Generalized Discontinuous Galerkin Scheme for Accurate Modeling of Binary Black Holes

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Class. Quant. Grav. (2009, 2013), Phys. Rev. D (2010) (also arXiv.org)

Outline

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Problem motivation

Numerics: Scheme, boundary conditions, asymptotic signal

Results

Broadly speaking...

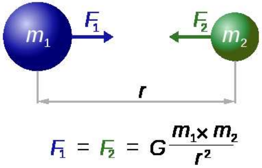
Gravitational wave astronomy: Observation of gravitational waves and parameter estimation

Gravitational wave physics: Modeling expected gravitational wave signals (PDEs, ODEs, closed-form expressions)

Computational relativity: Compute a gravitational wave signal given i) some model (e.g. PDEs such as Einstein's equation) plus ii) information about sources (e.g. 2 orbiting neutron stars)

First, what is gravity? Newton's answer

Gravitational potential: given by Poisson's equation $\nabla^2 \phi_{\rm grav} = 4\pi G \rho$
Gravitational force: produced by masses $F_{\rm grav} = m_1 \nabla \phi_2 = G \frac{m_1 m_2}{r^2} \hat{r}$

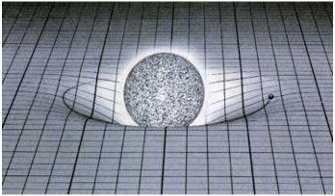


Mechanics: force changes motion $F_{\rm grav} = m_1 a_1 = m_1 \ddot{x}_1$
Gravitational waves? No, Poisson's equation *instantaneously* gives $\phi_{\rm grav}$ for the distribution ρ

First, what is gravity? Einstein's answer

Bending of spacetime: Given by Einstein's equation

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi \frac{G}{c^4} T_{\alpha\beta}$$



- $G_{\alpha\beta}\left(g_{\mu\nu}\right)$ is second order PDE for $g_{\alpha\beta}$
- Stress-energy tensor $T_{\alpha\beta}$ contains all matter fields (like m_1 and m_2)
- Solve for $g_{\alpha\beta}$, determines geometry (measurements of distances and durations)

Mechanics: Objects move according to geodesic equation absent of forces

Gravitational waves

Gravitational Waves? Yes! The solution $g_{\alpha\beta}$ obeys a finite speed of propagation. These radiative solutions are driven by moving masses.

- Observers on Earth will measure these solutions as "small" metric fluctuations, a stretching and squeezing of space

$$g_{\alpha\beta} = g_{\alpha\beta}^{\mathrm{Earth}} + h_{\alpha\beta} \implies \left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right) \hat{h}_{\alpha\beta} = 0$$

- 2 physical radiative degrees of freedom

$$h_{xy} = h_{yx} = h_{\times} \sin(\omega t - kz) \qquad h_{xx} = -h_{yy} = h_{+} \sin(\omega t - kz)$$

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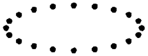
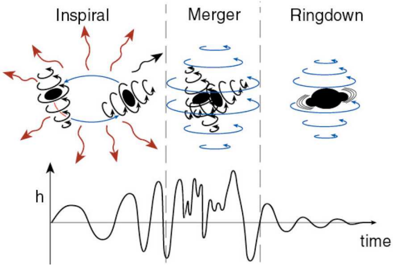
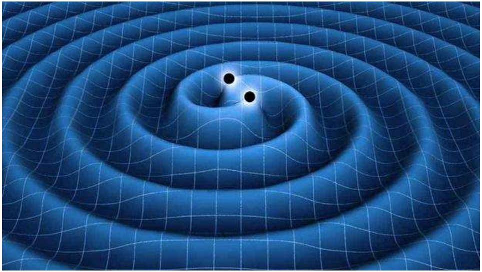


Astrophysical gravitational wave sources

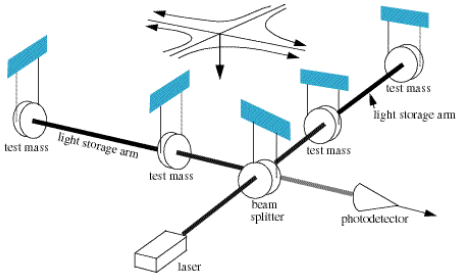
- Pair of orbiting black holes and/or neutron stars which inspiral, merge, and ringdown
- Observed GWs depend on the parameters of the binary system, and the objects' masses (2 parameters) are very important

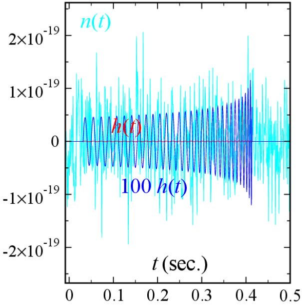




Gravitational Wave detectors

- A passing gravitational wave causes a path length change ΔL in the interferometer's arm L. Detector measures $h_{\alpha\beta} \propto \frac{\Delta L}{L} \leq 10^{-20}$





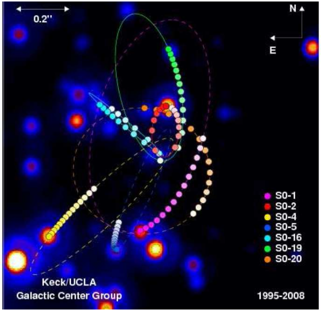
[1]

Requires inner product of data with templates (matched filtering)

[1] Fig. by Lee Lindblom

Extreme mass ratio binaries = EMRB

- We focus on astrophysical sources where a compact object, m_p, orbits a "massive" blackhole, M. Require $\mu = m_p/M \ll 1$
- Supermassive $M > 10^5\ M_\odot$ and stellar sized $m_p < 30 M_\odot$ black holes
- (Currently) impossible to model EMRBs with full GR equations due to disparity of length scales (open problem!).



