# Time-Domain Self-Force Computations Using Pseudospectral Methods

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# *Motivation*

*The metric perturbations generated by a point particle moving in a Schwarzschild black hole*

$$\Box_* \mathbf{U} + \mathbb{A}\,\partial_t \mathbf{U} + \mathbb{B}\,\partial_{r^*} \mathbf{U} + \mathbb{C}\,\mathbf{U} = \mathbf{F}\delta[r - r_p(t)]\,;$$

*Scalar charged particle falling in a geodesic of a Schwarzschild MBH spacetime.*

$$\Box\Phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = -\rho 4\pi$$

$$\rho = -4\pi q \int \delta_4(x - z(\tau))\,d\tau$$

*Problems: Distributional source term and divergence of the field at the particle location*

*The retarded field can be decomposed into spherical harmonics:*

$$\Phi^{ret} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Phi^{lm}(t,r) Y^{lm}(\theta,\varphi)$$

*The equation for each harmonic coefficient:*

$$(-\partial_t^2 + \partial_{r^*}^2 - V_l)\psi^{lm} - S^{lm} = 0$$

$$V_l = \left(1 - \frac{2M}{r}\right)\left[\frac{2M}{r^3} + \frac{l(l+1)}{r^2}\right] \qquad \Phi^{\ell m} = \frac{\psi^{\ell m}}{r}$$

$$\rho \rightarrow S^{lm} = A^{lm}\delta[r^* - r_p^*(t)]$$

# The Particle-without-Particle Scheme

*The mode-sum regularization scheme provides an analytic expression for the field singularities.*

Computed Analytically

$$\Phi^{ret} = \Phi^{S} + \Phi^{R} \quad \begin{cases} \Box\Phi^{S} = -4\pi q\, \delta(z) \\ \Box\Phi^{R} = 0 \end{cases}$$

Computed Numerically

$$\mathcal{F}_\alpha = q(\nabla_\alpha \Phi^{ret} - \nabla_\alpha \Phi^{S}) = q\nabla_\alpha \Phi^{R}$$

**To compute the $\Phi^{ret}$ using PSC methods we have developed a scheme that removes the singularity associated with the particle**

$$\mathcal{U} \Rightarrow \Phi^{ret}$$

## Numerical techniques to compute the field modes

### Frequency domain

- *Fourier harmonic decomposition: Solve ODEs*
- *Sum over the Fourier harmonics: Difficulties for handling high eccentric orbits*

### Time domain

- *Solve the PDE for each field mode*
- *Handles in the same way circular and eccentric orbits*
- *Numerically expensive due to the large scale variance in the solutions*

## *Three approaches for solving PDEs in time-domain*

- *Finite differences*
- *Finite elements (FE)*
- *Spectral methods (SM)*

→ *Easiest to code but are computationally expensive* (Finite differences)

→ *Both expand the solution in basis functions and use multidomain grids* (FE, SM)

- *FE  suited to irregular geometries.*
- *SM use fewer subdomains than FE and for sufficiently regular domains, are generally faster and/or more accurate.*

*[see e.g. H. P. Pfeiffer et al. 2003]*

# Outlook

- *Basics about pseudospectral methods*
- *Polynomial expansion and discretization points*
- *How to apply pseudospectral-collocation methods to the EMRI problem: Scalar case in Schwarzschild space-time*
- *Examples: Circular & eccentric case*
- *Results*

*Consider an arbitrary system of hyperbolic partial differential equations (PDEs) defined on* $\Omega \subset \mathbb{R}^d$

$$L[\mathcal{U}](x) = S(x) \qquad x \in \Omega$$

*with boundary conditions*

$$H[\mathcal{U}](x) = 0 \qquad x \in \partial\Omega$$

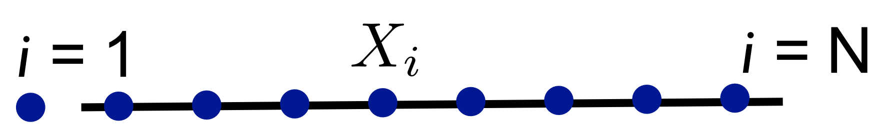
## The spectral representation

*In 1D spatial dimension pseudosectral collocation methods (PSCM) are based on expansions of every evolved field $\mathcal{U}(x)$ in terms of suitable basis functions $T_n(X)$ with (spectral) coefficients $a_n$*

$$\mathcal{U}(X) \equiv \mathcal{U}_N(X) = \sum_{n=0}^{N} a_n T_n(X)$$

*Known*

*We can derive* $a_n$ *from the values of* $T_n(X)$ *at the (discretization) collocation points* $X = X_i$ *(* $i = 1 \dots N$ *)*



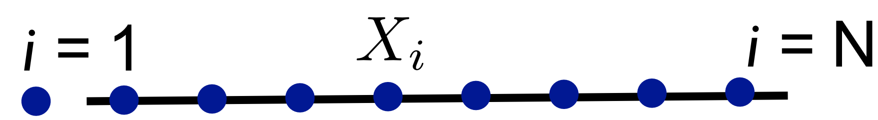

$$\mathcal{U}(X_i) = \sum_{n=0}^{N} a_n T_n(X_i) \longrightarrow a_n = \frac{[\mathcal{U}_n(X_i), T_n(X_i)]}{|T_n(X)|}$$

$$\mathbb{M}_{ij}\mathcal{U}(X_i) = a_j$$

## The physical representation

$\mathcal{U}(x)$ *is approximated employing the Lagrange Cardinal functions*

$\mathcal{C}_i(x_j) = \delta_{ij}$ *,associated with* $\{T_n(X)\}$.




$$\mathcal{U}_N(X) = \sum_{n=0}^{N} \mathcal{U}(X_i)\mathcal{C}_i(X)$$

*Change between representations employing matrix multiplication transformation*

$$\mathcal{U}(X_i) = \mathbb{M}_{ij}^{-1} a_j$$

The derivatives of $T_n(X)$ and $\mathcal{C}_i(X)$ are known analytically

$$\partial_X^{(m)} \mathcal{U}_N(X) = \sum_{n=0}^{N} a_n \partial_X^{(m)} T_n(X)$$

$$\partial_X^{(m)} \mathcal{U}_N(X_i) = \sum_{i=0}^{N} \sum_{j=0}^{N} \partial_X^{(m)} \mathcal{C}_i(Xj)\, \mathcal{U}_N(X_j) \mathcal{C}_i(X)$$

The derivatives are obtained through a matrix multiplication

## Interpolation Error

The error in interpolating the solution is given by Cauchy interpolation error

$$\mathcal{U} - \mathcal{U}_N(X) = \frac{1}{N+1!}\mathcal{U}^{N+1}(\xi)\prod_{i=0}^{N}(X - X_i)$$

controlled by changing the location of the collocation points,

$$\mathrm{Error} = \log_{10}|a_N|$$

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*controlled by changing the location of the collocation points,*

