Self-consistent motion of a scalar charge around a Schwarzschild black hole

Ian Vega[1] Peter Diener[2] Barry Wardell[3] Steve Detweiler[4]

[1]University of Guelph
[2]Louisiana State University
[3]University College Dublin
[4]University of Florida

12 June 2012
15th Capra Meeting on Radiation Reaction in General Relativity
College Park, Maryland

The problem

We wish to determine the self-forced motion and field (e.g. energy fluxes) of a particle.

2 general approaches:

- Compute enough “geodesic”-based self-forces and then use this to drive the motion of the particle. (Post-processing, fast, accurate self-forces, relies on adiabaticity)
- Compute the “true” self-force while simultaneously driving the motion. (Slow and expensive, less accurate self-forces)

Effective source approach ...

... is a general approach to self-force and self-consistent orbital evolution that doesn't use any delta functions.

Key ideas

- Compute a regular field, ψ^{R}, such that

$$(\text{self-force}) \propto \nabla\psi^{\mathsf{R}}$$

  where $\psi^{\mathsf{R}} = \psi^{\mathsf{ret}} - \psi^{\mathsf{S}}$, and ψ^{S} can be approximated via local expansions.

- The effective source, S, for the field equation for ψ^{R} is regular at the particle location.

$$\Box\psi^{\mathsf{R}} = S(x|z,u)$$

  where $S := \delta - \Box\psi^{\mathsf{S}}$

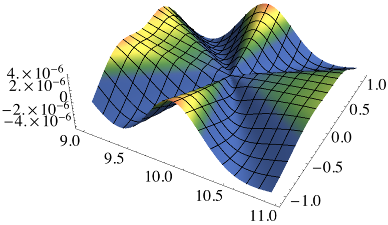
Effective source approach

Evolve the coupled particle-field dynamics:

$$\Box\psi^{\mathsf{R}} = S(x|z(\tau), u(\tau))$$

$$\frac{Du^\alpha}{d\tau} = \frac{q}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta\psi^{\mathsf{R}}$$

$$\frac{dm}{d\tau} = -qu^\beta\nabla_\beta\psi^{\mathsf{R}}$$



Effective source

- For constructing $S(x|z(\tau), u(\tau))$, we make use of the Haas-Poisson coordinate expression for the singular field. This has the form

$$\tilde{\Phi}_{\mathrm{S}} = \frac{a_{(2)} + a_{(3)} + a_{(4)} + a_{(5)}}{(b_{(2)} + b_{(3)} + b_{(4)} + b_{(5)})^{3/2}}$$

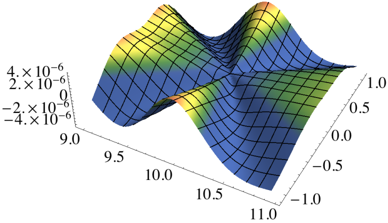
  where $a_{(n)} = a_{\alpha_1 \cdots \alpha_n} \Delta x^{\alpha_1} \cdots \Delta x^{\alpha_n}$ The coefficients $a_{\alpha_1 \cdots \alpha_n}$ contain all the worldline dependencies of the effective source, such as position, velocity and acceleration. $S = \Box \tilde{\Phi}_{\mathrm{S}}$.
- We set all acceleration terms to zero.
- To render the expression more manageable, we reexpand the denominator, keeping only the $O(\Delta x^2)$ dependence.
- We apply the d'Alembertian after substituting the particle's position and velocity.
- To make the source have compact spatial support, we make use of a smooth window function in r and θ.

"Geodesic" self-force with an effective source

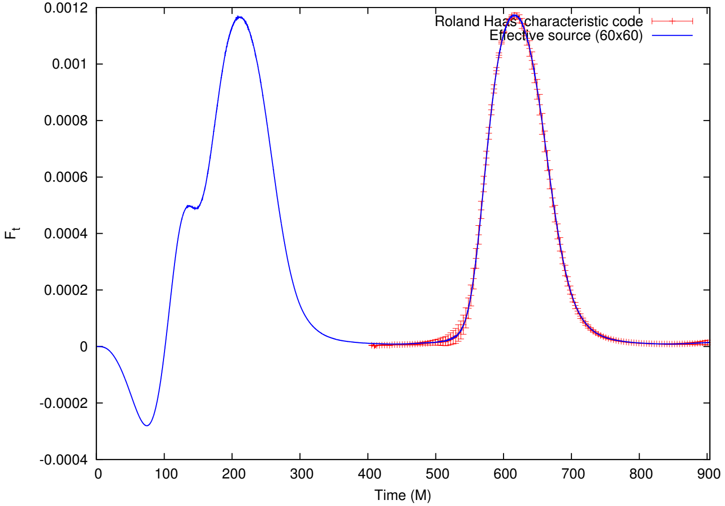
Solve the wave equation for a specified fixed geodesic; compute gradient of ψ^{R}:

$$\Box\psi^{\mathsf{R}} = S(x|z_0(\tau), u_0(\tau))$$

$$F^\alpha = q(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta\psi^{\mathsf{R}}$$