Perturbation equations (I)

Recall Einstein's equation $G_{\mu\nu}(g_{\alpha\beta}) = 8\pi\frac{G}{c^4}T_{\mu\nu}$

- Assume a *background* solution (i.e. spacetime metric) $\hat{g}_{\alpha\beta}dx^\alpha dx^\beta = -f dt^2 + f^{-1}dr^2 + r^2 d\Omega^2$, $f = 1 - 2M/r$.
- Assumption: small mass m_p causes small metric perturbations, $g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$.
- Stress energy tensor $T^{\mu\nu} = m_p \int d\tau(-g)^{-1/2}u^\mu u^\nu \delta^4(x - z(\tau))$
- Linearized Einstein equations...

Perturbation equations (II)

- Decompose perturbation equations into multipoles $\rightarrow$ 16 coupled PDE for each multipole
- **Key insight:** Introduce a "master function" $\Psi(h_{\alpha\beta})$

$$(-\partial_t^2 + \partial_x^2 - V(x))\Psi = G(x,t)\delta(x - x_p(t)) + F(x,t)\delta'(x - x_p(t))$$

- Potential V encodes supermassive black hole M, source terms encode small object m_p
- Caveat: tortoise coordinate $x = r + 2M\log(\frac{1}{2}r/M - 1)$
- Metric perturbations can be reconstructed everywhere
  - $[\Psi, \Psi', \Psi'', \dot{\Psi}] \iff [h_{\alpha\beta}]$ which carry (ℓ, m) multipole labels

Relevant quantities

From Ψ one can calculate...

- Gravitational wave signal

$$h_{+}^{\ell m} + i h_{x}^{\ell m} = \frac{1}{2r}\sqrt{\frac{(\ell+2)!}{(\ell-2)!}}\left[\Psi^{\mathrm{Polar}} + i\Psi^{\mathrm{Axial}}\right]{}_{-2}Y^{\ell m}$$

- Energy carried away by waves

$$\dot{E}_{\ell m} = \frac{1}{64\pi}\frac{(\ell+2)!}{(\ell-2)!}\left(\left|\dot{\Psi}_{\ell m}\right|^2\right), \qquad \dot{L}_{\ell m} = \frac{\mathrm{i}m}{64\pi}\frac{(\ell+2)!}{(\ell-2)!}\left(\bar{\Psi}_{\ell m}\dot{\Psi}_{\ell m}\right)$$

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Just a 1D wave-like equation?

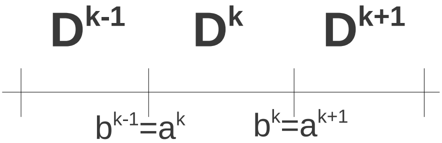
- Large errors due to distributional source terms
  - Previous methods approximate sources (e.g. by narrow Gaussian)
  - Our method effectively removes the particle. No accuracy loss
- Smooth fields to left and right of particle should be exploited
  - Previous methods use finite difference
- Applications require long time evolutions and good phase resolution
  - Our method is high order (similar to spectral element)
- Finite computational domain – artificial reflections and inaccurate waveforms
  - We employ *exact* outgoing BCs and waveform extraction techniques

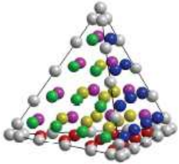
Discontinuous Galerkin Methods

Recipe for a DG scheme in 4 steps...

DG method: space (step 1 of 4)

- Approximate physical domain Ω by local subdomains D^k such that $\Omega \sim \Omega_h = \cup_{k=1}^{K} D^k$
- In general the grid is unstructured. We choose lines, triangles, and tetrahedrons for 1D, 2D, and 3D respectively.




[2]

[2]Figures from Jan Hesthaven's online lectures

DG Method: Solution (step 2 of 4)

- Local solution expanded in set of basis functions

$$x \in D^k : \Psi_h^k(x,t) = \sum_{i=0}^{N} \Psi_h^k(x_i,t) l_i^k(x)$$

- Numerical solution is a polynomial of degree at most N on D^k.
- Global solution is a direct sum of local solutions

$$\Psi_h(x,t) = \bigoplus_{k=1}^{K} \Psi_h^k(x,t)$$

- Solutions double valued along point, line, surface.

DG Method: Residual (step 3 of 4)

- Suppose our PDE is of the form $L\Psi = \partial_t \Psi + \partial_x f(\Psi) + V\Psi = 0$, where Ψ and f are vectors, and V a matrix.
- Integrate the residual $L\Psi_h$ against all basis functions D^k

$$\int_{D^k} (L\Psi_h)\, l_i^k(x) dx = 0 \qquad \forall i \in [0, N]$$

- We still must couple the subdomains D^k to one. Our choices will determine the scheme's stability...