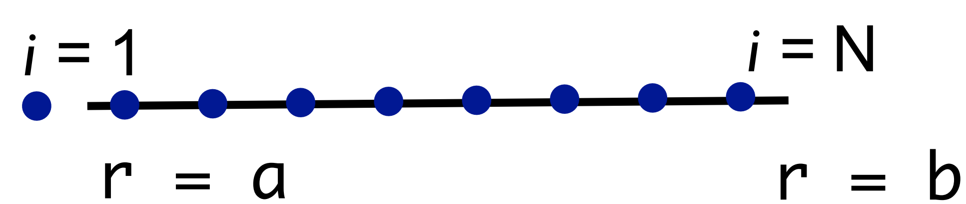
$$\text{Error} = \log_{10}|a_N|$$

**The solutions converge exponentially with N**

# Time step condition

*The domain of dependence of the system of hyperbolic equations evolve at finite velocity which leads to causality restrictions: Courant-Friedrichs-Lax conditions*

$$\Delta t_{CFL} \sim \frac{\pi^2 |b-a|}{4N^2}$$

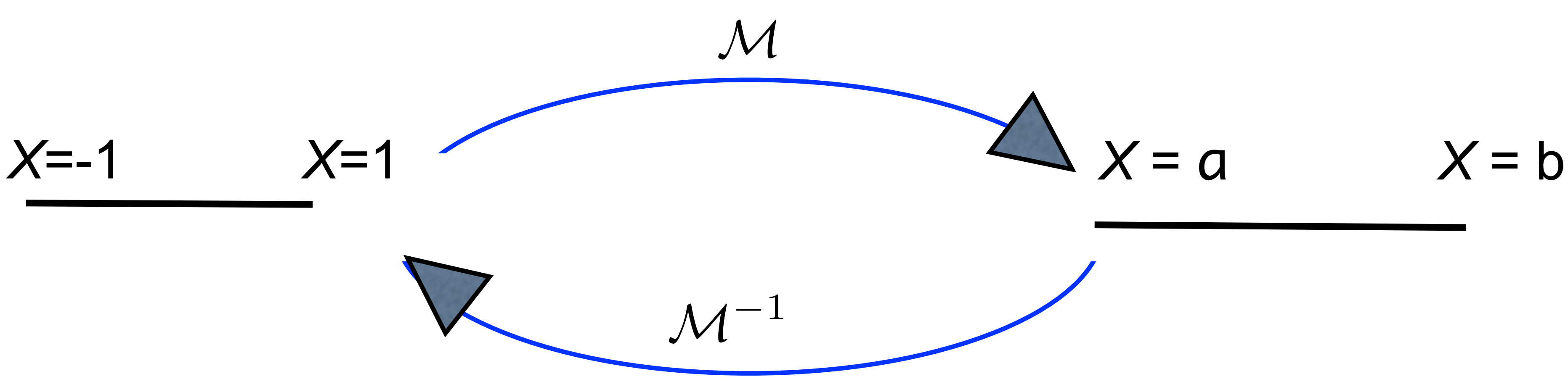


# *Expansion Basis & and discretization points*

*The solutions can be discretized using an expansion in a basis of Chebyshev polynomials*

$$T_n(X) = \cos\left(n \cos^{-1}(X)\right), \qquad X \in [-1, 1] \, .$$

*The domain of definition of $T_n(X)$ can always be mapped to the spatial (sub)domain of our problem*



## Chebyshev series can be expressed as a Fourier cosine series

$$X : \quad [0, 2\pi] \longrightarrow [-1, 1]$$
$$\theta \longrightarrow X(\theta) = \cos(\theta)$$

$$\mathcal{U}(X) = \sum_{n=0}^{\infty} a_n T_n(x) \Leftrightarrow \mathcal{U}(\cos\theta) = \sum_{n=0}^{\infty} a_n \cos(n\theta)$$

The Fourier series of $\mathcal{U}(\cos\theta)$ have exponential convergence, unless $\mathcal{U}(X)$ is singular.

*Matrix multiplications can be performed using a FFT algorithm:*

$\mathcal{O}(N \ln(N))$ *operations instead of* $N \times N$ *operations needed in a direct matrix multiplication*

$$\partial_{r^*} : \{\boldsymbol{U}_i\} \xrightarrow{FFT} \{\boldsymbol{a}_n\} \xrightarrow{\partial_{r^*}} \{\boldsymbol{b}_n\} \xrightarrow{FFT} \{(\partial_{r^*}\boldsymbol{U})_i\}$$

The set of collocation points that minimises the Cauchy interpolation error corresponds to the zeros of the Chebyshev polynomial or alternatively to the extrema of its derivative: Lobatto-Chebyshev grid

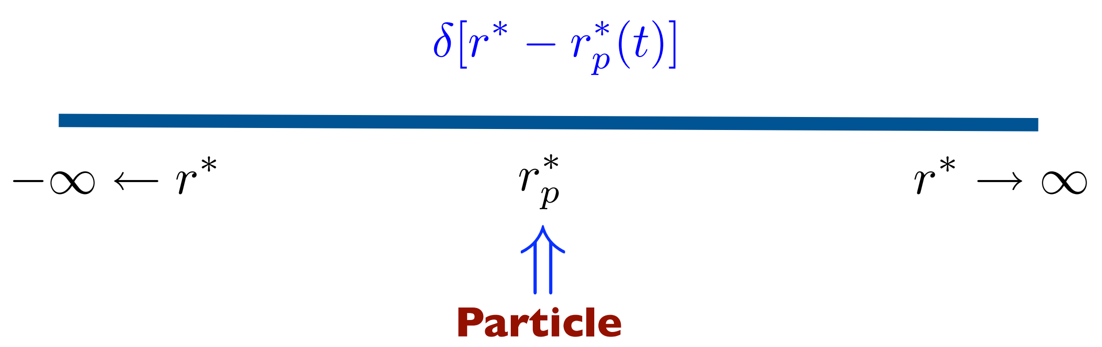
$$(1 - X^2)T'_N(X) = 0$$

$$X_i = -\cos\left(\frac{\pi\, i}{N}\right) \qquad (i = 0, 1, \ldots, N)$$

# PSCM & The EMRI problem

# The Particle-without-Particle Scheme

$$(-\partial_t^2 + \partial_{r^*}^2 - V_\ell)\psi^{\ell m} = A^{\ell m}\delta[r^* - r_p^*(t)]$$




*The computational domain is split and the particle is set at the interface between 2 subdomains*