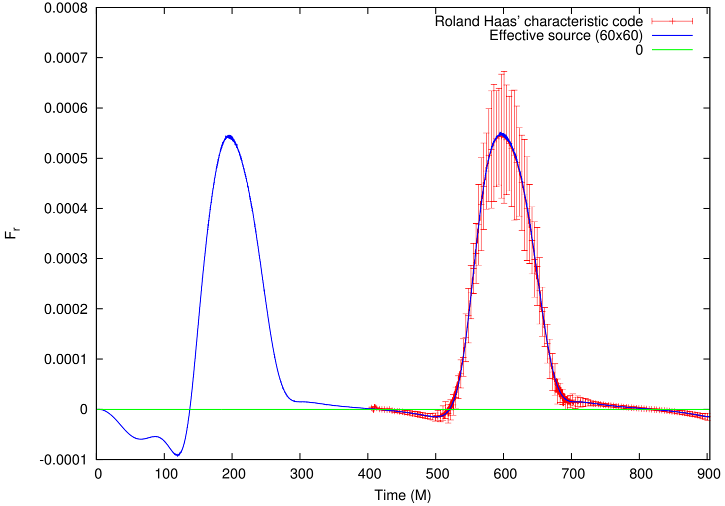


Comparison with (1+1) results



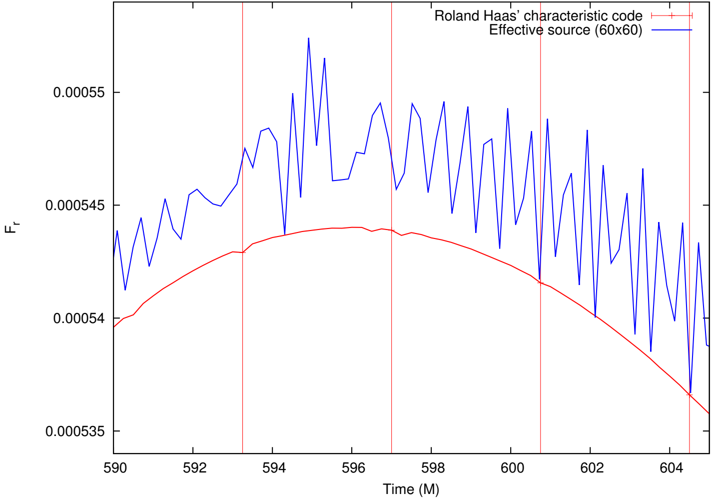
$$e = 0.5, p = 7.2$$

Comparison with (1+1) results



$$e = 0.5, p = 7.2$$

Self-force accuracy: worst case $\sim 2\%$



$$e = 0.5, p = 7.2$$

Ignoring acceleration terms: Are we still faithful to the Quinn-Wald equation?

In principle, the effective source depends on the acceleration of the particle, but we choose to set all acceleration terms to zero.

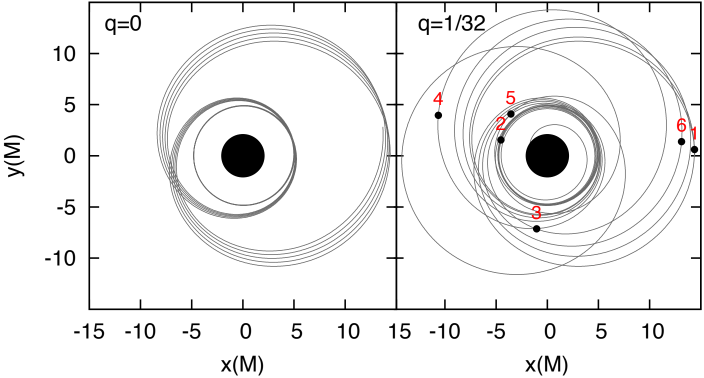
$$S(x|x_0, u_0, a_0, \dot{a}_0, \ddot{a}_0, \ldots) \Longrightarrow S(x|x_0, u_0, a_0 = 0, \ldots)$$

What this means is that the equations we solve are approximations to the Quinn-Wald equations of motion for a scalar charge.

Because test-particle motion (zeroth order in q) is geodesic in our case, the acceleration terms enter at order $O(q^2)$.

This means that we are ignoring contributions to the self-force of order $O(q^3)$.

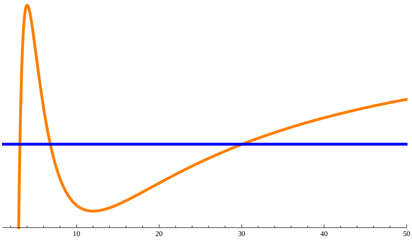
Self-forced orbit



$$r_1 = pM/(1+e), r_2 = pM/(1-e)$$

$$p = 7.2, e = 0.5,\ r_1 = 4.8M,\ r_2 = 14.4M$$

e-p parametrization of the motion



- A bound orbit can be specified by its eccentricity (e) and semi-latus rectum (p):

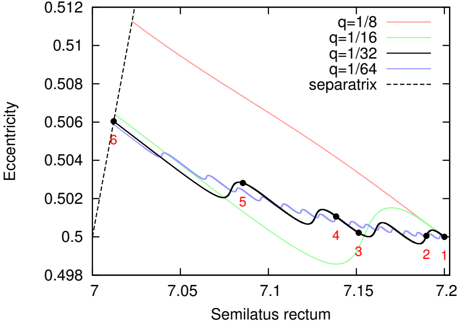
$$r_1 = \frac{pM}{1+e}, \;\; r_2 = \frac{pM}{1-e}$$

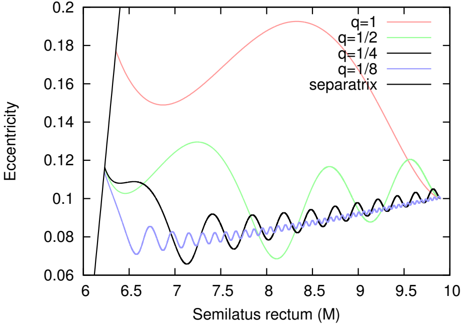
where r_1 and r_2 are the turning points of the radial motion.