DG method: Numerical flux (step 4 of 4)

- To couple elements first perform IBPs

$$\int_{D^k} \left(l_i^k \partial_t \Psi_h - f\left(\Psi_h\right) \partial_x l_i^k + V \Psi_h l_i^k \right) dx = - \oint_{\partial D^k} l_i^k \hat{n} \cdot f^* \left(\Psi_h\right)$$

where the *numerical* flux is $f^* \left(\Psi_h\right) = f^* \left(\Psi^+, \Psi^-\right)$

- Ψ^+ and Ψ^- are the solutions exterior and interior to subdomain D^k, restricted to the boundary
- **Example**: Central flux $f^* = \frac{f(\Psi^+)+f(\Psi^-)}{2}$
- Passes information between elements, implements boundary conditions, and ensures stability of scheme
- Choice of f^* is, in general, problem dependent

Summary so far

We now have a useful numerical scheme. For sufficiently smooth solutions the error decays like

$$\|\Psi - \Psi_h^k\|_{\mathsf{D}^k} \leq C(t)\left(|\mathsf{D}^k|\right)^{N+1}$$

What about the δ-type source terms?

Discontinuous Galerkin Method: the δ

- Generalized dG (GDG) Method extends dG to solutions (analytically) discontinuous at an interface[3]
- **Key idea:** treat the δ function as an additional numerical flux term
  - Let the global test function be $v(x) = \bigoplus_{i=1}^{K} v^i(x)$ and require the usual δ property over Ω

$$\int_\Omega \delta(x) v(x) dx = v(0)$$

  - Freedom to choose how to "split it" between adjacent elements

$$\int_{D^k \cup D^{k+1}} \delta(x) v(x) dx =$$

$$\int_{D^k} \delta(x) v^k(x) dx + \int_{D^{k+1}} \delta(x) v^{k+1}(x) dx = a v^k(0) + b v^{k+1}(0) = v(0)$$

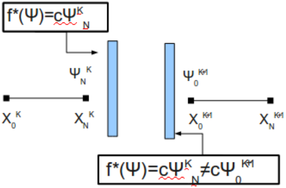
[3] K. Fan, W. Cai, X. Ji. J. Comp. Phys., 227 (2008) 2387-2410.

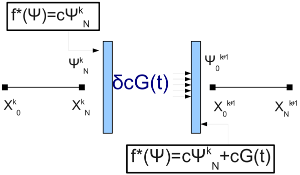
Discontinuous Galerkin Method: the δ

For hyperbolic problems we find the splitting is motivated by how information is flowing. Consider

$$\frac{1}{c}\partial_t \Psi + \partial_x \Psi = G(t)\delta(x)$$

A standard numerical flux choice is upwinding, given schematically by





- What about our problem?

As a first order system (I)

Specializing to circular orbits ($x_p(t) = x_p$)

Compute the jumps...

$$(-\partial_t^2 + \partial_x^2 - V)\Psi = G(x,t)\delta(x - x_p) + F(x,t)\delta'(x - x_p)$$

$$[[\Psi]](t) \equiv \lim_{\epsilon \to 0^+} \left[\Psi(t, x_p + \epsilon) - \Psi(t, x_p - \epsilon)\right]$$

$$[[-\partial_t \Psi]]_{x_p} = J_\Pi(t; G, F) \qquad [[\partial_x \Psi]]_{x_p} = J_\Phi(t; G, F)$$

...suggesting the first order system [recall $\partial_x H(x) = \delta(x)$]

$$\begin{aligned} \partial_t \Psi &= -\Pi \\ \partial_t \Pi &= -\partial_x \Phi + V\Psi + J_\Phi \delta(x - x_p) \\ \partial_t \Phi &= -\partial_x \Pi + J_\Pi \delta(x - x_p), \end{aligned}$$

As a first order system (II)

Notice that

$$\begin{aligned}\partial_t \Psi &= -\Pi \\ \partial_t \Pi &= -\partial_x \Phi + V\Psi + J_\Phi \delta(x - x_p) \\ \partial_t \Phi &= -\partial_x \Pi + J_\Pi \delta(x - x_p),\end{aligned}$$

is equivalent to the original system

- Subject to the constraint $\Phi = \partial_x \Psi - [[\Psi]]\delta(x - x_p)$
- $\Phi - \partial_x \Psi = 0$ away from x_p – estimate of method error

GDG for the first order system

To incorporate the effect of the δ functions for the system

1. Diagonalize ($W = -\Pi - \Phi$ and $X = -\Pi + \Phi$)

$$
\begin{aligned}
\partial_t \Psi &= \frac{1}{2}(W + X) \\
\partial_t W &= -\partial_x W - V\Psi - (J_\Phi + J_\Pi)\delta(x - x_p) \\
\partial_t X &= \partial_x X - V\Psi + (J_\Pi - J_\Phi)\delta(x - x_p),
\end{aligned}
$$