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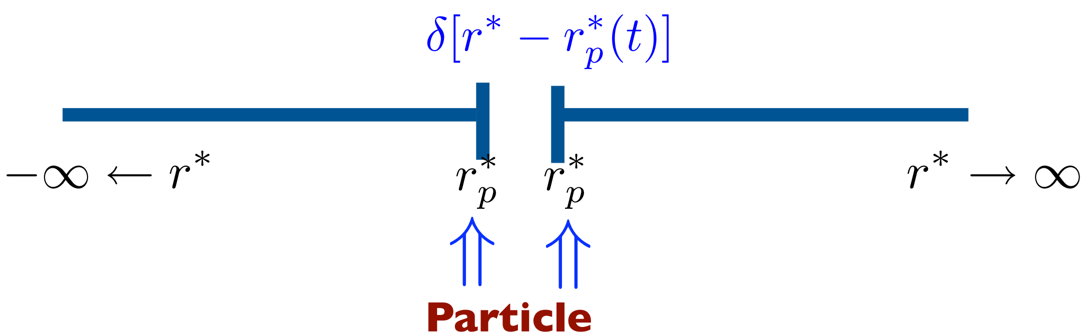


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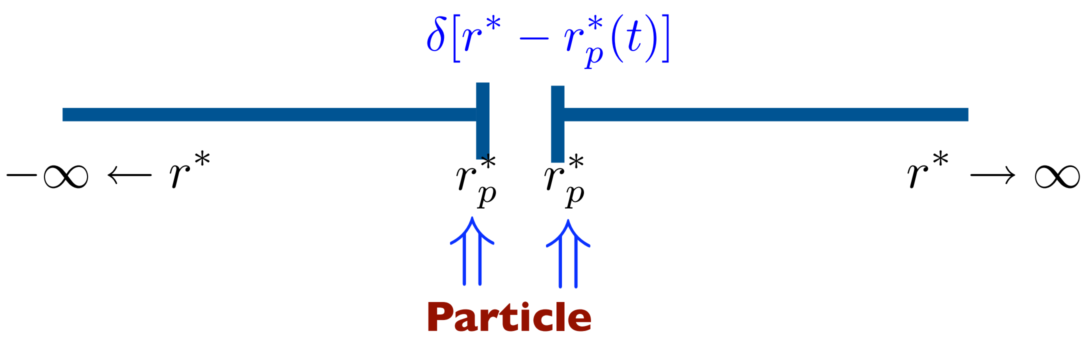



*The computational domain is split and the particle is set at the interface between 2 subdomains*

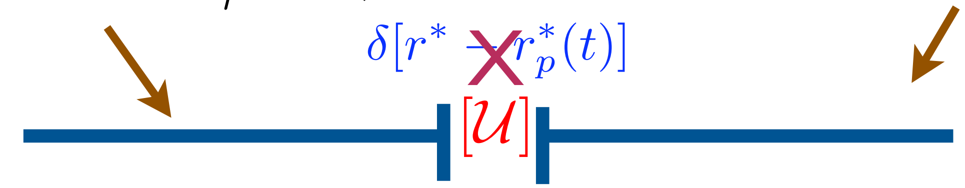
$$\mathcal{U} = \mathcal{U}_- \Theta(r_p^* - r^*) + \mathcal{U}_+ \Theta(r_p^* - r^*)$$

$$\partial_t \mathcal{U}_\pm = \mathbb{A}\cdot\partial_{r^*}\mathcal{U}_\pm + \mathbb{B}\cdot\mathcal{U}_\pm$$

**+**

$$[\mathcal{U}] = \lim_{r^*\to r_p^*}\mathcal{U}_+ - \lim_{r^*\to r_p^*}\mathcal{U}_-$$

$$\partial_t \mathcal{U}_- = \mathbb{A}\cdot\partial_{r^*}\mathcal{U}_- + \mathbb{B}\cdot\mathcal{U}_- \qquad \partial_t \mathcal{U}_+ = \mathbb{A}\cdot\partial_{r^*}\mathcal{U}_+ + \mathbb{B}\cdot\mathcal{U}_+$$




***The PwP technique ensures smooth solutions***

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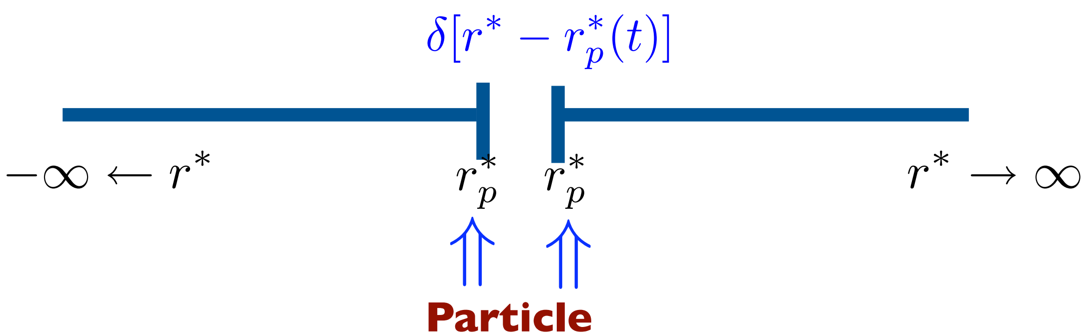


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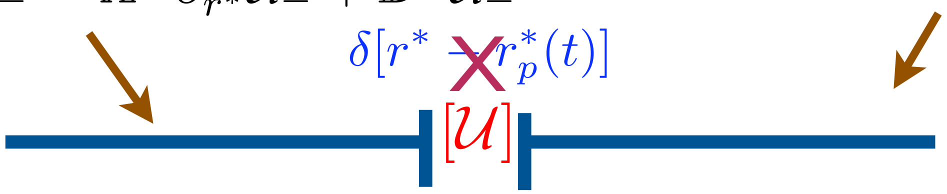


***The PwP technique ensures smooth solutions***

*The discontinuities on hyperbolic equations propagate along the characteristics.*

$$\mathcal{U} = (\psi^{\ell m}, \phi^{\ell m}, \varphi^{\ell m})$$

$$\partial_t\mathcal{U} = \mathbb{A}\cdot\partial_{r^*}\mathcal{U} + \mathbb{B}\cdot\mathcal{U}$$

$$\psi^{\ell m} = r\,\Phi^{\ell m}$$
$$\phi^{\ell m} = \partial_t\psi^{\ell m}$$
$$\varphi^{\ell m} = \partial_{r^*}\psi^{\ell m}$$

$$\mathbb{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbb{B} = \begin{pmatrix} 0 & 1 & 0 \\ -V_\ell & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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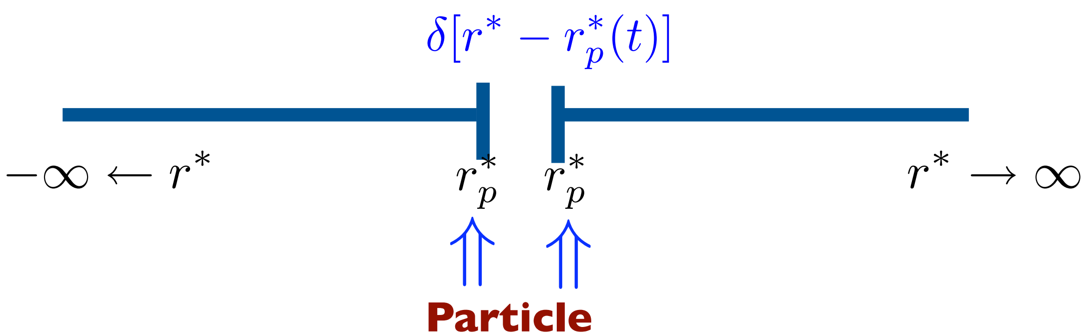


