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Twin Paradox

Complete Solution

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Paradox:

Suppose there is a pair of twin brothers.Once,One brother goes on a near-light-speed rocket while the other one stays on earth.Caused by the time dilation according to the special theory of relativity,the twin on earth would conclude the other one`s clock running slower as he is traveling at near light-speed.He would find his brother younger than him when they meet again;His brother on the rocket would also see the earth traveling backward at high speed,supposing the one on earth would be younger when they meet again.Nevertheless they could not be both younger than the other one,which ends up in a paradox.

Answer:

No,this is not a paradox.The twin on rocket would definitely be younger when they meet again due to the necessary presence of acceleration.

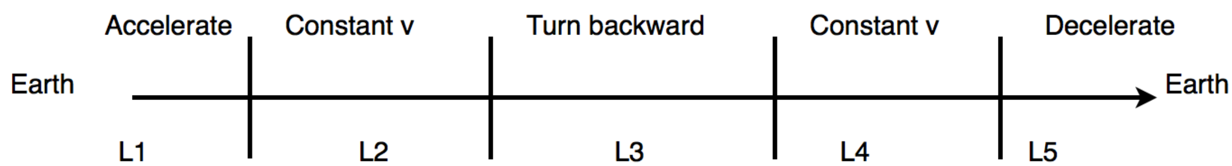
Proof:

To initially speed up from the earth,turning backward at some point and finally slow down to stop on earth,the rocket must experience at least one period of acceleration(In this case it is in centripetal motion),which only shortens the time of the coordinate system that is suffering from a force.Hence,the correct time measurement should come from only those inertial observers.Namely,the claim of the twin on earth is correct.

Notice that due to the same motional state of the starting and ending points of the journey,the integral displacement,velocity and acceleration should all add up to zero,which means that whichever motion satisfied this requirement should give the same result from the experiment.In addition,the direction of these vectors do not count into time measurement in the theory of relativity,leaving the freedom to set the motion in one dimension.

The special case of these motions is to set the rocket accelerate uniformly with acceleration *a* to reach *v1*,at which speed it travels through a distance *L2* before decelerate with (-*a*) to turn backward at velocity (-*v1*) and make the exact same way back to the earth.(The following proof would use only a`(the acceleration in rocket`s view,v1,L2 and light speed c as perimeter in final answer)

According to basic knowledges of acceleration and velocity,the time it takes to decelerate from *v1* to (-*v1*) with the same magnitude of acceleration should be twice of the initial accelerating time;the the final decelerating time should be exactly the same as the initial accelerating in order to stop the rocket on earth`s surface.Thereafter,,the one-dimensional accelerate-constant-backward-constant-decelerate example of the twin paradox could be shown in the following diagram:



Where *L1*=0.5**L3*=*L5* and *L2*=*L4*.

the whole journey could be separated into two parts:four period of uniform acceleration motion,and two periods of constant velocity motion.

Uniform acceleration periods:

To compute the four accelerating periods requires much work.Primarily,to solve for the relationship between *a* and *a*`,where *a*` stands for the acceleration in the rocket`s view,is necessary.Notice that only *a*` is meant to be constant in this so-called uniform accelerating motion.According to Lorentz transform,there are:

$$x` = \gamma(x - vt) \ , \ dx` = \gamma(dx - vdt)$$

$$t` = \gamma\left(t - \frac{xv}{c^2}\right) \ , \ dt` = \gamma\left(dt - \frac{vdx}{c^2}\right)$$

Resulting in:

$$u`=\frac{dx`}{dt`}=\frac{dx\text{-}vdt}{dt\text{-}\frac{vdx}{c^2}}=\frac{u\text{-}v}{1\text{-}\frac{uv}{c^2}}$$

which is the formula for relativistic velocity addition.

Notice that from now on whenever any perimeter is put with ` ,it means that the perimeter is measured in K` coordinate system,which is in the rocket`s view.

