First order BSSN formulation of Einstein's field equations

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April 3, 2012

arXiv:1202.1038 (to appear in PRD) APS April meeting

Outline

Binary black hole evolution codes

Formulations and numerical methods comprised of

- Generalized Harmonic with finite differences
- Generalized Harmonic with spectral methods
- Baumgarte-Shapiro-Shibata-Nakamura (BSSN) with finite differences

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- GH: Uses black hole excision, and thus requires horizon tracking. Significant effort for stable evolution through merger
- BSSN: "Easier" to use. With standard 1+log and gamma-driver shift very robust. No need for horizon tracking or special tricks at merger

Binary black hole evolution codes

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Due to their exponential convergence, spectral methods achieve higher accuracy than finite difference methods for the same computational cost (degrees of freedom count).

Best of both worlds: (FO)BSSN + spectral

- Can we combine the best of both worlds? A spectral BSSN solver.
- Spectral methods, and discontinuous Galerkin methods which we will consider here, are well developed for fully first order PDE systems

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Outline of the talk's remainder

- Rewrite BSSN as a fully first order BSSN (FOBSSN) system
- Discretize with discontinuous Galerkin and finite difference methods

Outline

Notable differences

Our second order BSSN system differs slightly from conventional choices

- The evolution equations are spatially-covariant
- All system variable will be true (weightless) tensors
    - Different in choice of evolution variables

Metric in ADM form

We may write the full spacetime metric metric as

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = -(\alpha^2 - \gamma_{ij}\beta^i\beta^j)dt^2 + 2\gamma_{ij}\beta^j dt dx^i + \gamma_{ij} dx^i dx^j,$$

Lapse α, shift β^i, and spatial metric γ_{ij}

- **Conformal spatial metric** ($\mathrm{e}^{-4\phi}$ weight to be specified)

$$\tilde{\gamma}_{ij} \equiv \mathrm{e}^{-4\phi}\gamma_{ij}$$

A choice for $\mathrm{e}^{-4\phi}$

Conventional BSSN requires $\tilde{\gamma} = 1$, thus $\phi = \frac{1}{12}\ln\gamma$ and $\mathrm{e}^{-4\phi}$ is of weight $-2/3$

- Thus the conformal metric is of weight $-2/3$

A choice for $e^{-4\phi}$

Conventional BSSN requires $\tilde{\gamma} = 1$, thus $\phi = \frac{1}{12}\ln\gamma$ and $e^{-4\phi}$ is of weight $-2/3$

- Thus the conformal metric is of weight $-2/3$

Instead introduce the *scalar* $\phi = \frac{1}{12}\ln(\gamma/\overline{\gamma})$

- $\overline{\gamma}$ is a scalar density of weight 2 (remains to be specified)
- The conformal metric is a usual tensor
- Not necessarily unit determinant

We will shortly return to $\overline{\gamma}$

The conformal connection functions

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For our BSSN system we instead introduce the "conformal connection function"

$$\tilde{\Lambda}^i = \tilde{\gamma}^{jk}\left(\tilde{\Gamma}^i{}_{jk} - \overline{\Gamma}^i{}_{jk}\right)$$

which is a tensor of no weight.

- We assume $\overline{\Gamma}^i{}_{jk}$ to be constructed from a "fiducial metric" $\overline{\gamma}_{ij}$ whose determinant is $\overline{\gamma}$

The fiducial metric $\overline{\gamma}_{ij}$

The role of $\overline{\Gamma}^i{}_{jk}$ and $\overline{\gamma}$, and hence $\overline{\gamma}_{ij}$, is to restore spatial covariance to the BSSN system. It is our job to specify what $\overline{\gamma}_{ij}$ is...

- Assume $\overline{\gamma}_{ij}$ is time-independent
- Note: traditional BSSN recovered when $\overline{\gamma}_{ij} = \mathrm{diag}(1,1,1) \rightarrow \overline{\gamma} = 1$ and $\overline{\Gamma}^i{}_{jk} = 0$
- Covariant BSSN permits direct reduction to spherical symmetry

A few of the evolution equations

Usual time derivative operator $\partial_\perp \equiv \partial_t - \mathcal{L}_\beta$

$$\partial_\perp \tilde{A}_{ij} = -\frac{2}{3}\tilde{A}_{ij}\overline{D}_k\beta^k + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right) + e^{-4\phi}\left[\alpha R_{ij} - D_i D_j \alpha\right]^{\mathrm{TF}} \ , \quad (1a)$$

$$\partial_\perp \tilde{\Lambda}^i = \tilde{\gamma}^{k\ell}\overline{D}_k\overline{D}_\ell\beta^i + \frac{2}{3}\tilde{\gamma}^{jk}\left(\tilde{\Gamma}^i{}_{jk} - \overline{\Gamma}^i{}_{jk}\right)\overline{D}_\ell\beta^\ell + \frac{1}{3}\tilde{D}^i(\overline{D}_k\beta^k) - 2\tilde{A}^{ik}\overline{D}_k\alpha + 2\alpha\tilde{A}^{k\ell}\left(\tilde{\Gamma}^i{}_{k\ell} - \overline{\Gamma}^i{}_{k\ell}\right) + 12\alpha\tilde{A}^{ik}\overline{D}_k\phi - \frac{4}{3}\alpha\tilde{D}^i K \ , \quad (1b)$$