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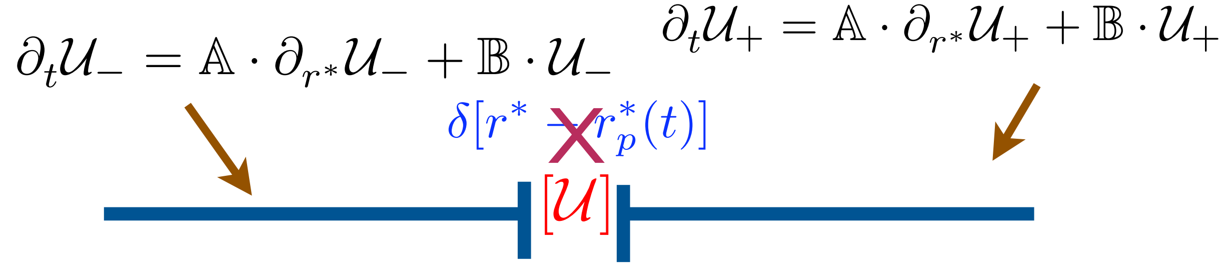


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The jumps are enforced employing two different methods:

1. The penalty method.
2. Communication of the characteristic fields

## 1. **The penalty method**:

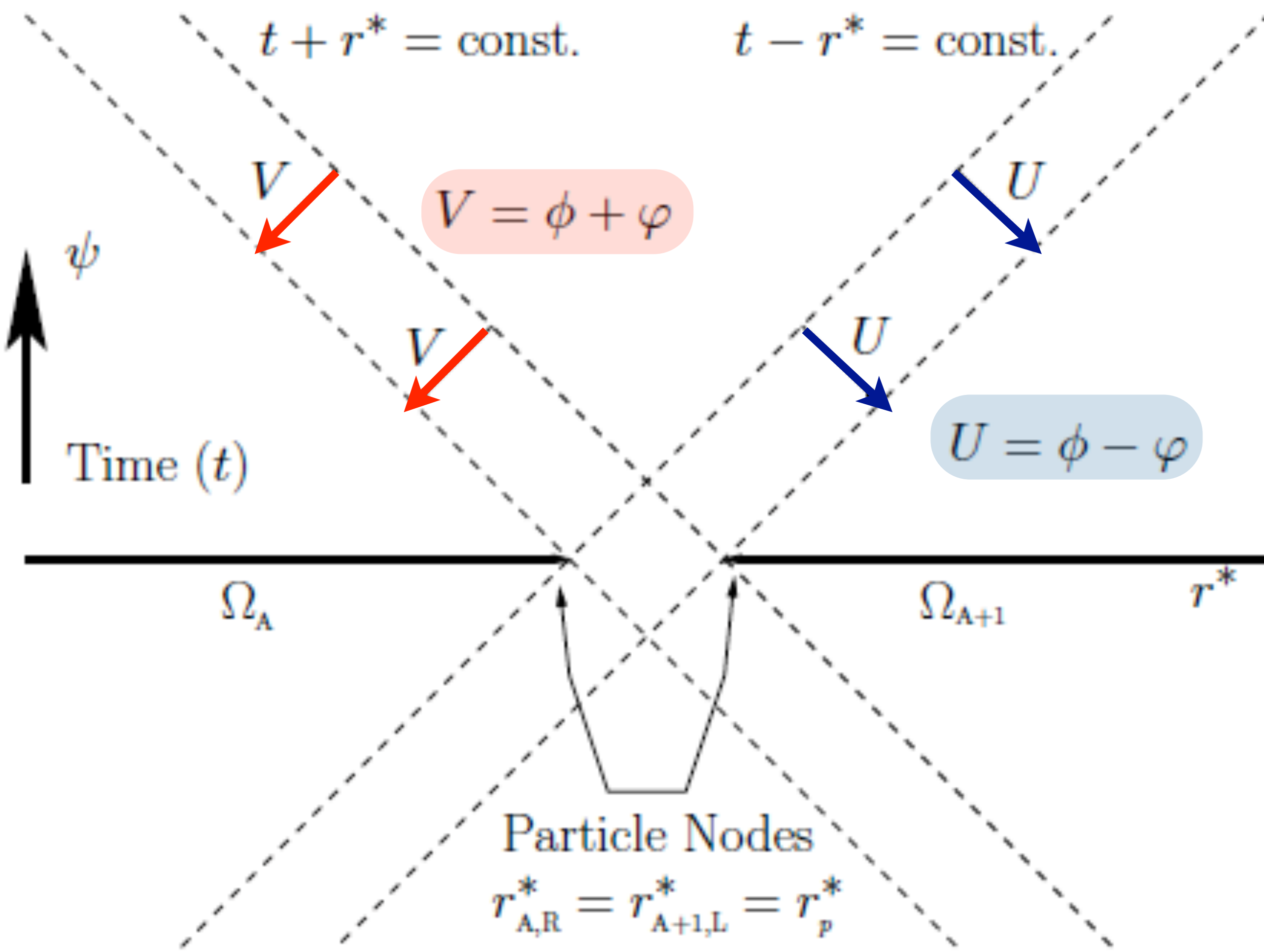
The system is dynamically driven to fulfil a set of additional conditions.

$$\partial_t \mathcal{U}_\pm = \mathbb{A} \cdot \partial_{r^*} \mathcal{U}_\pm + \mathbb{B} \cdot \mathcal{U}_\pm + \eta(\tau_{\mathcal{U}}[\mathcal{U}])$$