Differentiate again to get:

$$du`=\frac{\left(1\text{-}\frac{uv}{c^2}\right)du+\frac{(u\text{-}v)v}{c^2}du}{\left(1\text{-}\frac{uv}{c^2}\right)^2}=\frac{1\text{-}\frac{v^2}{c^2}}{\left(1\text{-}\frac{uv}{c^2}\right)^2du}=\frac{du}{\gamma^2\left(1\text{-}\frac{uv}{c^2}\right)^2}$$

$$dt`=\gamma dt\left(1\text{-}\frac{uv}{c^2}\right)$$

In this case,it is the K` system itself being in motion,which means that u=v:

$$\frac{du`}{dt`}=\gamma^3\cdot\frac{du}{dt}\ ,a`=\gamma^3a$$

Such formula is a differential equation with respect to v,where *a*` remains a constant.

$$\frac{dv}{dt}=\left(1\text{-}\frac{v^2}{c^2}\right)^{1.5}a`$$

Solve this equation by separating variables:

$$\int\left(1-\frac{v^2}{c^2}\right)^{-1.5}dv=\int a`\ dt$$

$$Let\ \frac{v}{c}\sin p,dv=c\cdot\cos p\ dp$$

Then

$$c\int(\cos p)^{-3}\cdot\cos p\ dp=a`t$$

$$a`t=c\int(\sec p)^2dp=c\cdot\tan p=c\cdot\tan\left(\arcsin\left(\frac{v}{c}\right)\right)=c\cdot\frac{\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$a`t=\gamma v=\frac{v}{\sqrt{1-\frac{v^2}{c^2}}}$$

In the question,to reach *v1*,the time it takes would be

$$t1=\frac{v1}{a`\sqrt{1-\frac{v1^2}{c^2}}}$$

Furthermore,to calculate the entire distance out,v(t) should be solved from the equation.

$$v(t)=\frac{a`t}{\sqrt{1+\frac{(a`t)^2}{c^2}}},\quad x(t)=\int_0^{t1}\frac{a`t}{\sqrt{1+\left(\frac{a`t}{c}\right)^2}}\ dt$$

$$Let\ \frac{a`t}{c}=\tan q,\ dt=\frac{c}{a`}\sec^2 q\ dq$$

$$x(t)=\int_0^{t1}\frac{c\tan q}{\sec q}\cdot\frac{c}{a`}(\sec q)^2 dq$$

$$=\frac{c^2}{a}\int_0^{t1}\tan q\sec q\ dq=\frac{c^2}{a}\sec q\ (0,t1)\ =\sec\left(\arctan\left(\frac{a`t}{c}\right)\right)\ (0,t1)$$

$$=\frac{c^2}{a`}\sqrt{\left(\frac{a`t}{c}\right)^2+1}+\frac{c^2}{a`}$$

In summary,The uniform accelerating period should cover a distance that is:

$$L1=\frac{c^2}{a`}\sqrt{\left(\frac{a`t1}{c}\right)^2+1}+\frac{c^2}{a`}$$ ,where t1 is defined by $$t1=\frac{v1}{a`\sqrt{1-\frac{v1^2}{c^2}}}$$

Plug in to get

$$L1=\frac{c^2}{a`}\sqrt{\left(\frac{v1}{c\sqrt{1-\frac{v1^2}{c^2}}}\right)^2+1}+\frac{c^2}{a`}$$

Another thing to do is to know the time it takes for this period,which could be derived from x(t) to be $$t=\frac{c}{a`}\sqrt{\left(\frac{a`L1}{c^2}-1\right)^2-1}$$

$$=\frac{c}{a`}\sqrt{\left(\frac{a`\left(\frac{c^2}{a`}\sqrt{\left(\frac{v1}{c\sqrt{1-\frac{v1^2}{c^2}}}\right)^2+1}+\frac{c^2}{a`}\right)}{c^2}-1\right)^2-1}$$

$$=\frac{c}{a}\sqrt{\frac{v1^2}{c^2-v1^2}}$$

What about the time in the rocket`s view,t1`?This is only a matter of time relationship in uniform acceleration.

From $\Delta t`=\frac{\Delta t}{\sqrt{1-\frac{v1^2}{c^2}}}=\gamma\Delta t$ the differentiated relationship between the two system could be defined by $t`=\int_0^t\sqrt{1-\frac{v^2}{c^2}}dt$.