- $e = 0$, stable circular orbits
  $p = 6 + 2e$, (separatrix), unstable circular orbits
  $0 \leq e < 1, \;\; p \geq 6 + 2e$, bound orbit

Self-forced orbit: e-p space

Some features: p monotonically decreases, while e oscillates. e decreases secularly far from the separatrix (e.g. weak field regime), but then enters an increasing phase as the particle nears plunge.





Self-forced evolution in e-p space for an orbit starting at $p = 7.2, e = 0.5$ (left plot) and $p = 10.0, e = 0.1$ (right plot).

Evolution code

- A 3D multi-block scalar wave equation code.
- Kerr background spacetime in Kerr-Schild coordinates.
- Spherical inner boundary placed inside the black hole.

Equations:

$$\begin{aligned}
\Box\psi^{\mathrm{R}} &= S(x|z^{\alpha}(\tau), u^{\alpha}(\tau)), \\
\frac{Du^{\alpha}}{d\tau} &= \frac{q}{m(\tau)}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\nabla_{\beta}\psi^{\mathrm{R}}, \\
\frac{dm}{d\tau} &= -qu^{\beta}\nabla_{\beta}\psi^{\mathrm{R}}.
\end{aligned}$$

- The field and the particle are evolved together.
  - The particle location $z^{\alpha}(\tau)$ and four-velocity $u^{\alpha}(\tau)$ gives the effective source that determines ψ^{R}.
  - $\nabla_{\beta}\psi^{\mathrm{R}}$ at the location of the particle in turn affects the orbit.
- We use 8th order summation by parts finite differencing and penalty boundary conditions at patch boundaries.
- We can evolve the orbit using the geodesic equations directly as well as using the osculating orbits framework.

Hyperboloidal slicing

We compactify in the radial direction

$$r = \frac{\rho}{\Omega}, \qquad \text{with} \qquad \Omega = \Omega(\rho).$$

Where $r \to \infty$ corresponds to $\Omega = 1 - \rho/S = 0$.
In addition we perform a transformation of the time coordinate

$$\tau = t - h(r)$$

in order to have the spatial slices asymptote to $\mathscr{I}^+$.
Choosing $H = dh/dr$ as

$$H = 1 + \frac{4M\Omega}{\rho} + \frac{(8M^2 - C^2)\Omega^2}{\rho^2}$$

ensures that the metric is regular at $\rho = S$ and that the characteristic speeds at $\rho = S$ are

$$c_- = 0, \qquad c_+ = S^2/C^2.$$

Hyperboloidal slicing

We still want to use standard spatial slices in the interior so we use a smooth transition

$$\Omega(\rho) = \begin{cases} 1 & \text{for} \quad \rho \le \rho_{\rm int} \\ 1 - f + (1-\rho/S)f & \text{for} \quad \rho_{\rm int} < \rho < \rho_{\rm ext}\,, \\ 1 - \rho/S & \text{for} \quad \rho \ge \rho_{\rm ext} \end{cases}$$

$$H(\rho) = dh/dr = \begin{cases} 0 & \text{for} \quad \rho \le \rho_{\rm int} \\ \left(1 + \frac{4M\Omega}{\rho} + \frac{(8M^2 - C^2)\Omega^2}{\rho^2}\right) f & \text{for} \quad \rho_{\rm int} < \rho < \rho_{\rm ext}\,. \\ 1 + \frac{4M\Omega}{\rho} + \frac{(8M^2 - C^2)\Omega^2}{\rho^2} & \text{for} \quad \rho \ge \rho_{\rm ext} \end{cases}$$

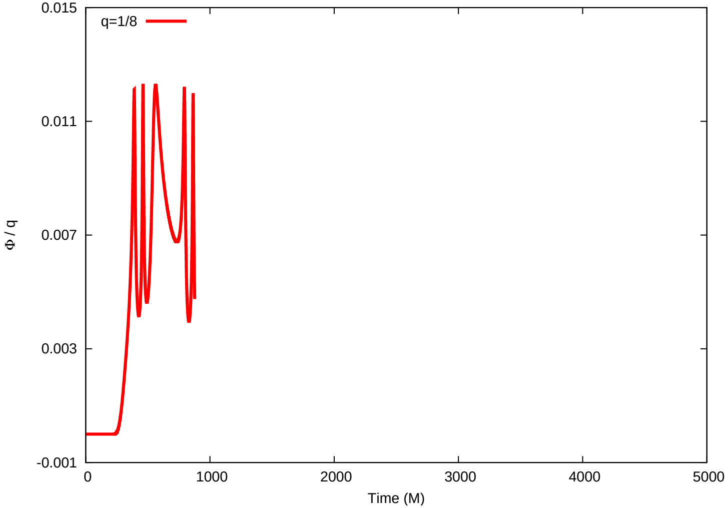
Here $f = 0$ for $\rho \le \rho_{\rm int}$, $f = 1$ for $\rho \ge \rho_{\rm ext}$ and f varies smoothly from 0 to 1 between $\rho_{\rm int}$ and $\rho_{\rm ext}$.