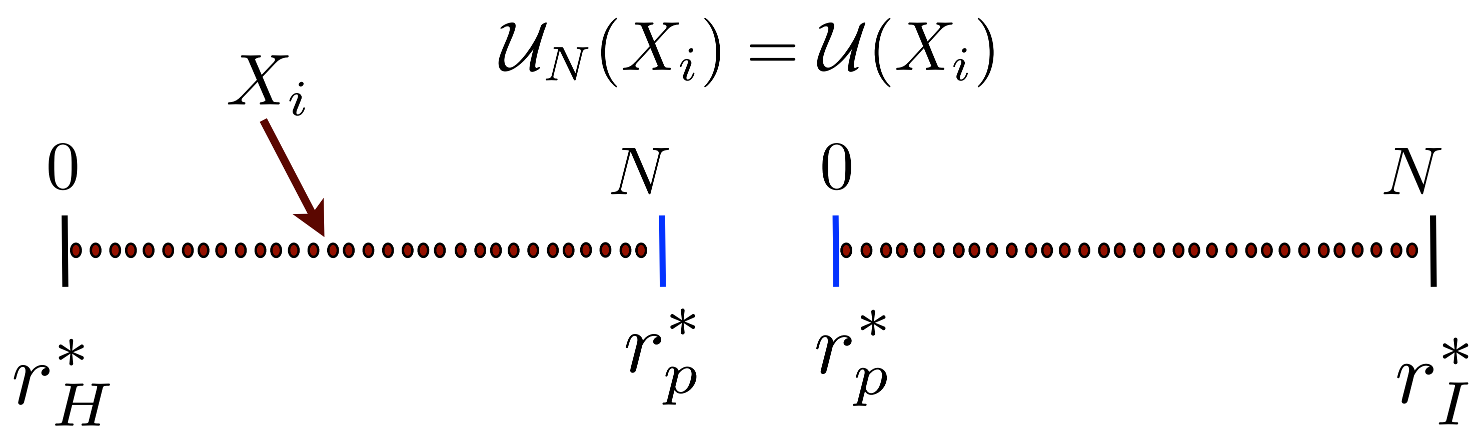
## 2. **The direct communication of the characteristic fields**:

We pass the value of the characteristic fields.

$$\psi^{\ell m} = r\,\Phi^{\ell m}$$
$$U^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$
$$V^{\ell m} = \phi^{\ell m} - \varphi^{\ell m}$$
$$\downarrow$$
$$\mathcal{U} = (\psi^{\ell m}, U^{\ell m}, V^{\ell m})$$



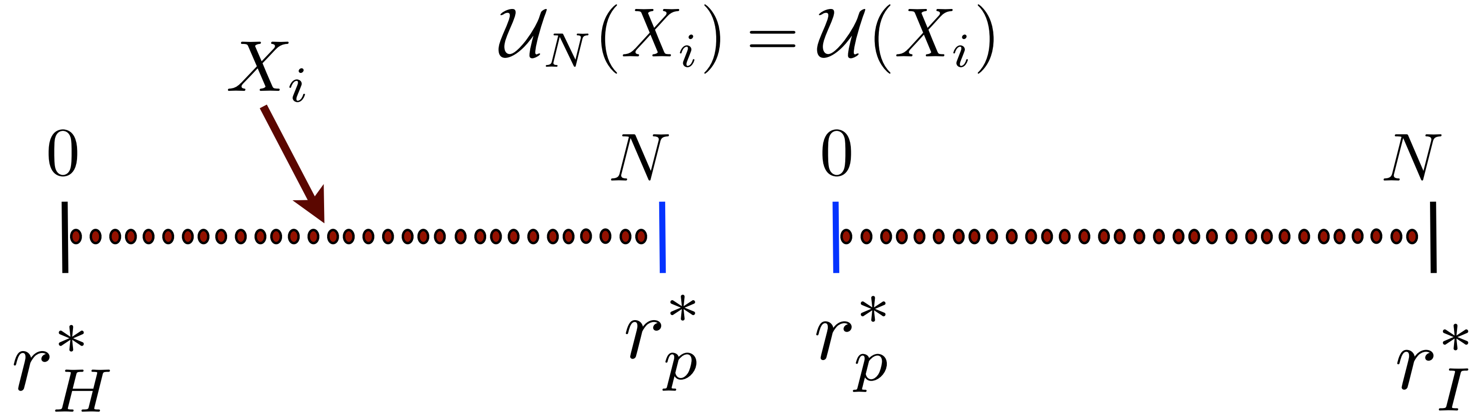
- *To implement the PwP scheme numerically we use **PSCM**. Each subdomain is discretised with a number N of collocation points of a Lobatto-Chebyshev grid:*



# The Particle-without-Particle Scheme

- *To implement the PwP scheme numerically we use* ***PSCM****. Each subdomain is discretised with a number N of collocation points of a Lobatto-Chebyshev grid:*

$$\mathcal{U}_N(X_i) = \mathcal{U}(X_i)$$




*To prevent incoming signals from outside the physical domain*

$$\begin{aligned} \phi^{\ell m}(t, r^*_{\rm H}) - \varphi^{\ell m}(t, r^*_{\rm H}) &= 0 = U^{\ell m}(t, r^*_{\rm H}) \qquad r^*_{\rm H} \to -\infty \\ \phi^{\ell m}(t, r^*_{\rm I}) + \varphi^{\ell m}(t, r^*_{\rm I}) &= 0 = V^{\ell m}(t, r^*_{\rm I}) \,. \qquad r^*_{\rm I} \to \infty \end{aligned}$$

# The Particle-without-Particle Scheme

*Employing a Chebyshev basis there are some paybacks:*

- *Physical representation* $\overset{\textbf{FFT}}{\longleftrightarrow}$ *Spectral representation*

$$\{\mathcal{U}_i\} \qquad \{a_i\}$$

- *Differentiation is cheaper in the spectral domain:*

$$\mathcal{U}'_N = \sum_{j=0}^{N} D_{ij}\,\mathcal{U}_j(X) \qquad \mathcal{U}'_N = \sum_{j=0}^{N} b_j\,T_j(X)$$

***N²*** *operations* $\qquad$ ~ ***NLn(N)*** *operations*

*Employing a Chebyshev basis there are some paybacks:*

**FFT**

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$$\mathcal{U}'_N = \sum_{j=0}^{N} D_{ij}\,\mathcal{U}_j(X)$$

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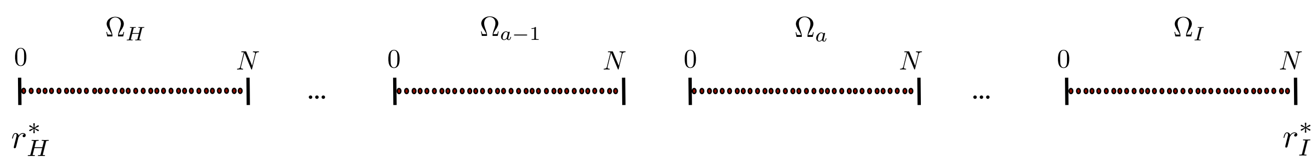
$N^2$ *operations* ~ **NLn(N)** *operations*