Plug in the v(t) equation to get:

$$t`=\int_0^t\sqrt{1-\frac{\left(\frac{a`t}{\sqrt{1+\left(\frac{a`t}{c}\right)^2}}\right)^2}{c^2}}dt$$

$$=\int_0^t \sqrt{1-\frac{(a`t)^2}{c^2+(a`t)^2}}\,dt=\int_0^t \sqrt{\frac{c^2}{c^2+(a`t)^2}}\,dt$$

Recall from hyperbolic functions that

$$\frac{d}{dx}(\operatorname{arcsinh}h(x))=\frac{1}{\sqrt{1+x^2}}.$$

Hence,after some substitution and abbreviation,the final answer turns out to be:

$$t`=\frac{c}{a`}\operatorname{arcsinh}\left(\frac{a`t}{c}\right).$$

Reflect the equation,t(t`) could be simply known as:

$$t=\frac{c}{a`}\sinh\left(\frac{a`t`}{c}\right).$$

Plug in the time it takes in the earth view derived earlier,

$$t`=\frac{c}{a`}arcsinh\left(\frac{a`}{c}\cdot\frac{c}{a`}\sqrt{\left(\frac{a`}{c^2}L1\right)^2-1}\right)$$

$$=\frac{c}{a`}arcsinh\left(\sqrt{\left(\frac{a`}{c^2}L1\right)^2-1}\right)$$

Here we need another knowledge of the hyperbolic function:

$$\operatorname{arcsinh}(x)=\ln\left(x+\sqrt{x^2+1}\right) \quad ar\cosh(x)=\ln\left(x+\sqrt{x^2-1}\right)$$

Which could lead to the final answer to be:

$$t`=\frac{c}{a`}arccosh\left(\frac{a`L1}{c^2}+1\right)$$

$$t` = \frac{c}{a`} \operatorname{arccosh}\left(\frac{a`\left( \frac{c^2}{a`}\sqrt{\left(\frac{v1}{c\sqrt{1-\frac{v1^2}{c^2}}} \right)^2 +1} + \frac{c^2}{a`} \right)}{c^2} + 1 \right)$$

$$t` = \frac{c}{a`} \operatorname{arcsinh}\left(\sqrt{\frac{v1^2}{c^2 - v1^2}} \right)$$

So far,we have solved for every useful perimeter in terms of *a`*, *v1* and c:
To accelerate from at rest to *v1*,the time it takes from the earth view is:

$$t = \frac{c}{a`}\sqrt{\frac{v1^2}{c^2 - v1^2}}$$

The distance involves in this period is:

$$L1 = \frac{c^2}{a`}\sqrt{\left( \frac{v1}{c\sqrt{1-\frac{v1^2}{c^2}}} \right)^2 +1} + \frac{c^2}{a`}$$

The time it takes from the rocket`s view is:

$$t`=\frac{c}{a`}\operatorname{arcsinh}\left(\sqrt{\frac{v1^2}{c^2-v1^2}}\right).$$

As described earlier,the time and distance it takes to decelerate from *v1* to 0 is exactly the same as these results,and the time to turn backward is twice of it.

Notice that t and t` is exactly not the same as each other.In addition,this difference could not be reversed in relativistic motion.Namely,the rocket`s measurement of the earth time is absolutely wrong.

Uniform velocity periods:

To solve for the time relationship in uniform velocity is easy:

Due to length contraction phenomenon described in special relativity,

$$L2`=\frac{1}{\gamma}L2=\sqrt{1-\left(\frac{v1}{c}\right)^2}\cdot L2$$

Which results in the time dilation:

$$t`=\frac{L2`}{v1}=\frac{1}{\gamma}\cdot\frac{L2}{v1}=\frac{1}{\gamma}t=\sqrt{1-\left(\frac{v1}{c}\right)^2}t$$

The whole journey:

The total traveling distance would be(Here only the earth`s distance seems to be useful):

$$L=4\cdot\frac{c^2}{a`}\sqrt{\left(\frac{v1}{c\sqrt{1-\left(\frac{v1}{c}\right)^2}}\right)^2+1}+4\cdot\frac{c^2}{a`}+2\cdot L2$$

The total time it takes from the earth view would be:

$$t=4\cdot\frac{c}{a`}\sqrt{\frac{v1^2}{c^2-v1^2}}+2\cdot\frac{L2}{v1}$$

And from the rocket`s view,the total journey time shall be:

$$t`=4\cdot\frac{c}{a`}arcsinh\left(\sqrt{\frac{v1^2}{c^2-v1^2}}\right)+2\cdot\frac{\sqrt{1-\left(\frac{v1}{c}\right)^2}\cdot L2}{v1}$$

Plug in any numbers for the perimeters *a`*, *v1* and L2,one could find that t` is significantly less than t,which explains that in twin paradox:when the two brothers meet again on earth,the one goes on rocket should be noticeably younger than the one on earth.

Space-time diagram:

From the function x(t) derived earlier,it is possible to draw the Minkovski Space-time diagram of this motion,at least one uniform acceleration period.Be aware that the space-time diagram uses not x and t as perimeter but ct and x,where ct is a function of x.Hence,it is necessary to derive the formula ct(x):

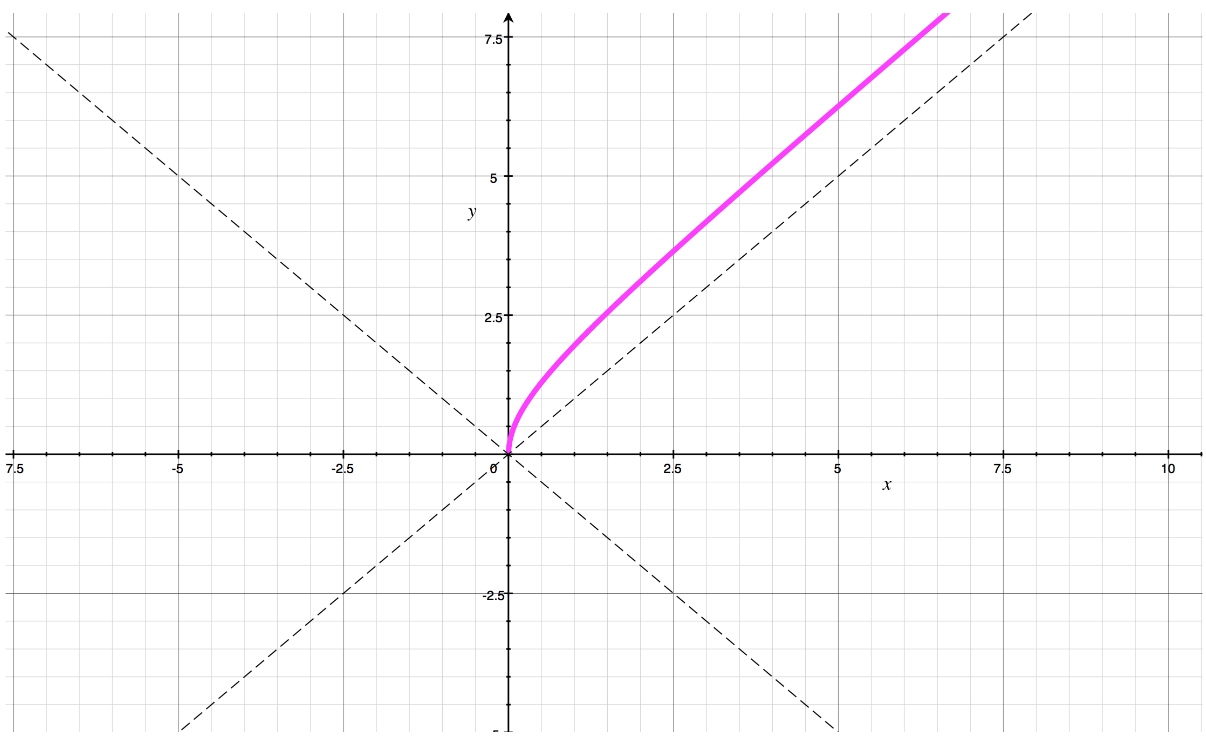
$$x=\frac{c^2}{a\text{`}}\sqrt{\left(\frac{a\text{`}t}{c}\right)^2+1}+\frac{c^2}{a\text{`}}$$

$$\left(x-\frac{c^2}{a\text{`}}\right)^2=\frac{c^4}{a\text{`}^2}\left(\left(\frac{a\text{`}t}{c}\right)^2+1\right)$$