2. “2 copies of advection equation”: Perform δ splitting according to characteristics
3. Transform back to system (Ψ, Π, Φ) variables

Summary of Scheme

- On each subdomain we interpolate with Lagrange polynomials at Legendre-Gauss-Lobatto nodal points
- In the implementation of a dG scheme we compute and store local mass and stiffness matrices

$$M_{ij}^k = \int_{D^k} l_i^k(x) l_i^k(x) \qquad S_{ij}^k = \int_{D^k} \frac{\partial l_i^k(x)}{\partial x} l_i^k(x)$$

- Upwind numerical flux is chosen, that is we pass information along characteristics
- δ's are split according to direction of characteristics
- Timestep with a classical 4^{th} order Runge-Kutta

Boundary Conditions

- Goal: non-reflecting BC, as if no boundary at all
- Reduces domain size, especially useful for long evolutions
- Sommerfeld BC works well near BH horizon as $V \sim 0$
- At the right boundary Sommerfeld fails as $V \sim r^{-2}$, instead consider

$$(\partial_t + \partial_x)\,\Psi = F(t, x_b, \Psi, V)$$

Brief history of exact boundary conditions

- Marcus Grote and Joseph Keller derived exact nonreflecting boundary condition for 3D wave equation (1995)
- Bradley Alpert, Leslie Greengard and Thomas Hagstrom showed how to "compress" these boundary kernels (2002)
- Stephen Lau generalized to wave propagation on curved geometry with AGH compression (2005)

Example: ordinary wave equation

We wish to solve...

$$(-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2)\psi = 0$$

Problem posed on spatially unbounded domain and with compactly supported initial data.

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We actually solve...

- For computational reasons the problem is solved on a spatially finite domain
- Outer *computational* boundary is a sphere located at $r = r_b$

GOAL: mimic open space problem by i) supplying correct non-reflecting boundary conditions and ii) recovering solution which escapes to infinity.

Example: ordinary wave equation (outgoing solutions)

- Flatspace wave equation for spherical harmonic modes:

$$\psi = \sum_{\ell m} \frac{1}{r} \Psi_{\ell m}(t,r) Y_{\ell m}(\theta,\phi) \rightarrow \quad \left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{r^2} \right] \Psi_{\ell m} = 0$$

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- Laplace transformed solution $\hat{\Psi}_{\ell m}(s,r) = \int_0^\infty \Psi_{\ell m}(t,r) \mathrm{e}^{-st} dt$ solves

$$\left[s^2 - \frac{\partial^2}{\partial r^2} + \frac{\ell(\ell+1)}{r^2} \right] \hat{\Psi}_{\ell m} = \frac{\partial \Psi_{\ell m}}{\partial t}(0,r) + s\Psi_{\ell m}(0,r)$$

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- General outgoing solution: $\boxed{\widehat{\Psi}_\ell(s,r) = a(s) s^\ell e^{-sr} W_\ell(sr)}$
- $W_\ell(sr) = (sr)^{-\ell} \sum_{k=0}^{\ell} c_{\ell k} (sr)^k$
  - Example $W_2(sr) = (sr)^{-2}\left[3 + 3sr + (sr)^2\right]$

Example: ordinary wave equation (BCs)

- We supply 1 piece of information: $(\partial_t + \partial_r)\Psi_\ell = ???$
- Apply Sommerfeld operator $s + \partial_r$ to $\widehat{\Psi}_\ell(s,r) = a(s)s^\ell e^{-sr} W_\ell(sr)$

$$s\widehat{\Psi}_\ell(s,r) + \partial_r \widehat{\Psi}_\ell(s,r) = \frac{1}{r}\left[sr\frac{W_\ell'(sr)}{W_\ell(sr)}\right]\widehat{\Psi}_\ell(s,r)$$
$$= \frac{1}{r}\left[\sum_{k=1}^{\ell} \frac{b_{\ell,k}/r}{s - b_{\ell,k}/r}\right]\widehat{\Psi}_\ell(s,r) \equiv \frac{1}{r}\widehat{\Omega}_\ell(s,r)\widehat{\Psi}_\ell(s,r)$$

- $b_{\ell,k}$ are zeros of $W_\ell(b_{\ell,k}) = 0$
- $\widehat{\Omega}_\ell(s,r)$ is the boundary kernel – evidently a sum-of-poles

Example: ordinary wave equation (BCs)

Using well known properties of inverse Laplace transforms...

$$\boxed{\partial_t \Psi_\ell + \partial_r \Psi_\ell = \frac{1}{r} \int_0^t \Omega_\ell(t - t', r) \Psi_\ell(t', r) dt'}$$

where $\Omega_\ell(t, r) = \sum_{k=1}^{\ell} \frac{b_{\ell,k}}{r} \exp\left(\frac{b_{\ell,k} t}{r}\right)$.