We typically use $S = C = 100M$, $\rho_{\rm int} = 25M$ and $\rho_{\rm ext} = 85M$.

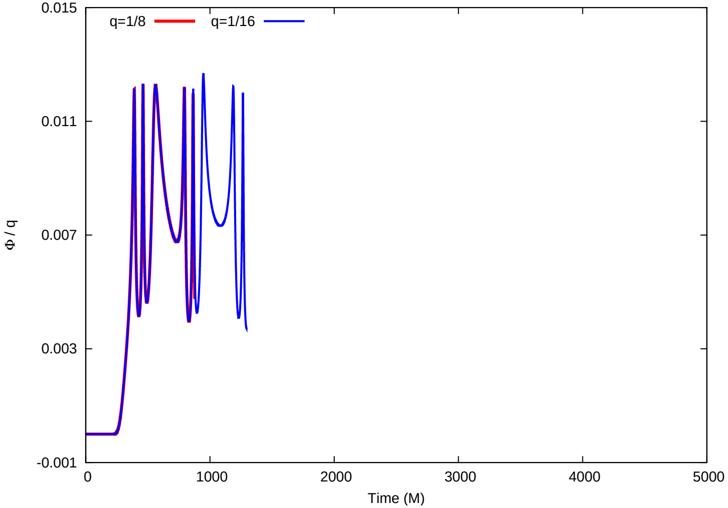
We can extract waveforms and energy fluxes at $\mathscr{I}^+$.

We have no problems with contaminations from our boundaries (outer or inner).

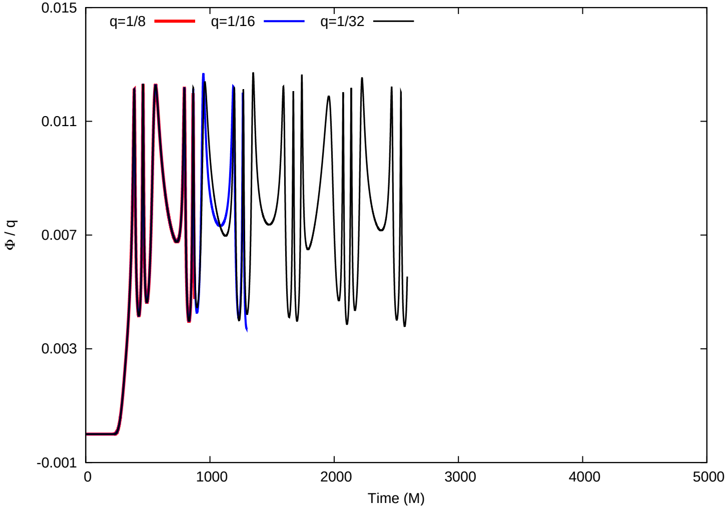
Waveform at $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



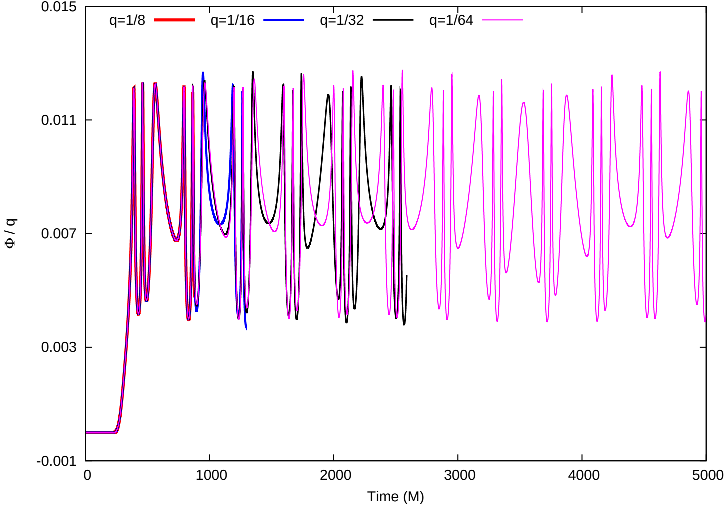
Waveform at $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



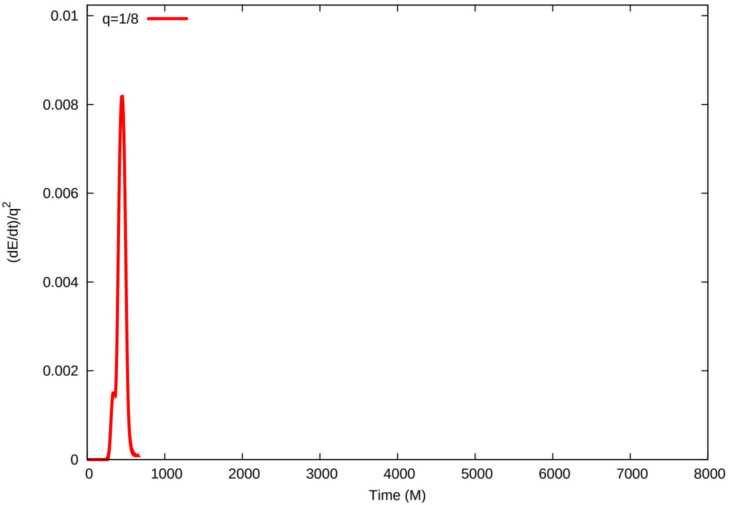
Waveform at $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