- Gauge conditions: Bona-Masso slicing with Gamma-driver shift

First order reduction

To write the system in fully first order form introduce new (covariant) variables such as

$$\tilde{\gamma}_{kij} = \overline{D}_k \tilde{\gamma}_{ij} \quad \rightarrow \quad \mathcal{D}_{kij} \equiv \tilde{\gamma}_{kij} - \overline{D}_k \tilde{\gamma}_{ij} = 0$$

leading to equations such as ($\overline{\partial}_0 \equiv \partial_t - \beta^j \overline{D}_j$)

$$\begin{aligned} \overline{\partial}_0 \tilde{\gamma}_{kij} \;\; = \;\; & -2\alpha \overline{D}_k \tilde{A}_{ij} + 2(\overline{D}_k \beta_{(i}{}^{\ell}) \tilde{\gamma}_{j)\ell} - \frac{2}{3} \tilde{\gamma}_{ij} \overline{D}_k \beta_\ell{}^\ell \\ & -2\alpha_k \tilde{A}_{ij} + \beta_k{}^\ell \tilde{\gamma}_{\ell ij} + 2\tilde{\gamma}_{k\ell(i} \beta_{j)}{}^\ell - \frac{2}{3} \tilde{\gamma}_{kij} \beta_\ell{}^\ell - \kappa^\gamma \mathcal{D}_{kij} \;, \end{aligned} \tag{2a}$$

- Outcome: Resulting system is strongly hyperbolic
  - Provided certain conditions are satisfied (e.g. sphere of ill-posedness)

Outline

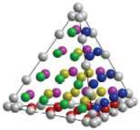
1. For spherically reduced system will consider discontinuous Galerkin implementation
   - Spectral convergence with order of polynomial approximation
   - Robust when matter fields are present (including shocks) [1]
2. Finite difference implementation of full equations
   - Numerics known to work with BSSN
   - Strong test that enlarged system won't lead to instabilities due to constraint violations

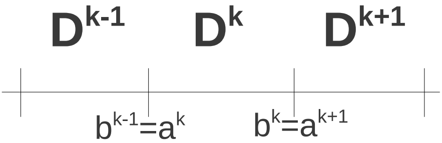
[1] David Radice and Luciano Rezzolla, arXiv: 1103.2426

Will develop the dG method in 4 steps, with 1 step per slide

DG method: space (step 1 of 4)

- Approximate physical domain Ω by subdomains D^k such that $\Omega \sim \Omega_h = \cup_{k=1}^{K} D^k$
- In general the grid is unstructured. We choose lines, triangles, and tetrahedrons for 1D, 2D, and 3D respectively.





DG method: solution (step 2 of 4)

- Local solution expanded in set of basis functions

$$x \in D^k : \Psi_h^k(x,t) = \sum_{i=0}^{N} \Psi_h^k(x_i,t) l_i^k(x)$$

- Polynomials span the space of polynomials of degree N on D^k.
- Global solution is a direct sum of local solutions

$$\Psi_h(x,t) = \bigoplus_{k=1}^{K} \Psi_h^k(x,t)$$

- Solutions double valued along point, line, surface.

DG method: residual (step 3 of 4)

- Consider a model PDE

$$L\Psi = \partial_t \Psi + \partial_x f = 0,$$

where Ψ and $f = f(\Psi)$ are scalars.

- Integrate the residual $L\Psi_h$ against all basis functions on D^k

$$\int_{D^k} (L\Psi_h)\, l_i^k(x) dx = 0 \qquad \forall i \in [0, N]$$

- We still must couple the subdomains D^k to one another...

DG method: numerical flux (step 4 of 4)

- To couple elements first perform IBPs

$$\int_{D^k} \left(l_i^k \partial_t \Psi_h - f\left(\Psi_h\right) \partial_x l_i^k \right) dx = - \oint_{\partial D^k} l_i^k \hat{n} \cdot f^*\left(\Psi_h\right)$$

where the *numerical* flux is $f^*\left(\Psi_h\right) = f^*\left(\Psi^+, \Psi^-\right)$

- Ψ^+ and Ψ^- are the solutions exterior and interior to subdomain D^k, restricted to the boundary
- **Example**: Central flux $f^* = \frac{f(\Psi^+)+f(\Psi^-)}{2}$
- Passes information between elements, implements boundary conditions, and ensures stability of scheme
- Choice of f^* is, in general, problem dependent

We have finished

Remark: The term 'nodal discontinuous Galerkin' should now be clear. We seek a global discontinuous solution interpolated at nodal points and demand this solution satisfy a set of integral (Galerkin) conditions.