Observations

- Exact outgoing boundary condition in time domain at any r_b
- Numerical solution computed with boundary at r_b and ∞ are *identical*

BC for EMRB equations

Similar to ordinary wave equation but with extra complications[4]

1. Numerically compute $\widehat{\Omega}_\ell(s, x_b; V)$ where $(s + \partial_x)\,\Psi = (1/r_b)\widehat{\Omega}\Psi$

[4] Lau, gr-qc/0401001

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3. AGH rational approximation to good agreement on $s \in \mathrm{i}\mathbb{R}$

$$\widehat{\Omega}_\ell(s, x_b; V) \approx \frac{\text{degree } d-1 \text{ polynomial}}{\text{degree } d \text{ polynomial}} = \sum_{i=1}^{d} \frac{\gamma_i}{s - \beta_i}$$

where γ_i and β_i are outputs

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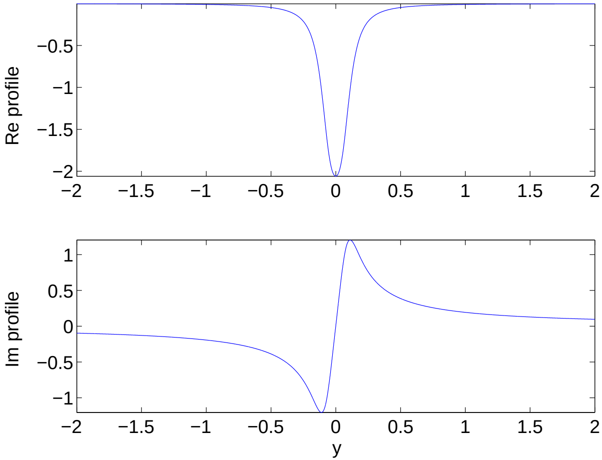
   where γ_i and β_i are outputs

4. Invert rationally approximated kernel

$$\boxed{\partial_t \Psi_\ell + \partial_x \Psi_\ell = \frac{1}{r}\int_0^t \Omega_\ell(t-t', r)\Psi_\ell(t', r)dt'}$$

[4] Lau, gr-qc/0401001

$\ell = 2$, $r_b = 30M$ boundary kernel evaluated along $s = iy$



Profiles shown on the left

$\Downarrow$

Rational approximation

$\Downarrow$

Poles and strengths

Access to asymptotic waveform

Problem: Short computational domain, but we need the signal at large distances (black holes are in other galaxies!)

Goal: From a signal (as a time-series) recorded at a fixed $r_b \approx 30$, recover the signal at (say) $r \approx 10^{15}$

Preview: Very similar to boundary condition approach

Signal "teleportation" for outgoing solution

- From the outgoing solution $\widehat{\Psi}_\ell(s,r) = a(s)s^\ell e^{-sr} W_\ell(sr)$

$$\widehat{\Psi}_\ell(s,r_2) = \mathrm{e}^{s(r_1-r_2)}\left[\frac{W_\ell(sr_2)}{W_\ell(sr_1)}\right]\widehat{\Psi}_\ell(s,r_1) \equiv \mathrm{e}^{s(r_1-r_2)}\widehat{\Phi}_\ell(s,r_1,r_2)\widehat{\Psi}_\ell(s,r_1)$$

- $\widehat{\Phi}_\ell(s,r_1,r_2)$ is the teleportation kernel[5]
- When $r_2 \approx \infty$, $\widehat{\Phi}_\ell(s,r_1,\infty)$ is the *asymptotic waveform kernel*

- Straightforward to show

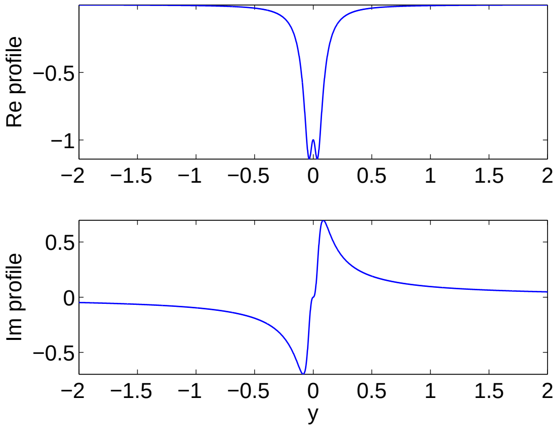
$$\widehat{\Phi}_\ell(s,r_1,r_2) = \frac{W_\ell(sr_2)}{W_\ell(sr_1)} = \exp\left[\int_{r_1}^{r_2} \frac{\widehat{\Omega}_\ell(s,\eta)}{\eta} d\eta\right]$$

Teleportation kernel is an integral over boundary kernels

[5]Disclaimer: must define $\widehat{\Phi}_\ell(s,r_1,r_2) = W_\ell(sr_2)/W_\ell(sr_1) - 1$ so that $\widehat{\Phi}_\ell \to 0$ along path of inverse Laplace transform. This amounts to offsetting by $\widehat{\Psi}_\ell(s,r_1)$

$\ell = 2$, $r_1 = 30M$, $r_2 = \infty$ extraction kernel along $s = iy$

Numerically compute $\widehat{\Phi}_2(s) = \exp\left[\int_{30M}^{\infty} \frac{\widehat{\Omega}_2(s,\eta)}{\eta} d\eta\right]$