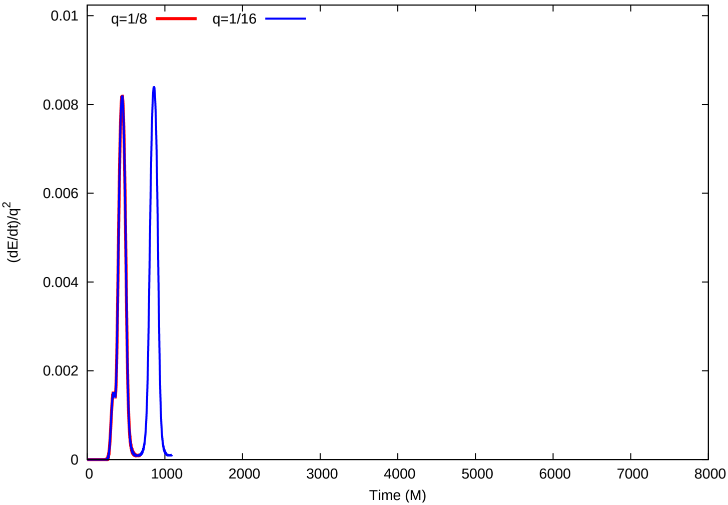
Waveform at $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



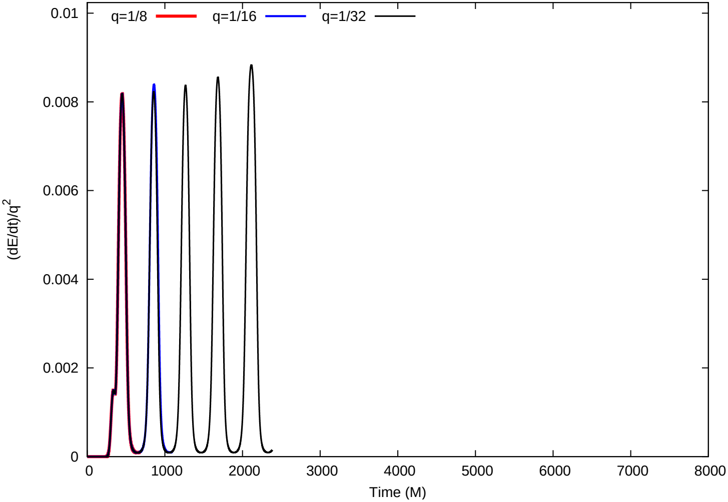
Energy flux through $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



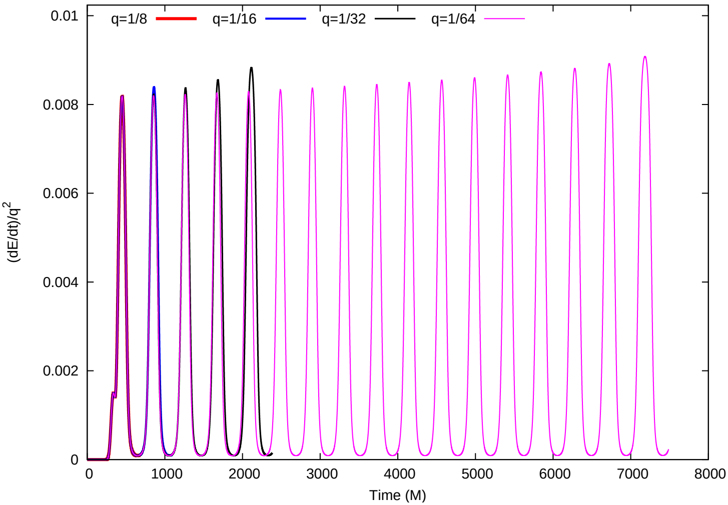
Energy flux through $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



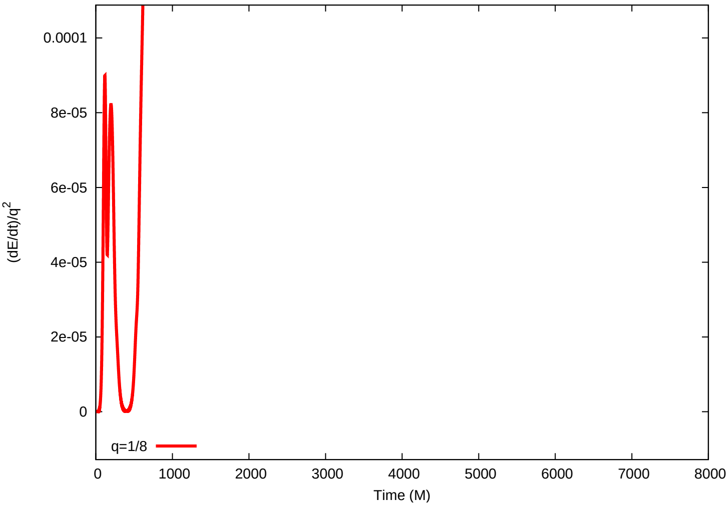
Energy flux through $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