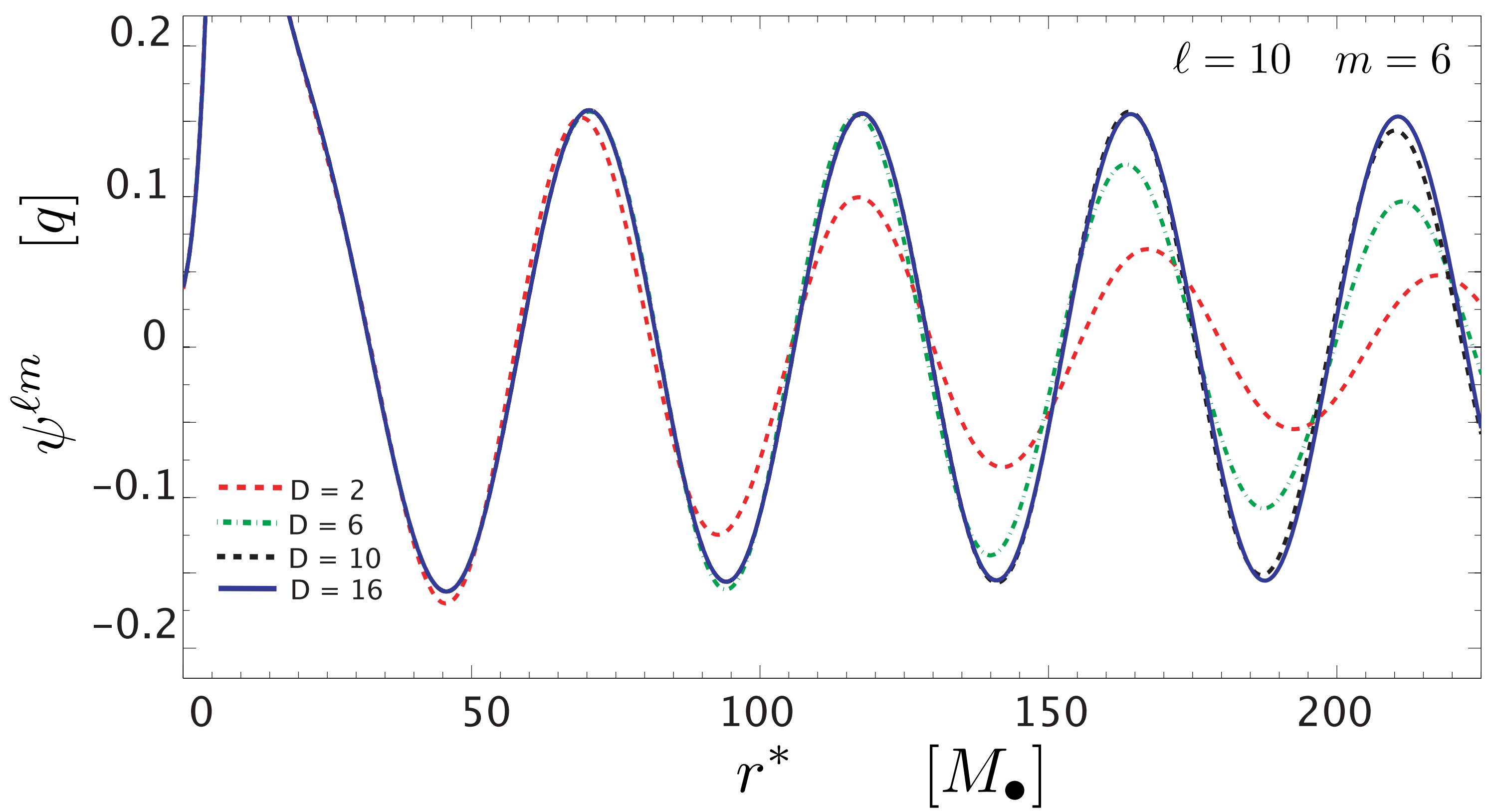
**Our (smooth) solutions converge exponentially with N**

## Multidomain flexibility

- *Covering the spatial domain with a given number of subdomains (D) we improve the field resolution with a relatively small N.*




$$\Delta r^*_a = r^*_N - r^*_0$$

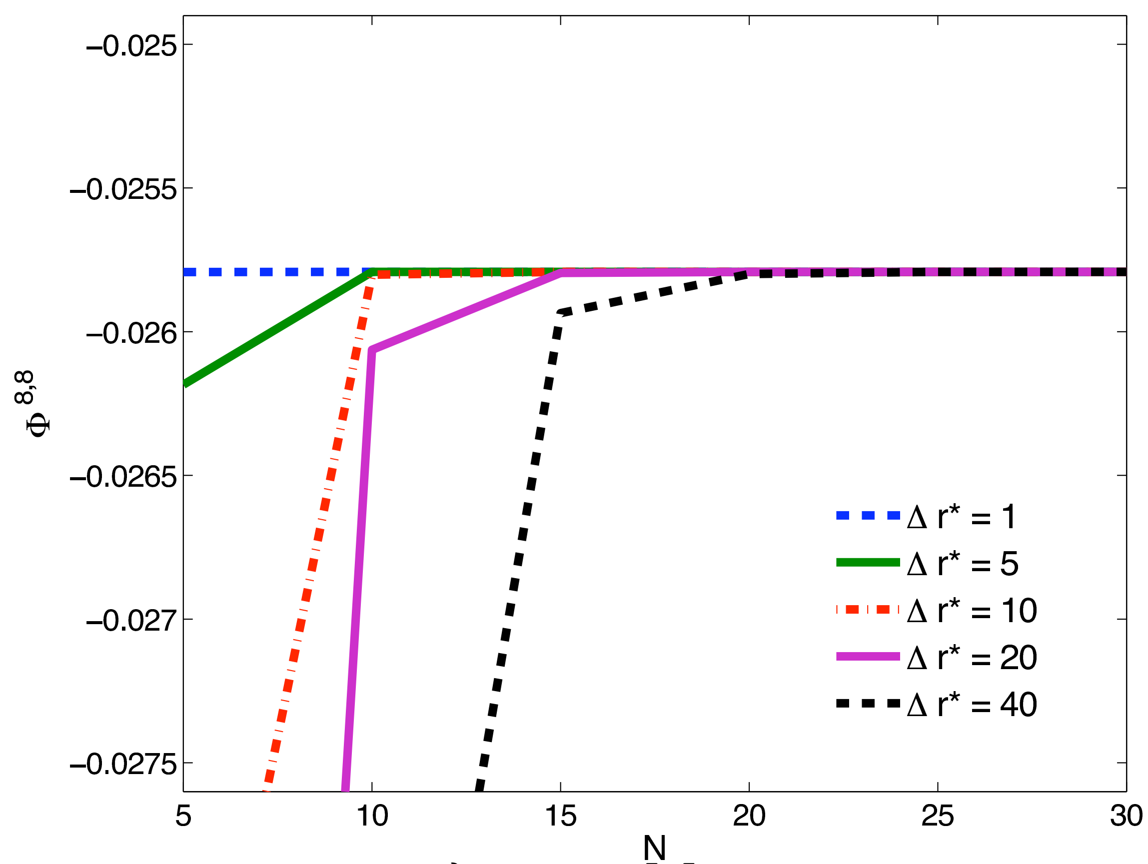



*Resolution* = { *D* ↑ , *N* ↓ ; *D* ↓ , *N* ↑ }

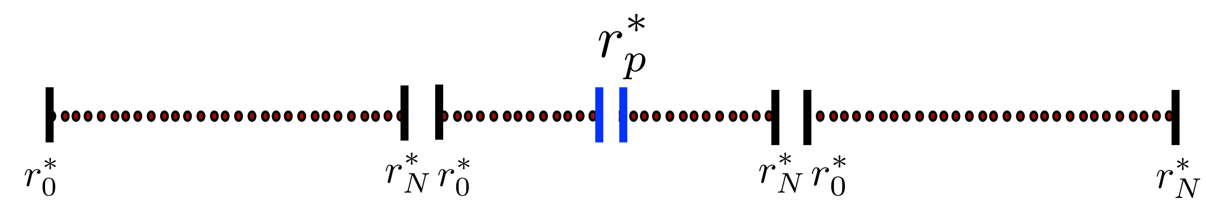
$$\Delta r^* = 50 M_\bullet \qquad N = 50$$