- Timestep with a classical 4^{th} order Runge-Kutta
- Robust for hyperbolic equations as we *directly* control the scheme's stability through a numerical flux choice
- For a smooth enough solution, numerical error decays exponentially with polynomial order N

FOBSSN with dG code

- After each timestep a filter is used to control alias driven instabilities
- 1+log slicing and Gamma-driver shift
- Analytic values for the incoming characteristic modes

A few observations

FOBSSN with dG code

- After each timestep a filter is used to control alias driven instabilities
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A few observations

- BUT, filtering the metric (or enforcing conformal metric determinant constraint) is unstable
- Must damp constraints which arise from new (auxiliary) variables for stability

Schwarzschild in conformal Kerr-Schild with excision

Radial domain $[0.4, 50]M$ covered by 100 equally sized domains

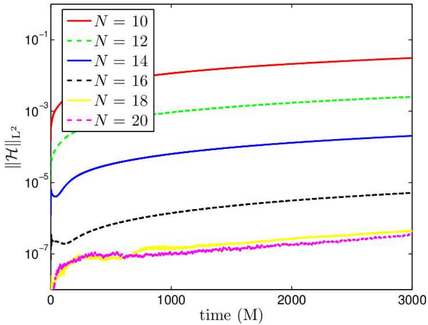


Figure: Long term stability

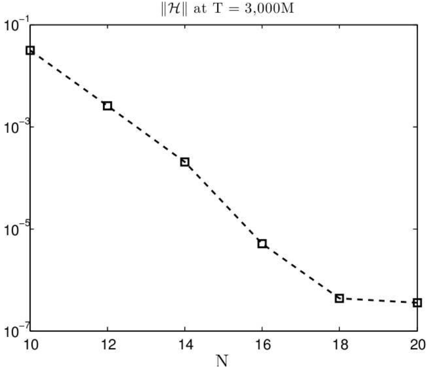


Figure: Exponential convergence

Results from dG code

- Other fields and constraints show similar convergence
- A variety of domain sizes and locations
- Perturbing all fields leads to a stable scheme
- **Main result:** We conclude that the scheme is stable in 1D

Overview of FD implementation

The code

- Cactus framework employing the Carpet adaptive mesh refinement driver
- Mathematica package Kranc to expand the FOBSSN equations to C code
- Both the Mathematica notebook and C code is available as part of the Einstein Toolkit under the name Carlile

The numerics

- Fourth order accurate stencils and fifth order Kreiss-Oliger dissipation
- Fourth order accurate Runge-Kutta time integrator
- Algebraic constraints $\tilde{\gamma}^{ij}\tilde{A}_{ij} = 0$ and $\tilde{\gamma}^{ij}\tilde{\gamma}_{kij} = 0$ are enforced
- $\tilde{\gamma} = 1 = \overline{\gamma}$ is not enforced

Single puncture black hole

- $M = 1$ and $a = .7$
- Eight levels of mesh refinement in a cubic domain, refinement boundaries at $x = [1, 2, 4, 8, 16, 64, 128]\,M$,
- The resolution on the finest level which encompasses the horizon at all times, is $h = 0.032\,M$.
- Outer boundary at $258.048\,M$.

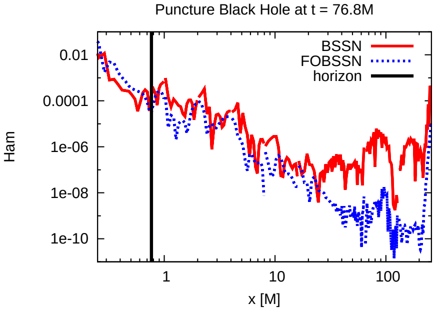
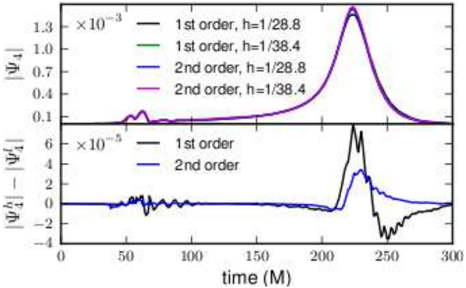


Figure: Hamiltonian constraint along x axis at t=77M

Binary black hole

- Nonspinning and equal mass
- Extracted $\ell = m = 2$ Weyl scalar
- Good agreement between BSSN and FOBSSN



Final remarks

- Fully first order spatially covariant BSSN system with constraint damping terms
- Complete hyperbolicity analysis
- Discretized with discontinuous Galerkin solver
    - Stable long time and exponentially convergent runs
- Discretized with finite differences using Cactus framework
    - For cases we considered, BSSN and FOBSSN behave similarly
    - Enlarged system shows no obvious signs of instability

QUESTIONS?