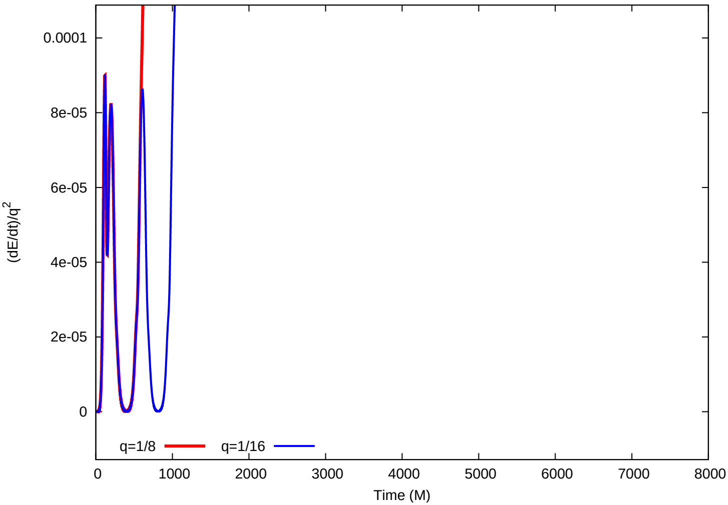
Energy flux through $\mathscr{I}^+$ ($e = 0.5$ and $p = 7.2$)



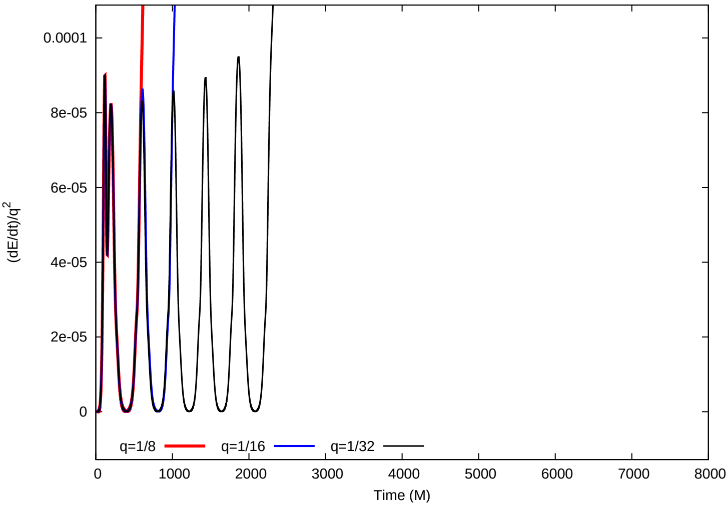
Energy flux through horizon ($e = 0.5$ and $p = 7.2$)



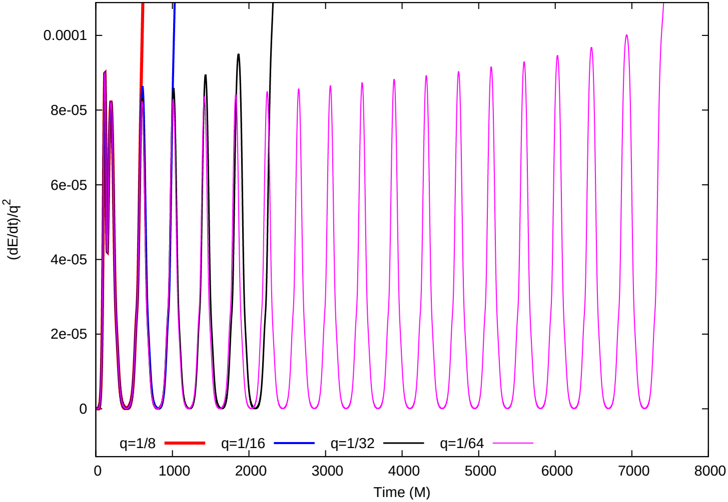
Energy flux through horizon ($e = 0.5$ and $p = 7.2$)



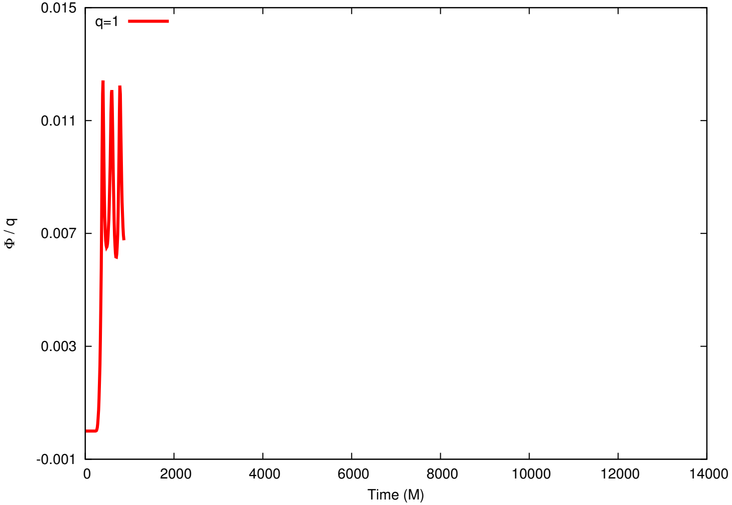
Energy flux through horizon ($e = 0.5$ and $p = 7.2$)