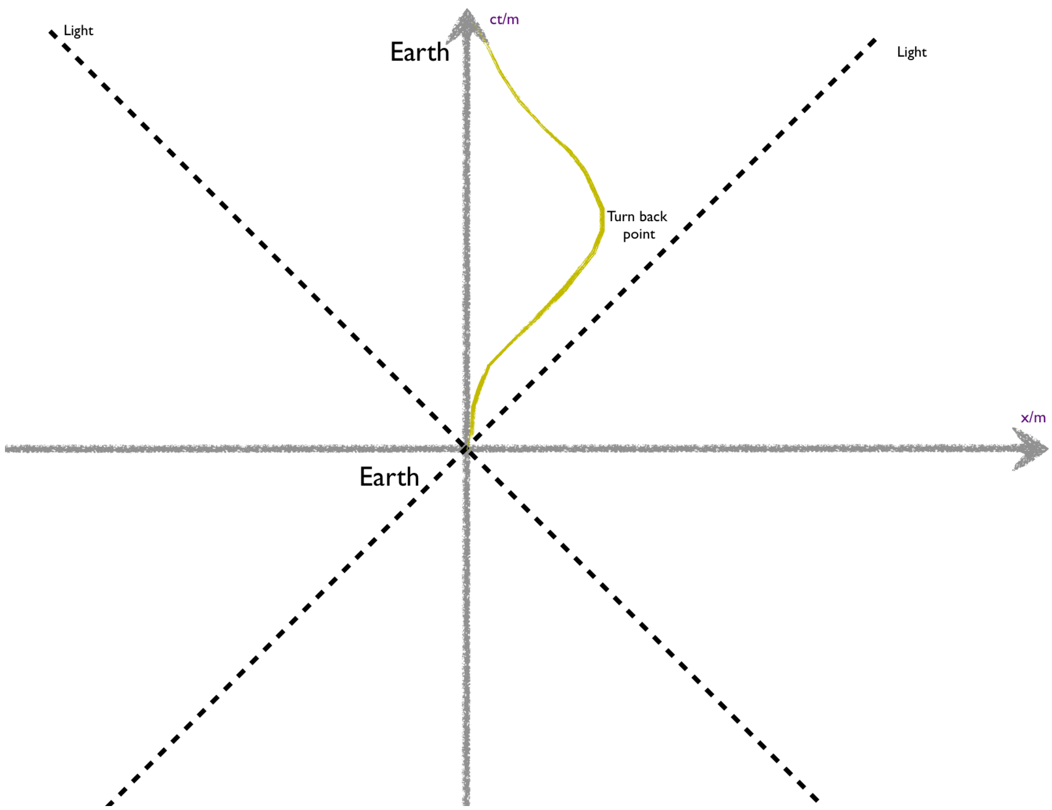
$$ct=\sqrt{x^2-\frac{2c^2}{a\text{`}}}.$$

With some minor adjustment,a simple diagram could be drawn:

The slop of the motion would approach the light gradually without getting over.



Nevertheless,it is difficult to draw an exact motion line of the rocket by plugging in functions into any grapher.Instead,a draft of the approximate motion could be drawn:



Possibility of twin paradox experiment:

To experimentally prove twin paradox is certainly not an easy job.There are several physical difficulties of making a rocket that could travel at near-light speed for years:

1)The fuel would be a huge problem,especially as shown before,the faster you travel,the harder it is to accelerate.According to Einstein`s theory,the largest possible energy of matter comes from $E=mc^2$,which means the best energy source would be matter-anti-matter power,whose production cost remains trillion-per-gram so far.