Profiles shown on the left

⇓

Rational approximation

⇓

Poles and strengths

Pole #	Gamma strengths	Beta locations
1	-1.7576263057e-08 + 0i	-5.4146529341e-01 + 0i
2	-6.4180514293e-08 + 0i	-4.1310954989e-01 + 0i
3	-6.2732971050e-06 + 0i	-3.1911338482e-01 + 0i
4	-6.9363117988e-05 + 0i	-2.4711219871e-01 + 0i
5	-5.7180637750e-04 + 0i	-1.9108163722e-01 + 0i
6	-2.7884247577e-03 + 0i	-1.4749601558e-01 + 0i
7	-5.8836792033e-03 + 0i	-1.1366299945e-01 + 0i
8	-3.6549136132e-03 + 0i	-8.6476935381e-02 + 0i
9	-1.0498746767e-03 + 0i	-6.4512065175e-02 + 0i
10	-2.4204781878e-04 + 0i	-4.7332374442e-02 + 0i
11	-5.5724464176e-05 + 0i	-3.4115775484e-02 + 0i
12	-1.2157296793e-05 + 0i	-2.4048935704e-02 + 0i
13	-2.6651813247e-06 + 0i	-1.6468632919e-02 + 0i
14	-4.8661708981e-07 + 0i	-1.0845690423e-02 + 0i
15	-8.6183677612e-08 + 0i	-6.7552918597e-03 + 0i
16	-9.3735071189e-09 + 0i	-3.8525630196e-03 + 0i
17	-8.7881787023e-10 + 0i	-1.8481215040e-03 + 0i
18	-9.1164536027e-02 -5.3953709155e-02i	-9.4779490815e-02 +5.9927979877e-02i
19	-9.1164536027e-02 +5.3953709155e-02i	9.4779490815e-02 -5.9927979877e-02i

$$\text{For } s \in i\mathbb{R},\ \widehat{\Phi}_2(s) \approx \sum_{i=1}^{19} \frac{\gamma_i}{s-\beta_i} \to \Phi_2(t) \approx \sum_{i=1}^{19} \gamma_i \exp\left(\beta_i t\right)$$

Implementation and features

- Suppose we have evolved the EMRB equations, recording a (discrete) time-series $\Psi^n = \Psi(t_n, x_b)$ at the outer boundary x_b
- Discrete times from the numerical scheme are $t^n = 0 + n\Delta t$
- From $\Psi(t_n, x_b)$ we compute $\Psi(t_n + \infty, x_b + \infty)$ by

$$\Psi(t+\infty, b+\infty) \simeq \sum_{q=1}^{d} \gamma_q \int_0^t e^{\beta_q(t-t')}\Psi(t', b)dt' + \Psi(t, b)$$

Key features of this technique

- With a time-series at *ANY* radial location one can *EXACTLY* teleport it to any other radial value
- Non-intrusive to existing code (possibly as post-processing step)

Outline

Problem motivation

Numerics: Scheme, boundary conditions, asymptotic signal

Results

Trivial Data

- One must provide initial conditions to solve the partial differential equation. Physically motivated initial conditions are presently unknown for this problem
- It is common to set all the fields to zero
- This is clearly wrong initial data since it
  - Does not capture information about physics in any way
  - Inconsistent with the PDE as

$$0 = G(x,t)\delta(x - x_p(t)) + F(x,t)\delta'(x - x_p(t))$$

Consequences of Trivial Data

Standard argument...

- Because we are solving a wave–like equation, the violations introduced will propagate away (perhaps difficult to verify?)

Recall our first order system

$$\partial_t \Psi = -\Pi$$
$$\partial_t \Pi = -\partial_x \Phi + V\Psi + J_\Phi \delta(x - x_p)$$
$$\partial_t \Phi = -\partial_x \Pi + J_\Pi \delta(x - x_p),$$

subject to the constraint $\Phi = \partial_x \Psi - [[\Psi]]\,\delta(x - x_p)$

Initial data is (distributionally) constraint violating. What to expect?

Development of Static Junk: 1+1 Example

- Consider the $V = 0$ case, corresponding to

$$-\partial_t^2\Psi + \partial_x^2\Psi = \cos(t)\delta(x) - \mathrm{i}\cos(t)\delta'(x)$$

- Subtract numerical solutions with (trivial data) and without constraint violating data
- Empirically: $\Psi_{\text{Junk}} = C_L\Theta(-x)\Theta(t+x) + C_R\Theta(x)\Theta(t-x)$

$$(-\partial_t^2 + \partial_x^2)\Psi_{\text{Jost}} = \left[\left[\Psi_{\text{Junk}}\right]\right]\Theta(t)\delta'(x) = (C_R - C_L)\,\Theta(t)\delta'(x)$$

- Constraint violating at $x = 0$ – NOT A SOLUTION!

