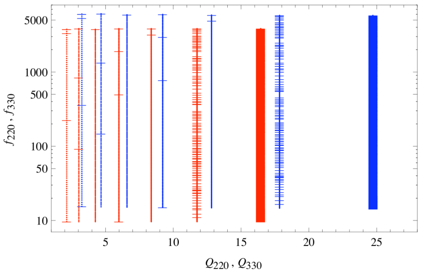


Figure: The metric-based training space (points) and parameter values selected by the greedy algorithm (bars), with (2, 2, 0) mode (red) and (3, 3, 0) mode (blue).

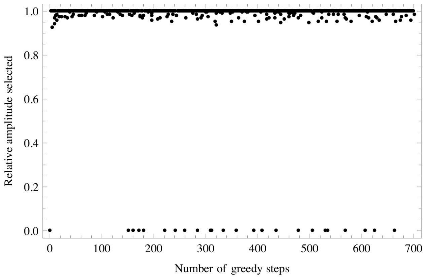


Figure: Values for the relative amplitude parameter A selected by the greedy algorithm. 1,000 samples were used for $\mathcal{A} \in [0, 1]$.

Constrained two-mode: Monte Carlo study of errors

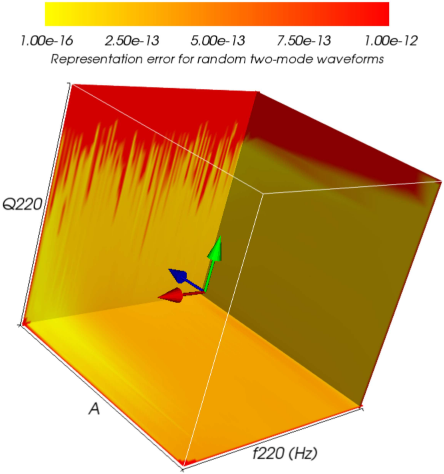


Figure: Error in representing *any* template (i.e. outside of the training space) using reduced basis waveforms. Axes defined by
$\mathcal{A} \in [0, 1]$
$f_{22} \in [10, 4000] Hz$
$Q_{22} \in [2.1187, 20]$

$1 - \mathrm{MM}$	2-mode, GR		2-mode	
	N_{metric}	N_{RB}	N_{metric}	N_{RB}
0.03	3.5×10^3	737	3.4×10^6	1,198
10^{-2}	1.8×10^4	751	5.3×10^7	1,237
10^{-3}	5.8×10^5	958	1.9×10^{10}	1,495
10^{-4}	1.8×10^7	1,007	5.3×10^{12}	1,567
10^{-5}	5.8×10^8	1,018	1.9×10^{15}	1,590

Table: Number of reduced basis waveforms (N_{RB}) needed to represent 2-mode training spaces with $(\ell, m, n) = (2,2,0)$ and $(3,3,0)$ for different minimal matches MM. Values of N_{metric} in the second column courtesy of V. Cardoso.

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- Demonstrated the space of waveforms can be **accurately represented** by a compact set of basis function
- Proposed an efficient algorithm for finding this space
- Applied the algorithm to chirp and multi-mode ringdown waveforms
    - Selects most relevant parameter points
- Significant compressions, especially high dimensional ones like multi-mode ringdown

Current work includes...

- Interpolation
- Waveforms with spin (anti-)aligned with orbital momentum

QUESTIONS?