2)The actual traveling time should be no longer than a decade as no human could survive through long-term physical and metal challenge on rockets.

3)The acceleration of the rocket would cause overweight phenomenon on the astronomer`s body,making it tough for the twin to live.Typically a human being should suffer from no more than five times of the earth gravitational acceleration.

In concern of these three questions,a maximum of the experiment could be set to have an acceleration of 5g,a constant velocity of 0.9 times light speed and a total traveling time of 8 years in rocket`s view.

Through calculation,the length of one constant velocity period,L2 comes out as 7.102light year,which makes the time on earth 17.3542 years.The result of this experiment could be noticeable if the twins are initially young kids.

To consider the energy use to accelerate and decelerate is also essential.According to mass-energy equation,any object at rest would have its internal energy as:

$\mathrm{E}=m0\cdot c^2$,where m0 is the stable mass of the object.In motion,the object gain energy to get $\mathrm{E}=\gamma m0\cdot c^2=\dfrac{m0\cdot c^2}{\sqrt{1-\left(\frac{v1}{c}\right)^2}}$.The subtraction of the two equation would give the energy required to accelerate the object to *v1* velocity: $\mathrm{E}=(1-\gamma)\cdot m0\cdot c^2$.Such energy comes from the combining of matter and anti matter in theory,which gives energy as: $\mathrm{E}=(Mmatter+Manti)c^2$.Since the amount of anti matter would always be the same as matter,the equation would be simplified as: $\mathrm{E}=2\cdot Mfuel\cdot c^2$.

The thing is,according to the description above,merely to accelerate the rocket is never enough for the experiment to be completed.In fact,four times of the acceleration shall be done,utilizing four times the energy.Hence,the mass of the fuel required should be:

$\mathrm{E}motion=\mathrm{E}fuel$

$$4\cdot\gamma m0\cdot c^2=2\cdot Mfuel\cdot c^2$$

$$Mfuel=2\gamma m0=\frac{2m0}{\sqrt{1-\left(\frac{v1}{c}\right)^2}}$$

In the case discussed earlier,the constant velocity *v1* should be 0.9 times light speed,making $Mfuel=4.588\cdot m0$.

The fuel should be 4.5 times the mass of the rocket itself.A modern-day rocket usually weights no less than 10 tons,making the amount of anti matter 40 tons,which is obviously not a good idea in financial consideration.

Theoretically improvement:

In theory,however,there is ways to lower the cost:To build a rocket with matter-anti-matter.This seems to be a rather crazy idea.Nevertheless,by creating vacuum environment between matter and anti-matter,the explosion could somehow be avoided.By doing so,the actual fuel would be reduced by approximately two ninths as the rocket itself becomes this part of fuel.Notice that as the trip going,the twin does not really need so many massive things to keep him alive.As a matter of fact,the final device to take the twin back to earth would be so tiny that its mass would be neglectable.

Additionally,instead of laughing the rocket in any base on earth,it would be a lot easier to start the trip in the space:Namely,to finish building the rocket in the space above the earth.In this way,the rocket needs not to be so tough in order to get through the atmosphere,enabling scientists to utilize some light polymer materials to lower the mass of the rocket.After development,the final weight of the rocket should be about five tons.

Last but not least,for energy consideration,the initial accelerating process could be done by external power sources,among which electron-magnetic acceleration does the job the best.theoretically,an electromagnetic gun could accelerate the rocket to near light speed in vacuum,only if it still needs little additional fuel.Hence,only three times of the energy is in needed to complete the journey.

After these technological advances,the recalculation of energy would give the answer as:

$Mfuel=\left(\frac{3}{2}\gamma-1\right)m0=12.206\ tons$,which is better but still not reasonable.

Conclusion:

The twin paradox could be explained clearly in special relativity,that the twin on rocket should experience acceleration,which slows his clock to make him younger when they meet again on earth.Nevertheless,to set up any feasible experiment seems to be impossible so far,only waiting for further technological development of the human society.

The End

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