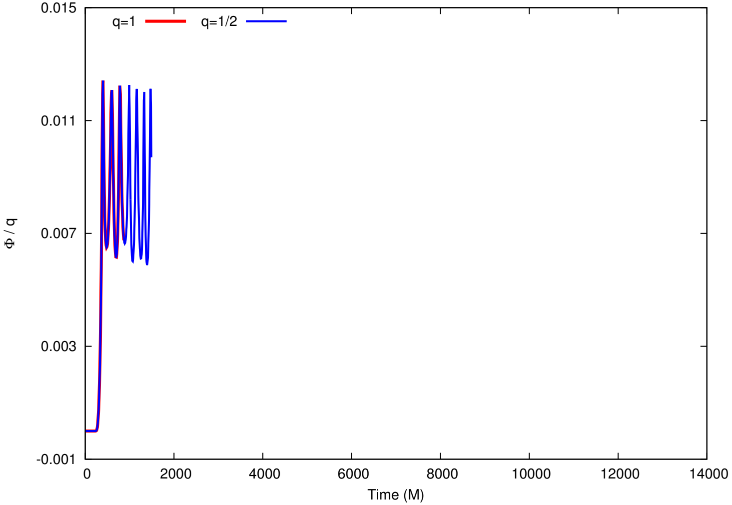
Energy flux through horizon ($e = 0.5$ and $p = 7.2$)



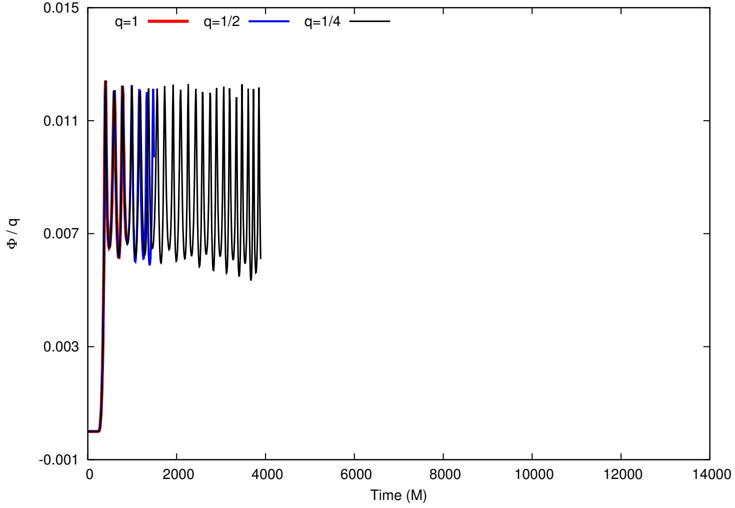
Waveform at $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



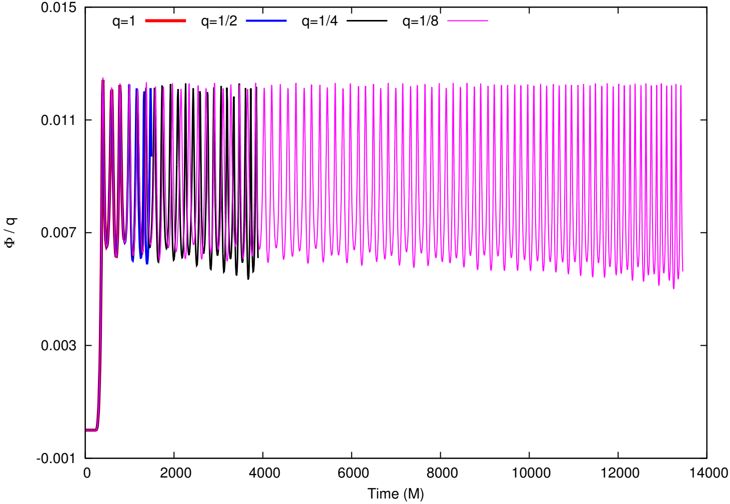
Waveform at $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



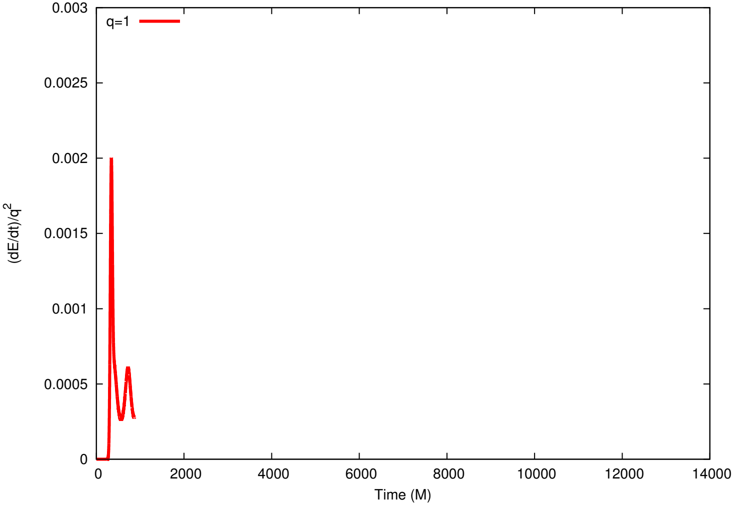
Waveform at $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