Alternative Source Description

- Without the exact initial data, we consider modifying the source terms such that they...
  - Are consistent with the choice of trivial initial data to machine precision
  - Become the physical sources in a finite (short) time
- "Switched on" the source terms smoothly by multiplying with a function that interpolates 0 and 1, we use

$$\begin{array}{rl} \frac{1}{2}[\mathrm{erf}(\sqrt{\delta}(t-\tau/2)+1] & \text{for } 0 \leq t \leq \tau \\ 1 & \text{for } t > \tau, \end{array} \tag{1}$$

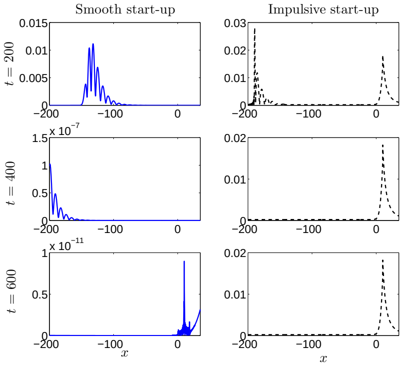
Observing Junk Solutions

- Define $\hat{\Psi} = \frac{1}{2}\partial_t \Psi$
- Equations for Ψ and $\hat{\Psi}$ have different distributional source terms, but the same potential
- They are related by

$$\hat{\Psi} - \frac{1}{2}\partial_t \Psi = 0$$

- Evolve 2 systems with trivial data, one for $\hat{\Psi}$ and one for Ψ
- Violations of above relationship necessarily due to numerical errors and/or incorrect initial conditions

$$\left|\hat{\Psi} - \frac{1}{2}\partial_t \Psi\right|$$



Summary of Static Junk

Features...

- Constraint violating solution has analytic solution in terms of Gauss-Hypergeometric functions
- Discontinuous at the particle
- Ψ_{Junk} decays faster than $1/r$
- Small effect on gravitational wave signal

Remedy...

- By slowly turning on sources the constraint violation is arbitrarily well suppressed

Convergence with Approximation Order

- For a fixed velocity v obeying $|v| < 1$ and $V = 0$

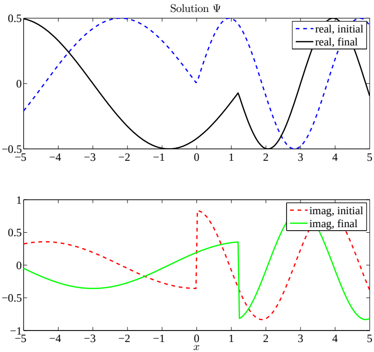
$$-\partial_t^2 \Psi + \partial_x^2 \Psi = \cos(t)\delta(x - vt) - \mathrm{i}\cos(t)\delta'(x - vt)$$

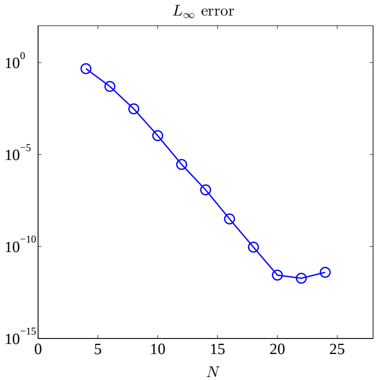
and the solution to the homogeneous problem is

$$\Psi(t, x) = -\frac{1}{2}\sin\vartheta + \frac{1}{2}\mathrm{i}\gamma^2[v + \mathrm{sgn}(x - vt)]\cos\vartheta$$
$$\vartheta = \gamma^2(t - xv - |x - vt|) \qquad \gamma = (1 - v^2)^{-1/2}$$

- $\Psi(t = 0, x)$ and $\partial_t \Psi(t = 0, x)$ supplies initial data.

Convergence with polynomial order

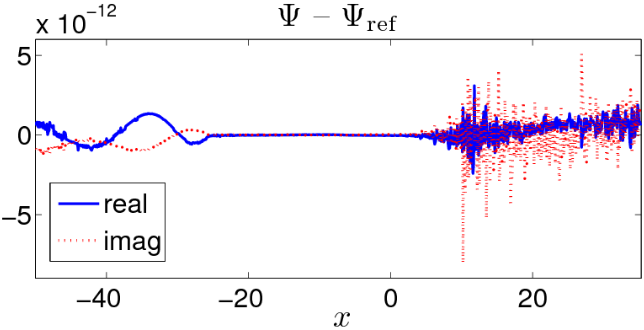




2 domain set-up, coordinate transformation keeps particle at interface.

BC Test for $(-\partial_t^2 + \partial_x^2 - V^\ell)\Psi = 0$ and $\ell = 2$

Use smooth compactly supported initial data.
Experiment: Generate Ψ_{ref} causally disconnected from outer boundary and a second solution Ψ with RBCs (time = 100)



- A small blackhole ($m_p \ll 1$) orbits a large blackhole (take $M = 1$)
- Ψ determines metric perturbation (GW signal $h_+ = \frac{1}{r}\left(\Psi_{\ell m} Z^{\ell m}_{\theta\theta}\right)$)

$$(-\partial_t^2 + \partial_x^2 - V(x))\Psi = G(x,t)\delta(x - x_p(t)) + F(x,t)\delta'(x - x_p(t))$$