## *Advantages of the Multidomain Framework*

- *Different harmonic modes need different resolution*
- *We adjust the size of the subdomain around the particle location to the smaller mode wavelength*




$\lambda_8 \sim 11 M_\bullet$




$$\Delta r^* = r_N^* - r_0^*$$

$$\boxed{\Delta r^* \sim \lambda_m}$$

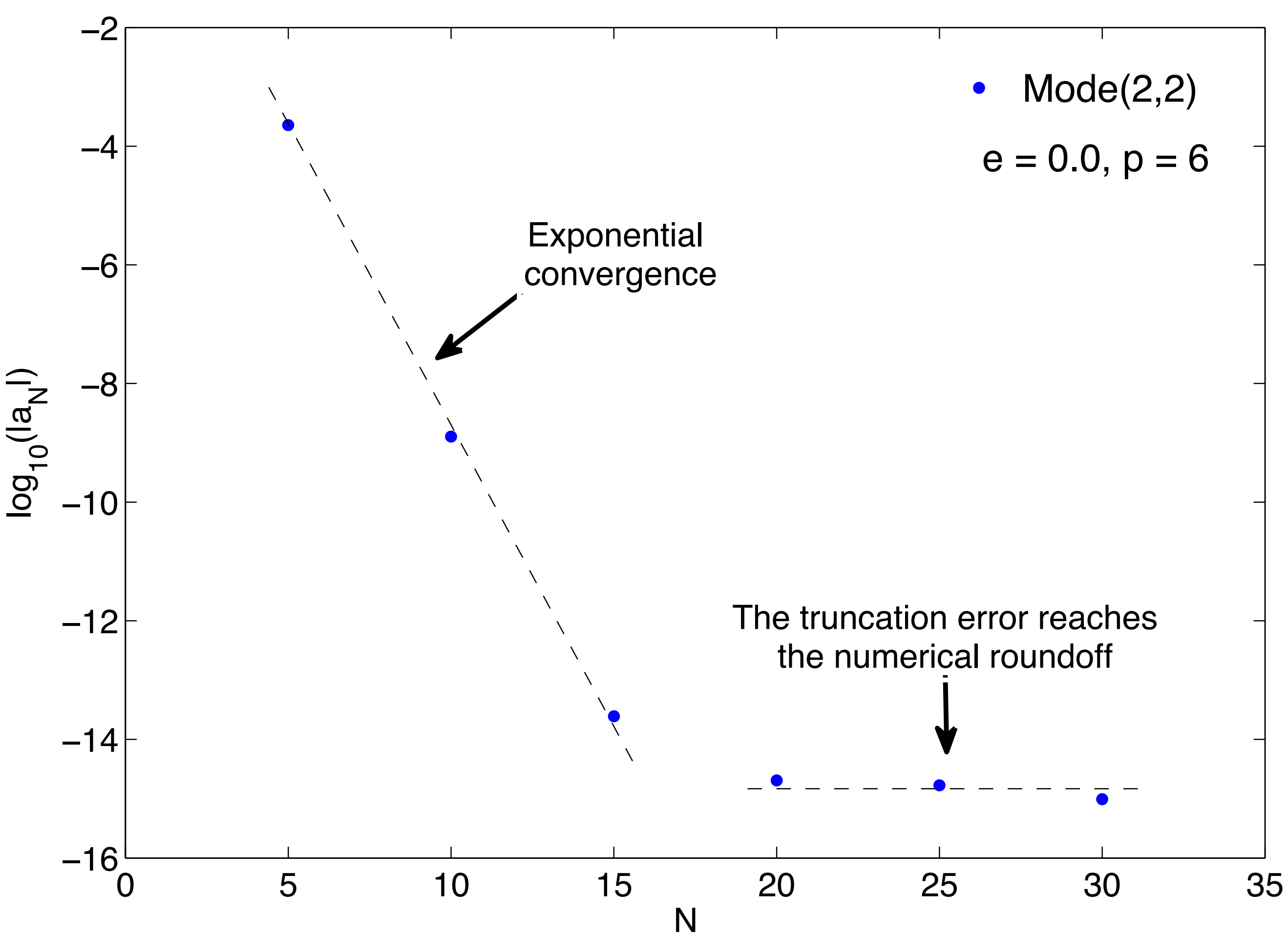
$$\rho \sim \frac{\Delta r^*}{N}$$

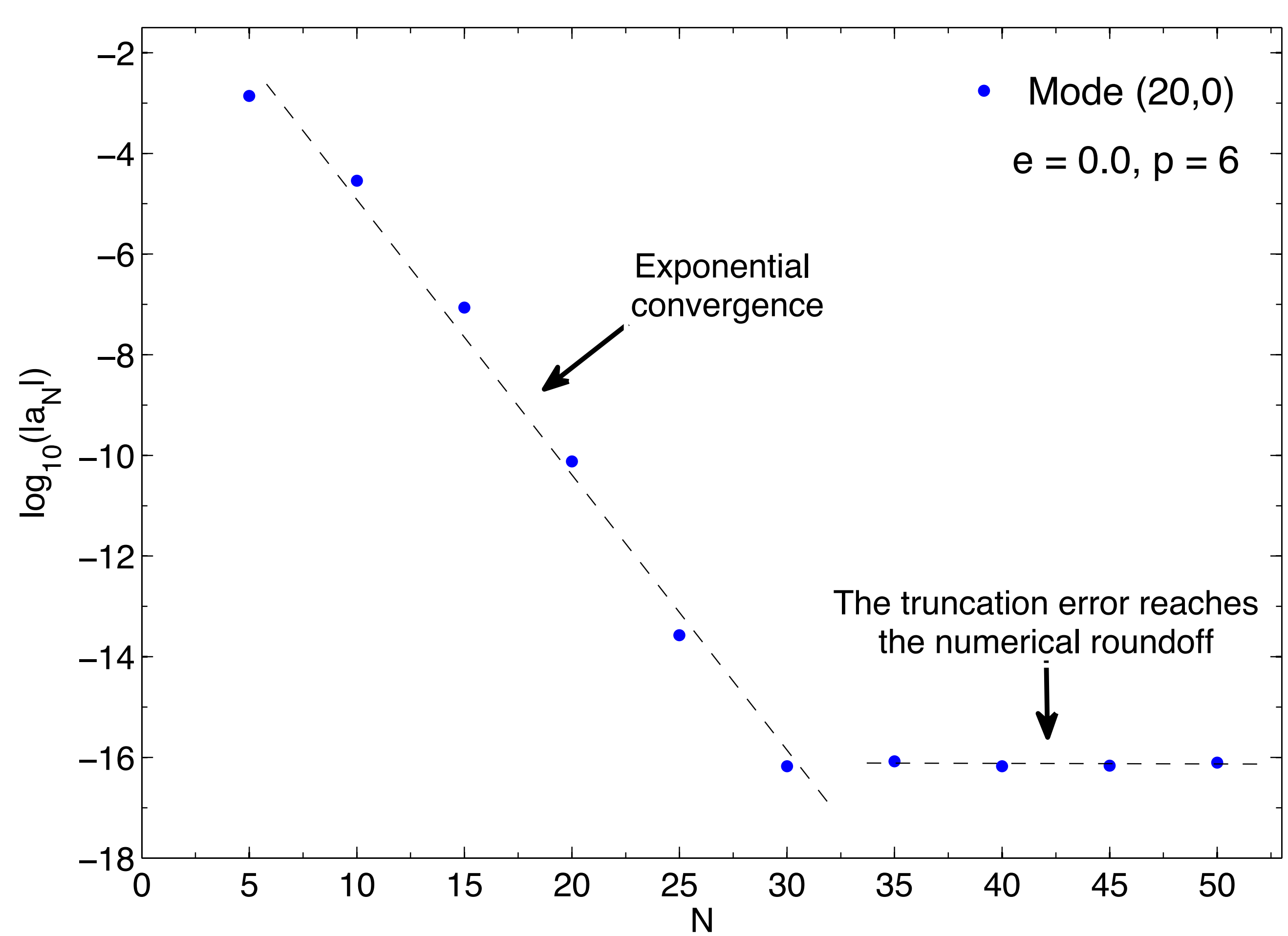
*[ Canizares & Sopuerta (2011)].*

# Convergence Test (circular case)

*The dependence of the truncation error (~ $|a_N|$ ) with respect increasing numbers of collocation points, N, give us an estimation of the exponential convergence of the code:* $e^{-N}$