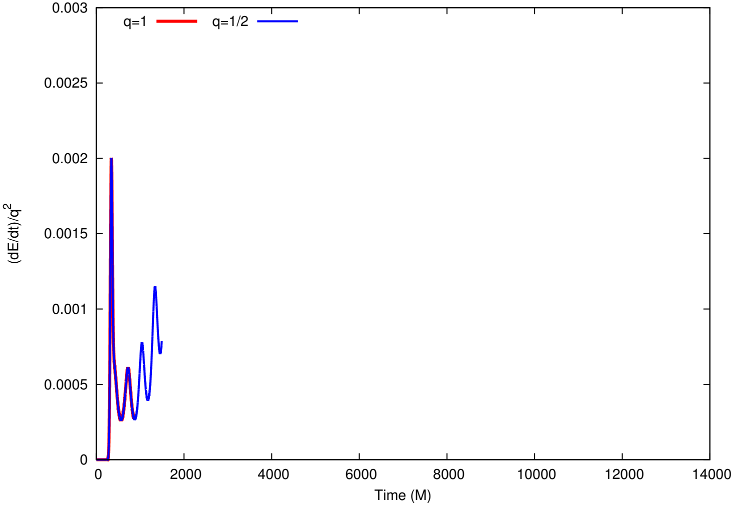
Waveform at $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



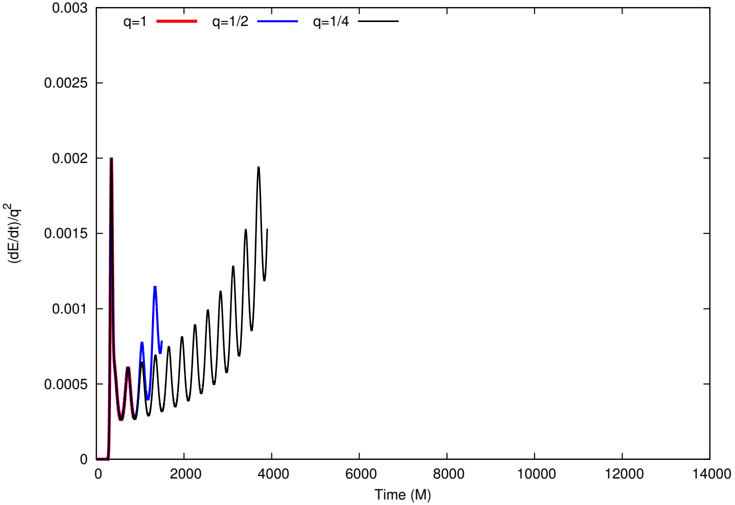
Energy flux through $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



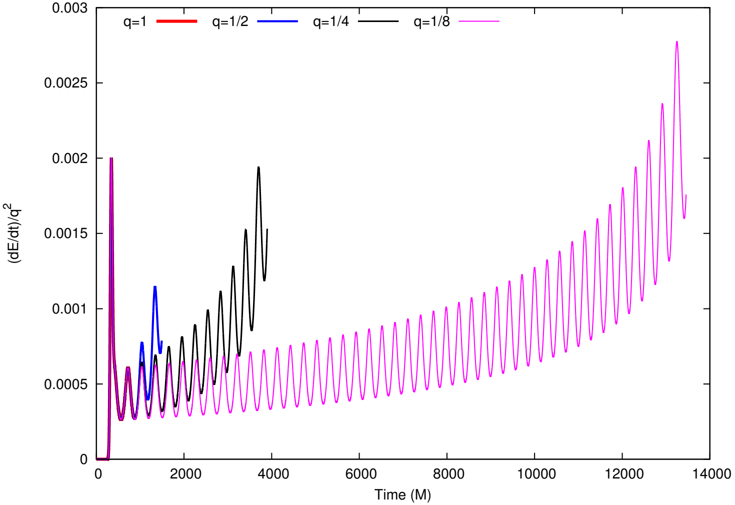
Energy flux through $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



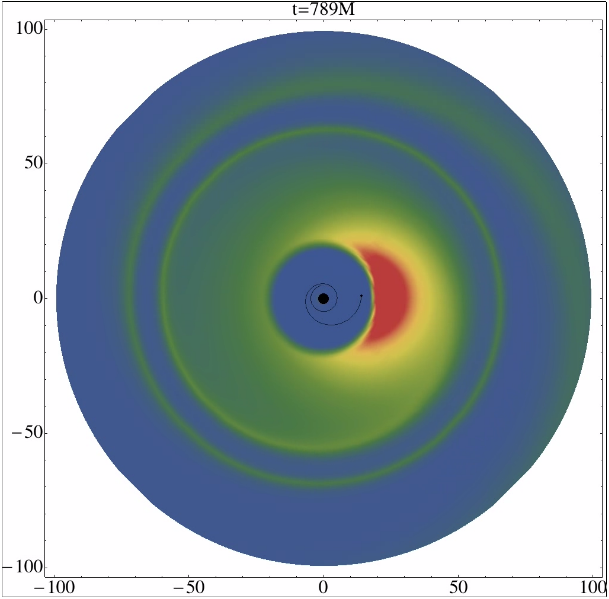
Energy flux through $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



Energy flux through $\mathscr{I}^+$ ($e = 0.1$ and $p = 10$)



Movie ($e = 0.5$, $p = 7.2$, $q = 1/32$)



Conclusions and future work

Conclusions

- We have computed the first self-consistent evolutions and waveforms of a scalar charge in orbit around Schwarzschild.
- The code is robust, well parallelized and fully generic.
- The main limitations are the expense of evaluating the effective source and the cost of evolving in 3D.

Future work.

- We plan to do self-consistent orbits in Kerr.
- We would like to compare evolutions based on the geodesic self-force.
- The extension of the method to the gravitational case is underway.