and

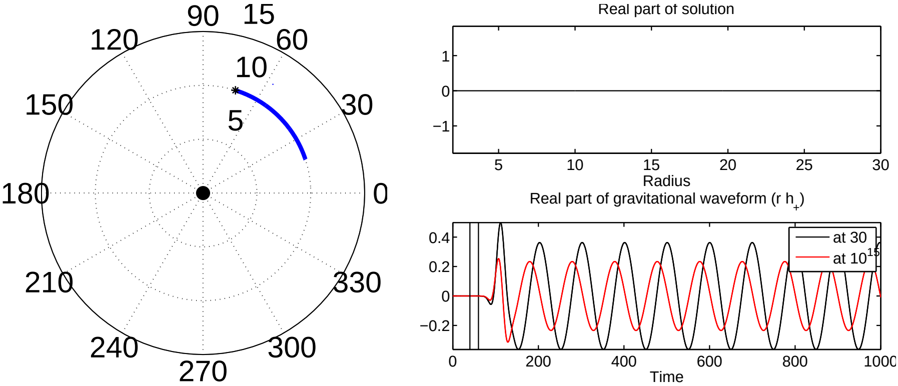
$$V^{\text{axial}}(r) = \frac{f(r)}{r^2}\left[\ell(\ell+1) - \frac{6M}{r}\right]$$

$$V^{\text{polar}}(r) = \frac{2f(r)}{(nr+3M)^2}\left[n^2\left(1 + n + \frac{3M}{r}\right) + \frac{9M^2}{r^2}\left(n + \frac{M}{r}\right)\right].$$

with $n = \frac{1}{2}(\ell - 1)(\ell + 2)$

- We are typically interested in $(\ell, m) = (2, 2)$ multipole solutions
- Orbital motion (must solve ODEs) specifies $G(x,t)\delta(x - x_p(t)) + F(x,t)\delta'(x - x_p(t))$

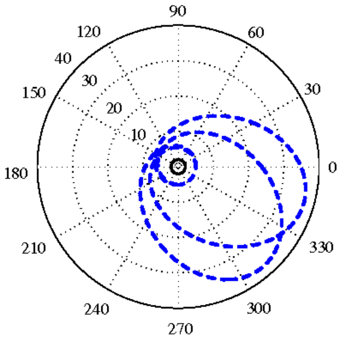
Circular Orbits: $M = 1$, $r_p = 10M$ (horizon at $r = 2M$)



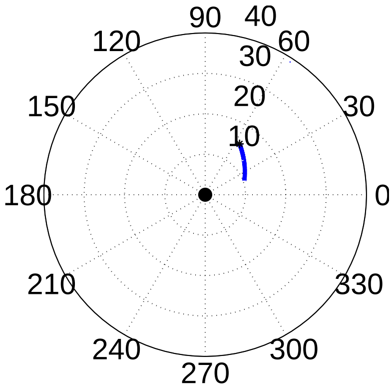
$(\ell, m) = (2, 2)$ perturbations. Scale Ψ by $m_p << 1$

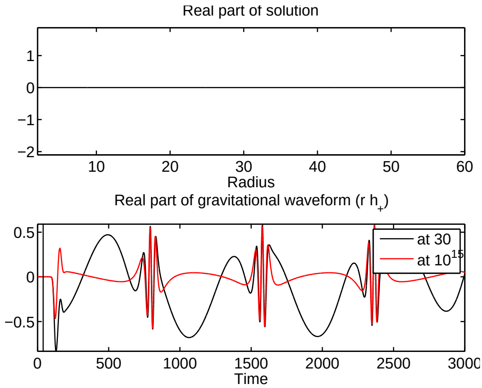
Eccentric Orbit

Same $(\ell, m) = (2, 2)$, $M = 1$, eccentricity $= 0.76412402$, semi-latus rectum $= 8.75456059$



Eccentric Orbit: eccentricity = 0.76412402, semi-latus rectum = 8.75456059





Eccentric Orbit

Selected energy and angular momentum flux calculated at null infinity. Averages computed according to

$$< \dot{E}_{\ell m} >= \frac{1}{T_f - T_0} \int_{T_0}^{T_f} \dot{E}_{\ell m} dt \qquad T_f - T_0 = 4T_{radial}$$

Total $\ell = 2$ energy luminosity $m_p^{-2} \sum_{m=-2}^{2} \langle \dot{E}_{2m} \rangle$		
Orbit parameters	dG	FR
$e = 0.76412402,\ p = 8.75456059$	1.57120×10^{-4}	1.57131×10^{-4}

Total $\ell = 2$ angular momentum luminosity $m_p^{-2} \sum_{m=-2}^{2} \langle \dot{L}_{2m} \rangle$		
Orbit parameters	dG	FR
$e = 0.76412402,\ p = 8.75456059$	2.09220×10^{-3}	2.09221×10^{-3}

Summary...

- Introduced discontinuous Galerkin method for EMRB modeling
- Particular attention to treatment of delta functions, boundary conditions and asymptotic signal
- Observed static junk solution seeded by constraint violating initial data
- Taken together, scheme is *very* accurate and most sources of error have been isolated
- Future prospects: Would like to look at improved waveform extraction techniques, waveform compression techniques
- Would be interesting to apply similar numerical techniques to Kerr case

Future work...

- Apply method to so-called Lorentz formulation (similar, better suited at adding in relevant physics)
- Would be interesting to apply similar numerical techniques to Kerr case (spinning black holes)