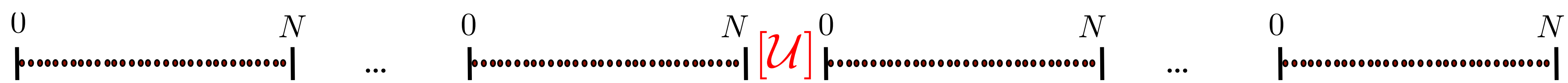
$(\ell, m) = (2, 2)$ $(\ell, m) = (20, 0)$

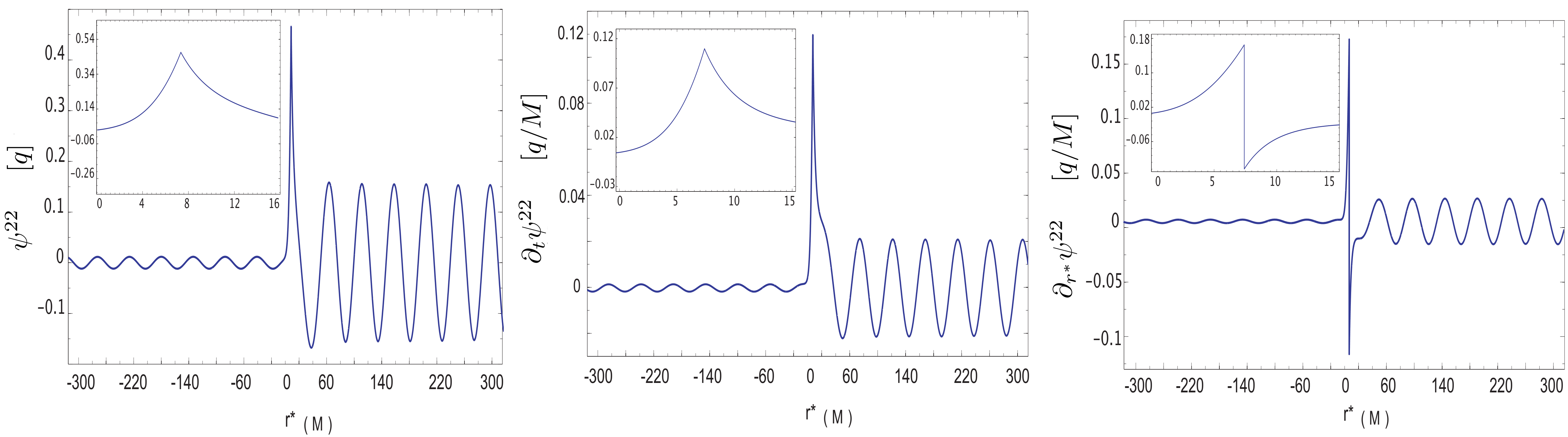



*[ Canizares & Sopuerta (2009)] [ Canizares & Sopuerta (2011)].*

## *Snapshots from the Circular case*
## *(D=12, N=50)*




$$[\psi^{\ell m}]_p = 0 \ ,$$

$$\left[\partial_t \psi^{\ell m}\right]_p = 0$$

$$\left[\partial_{r^*} \psi^{\ell m}\right]_p = A^{\ell m}$$

*[ Canizares & Sopuerta (2009)]*

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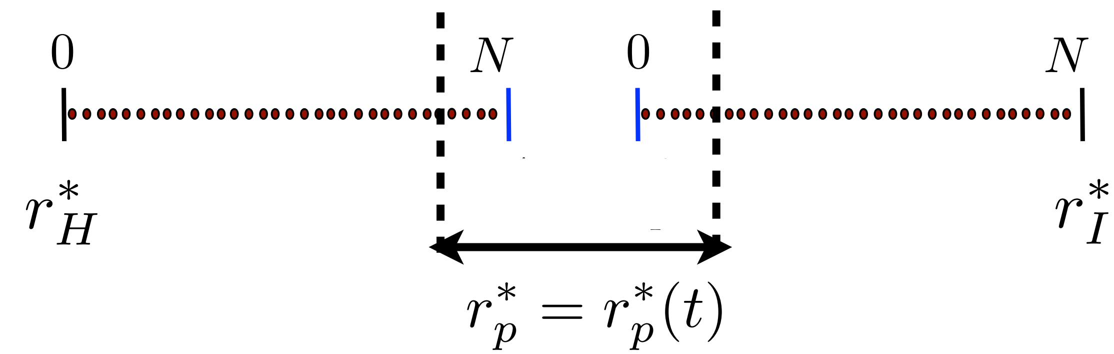
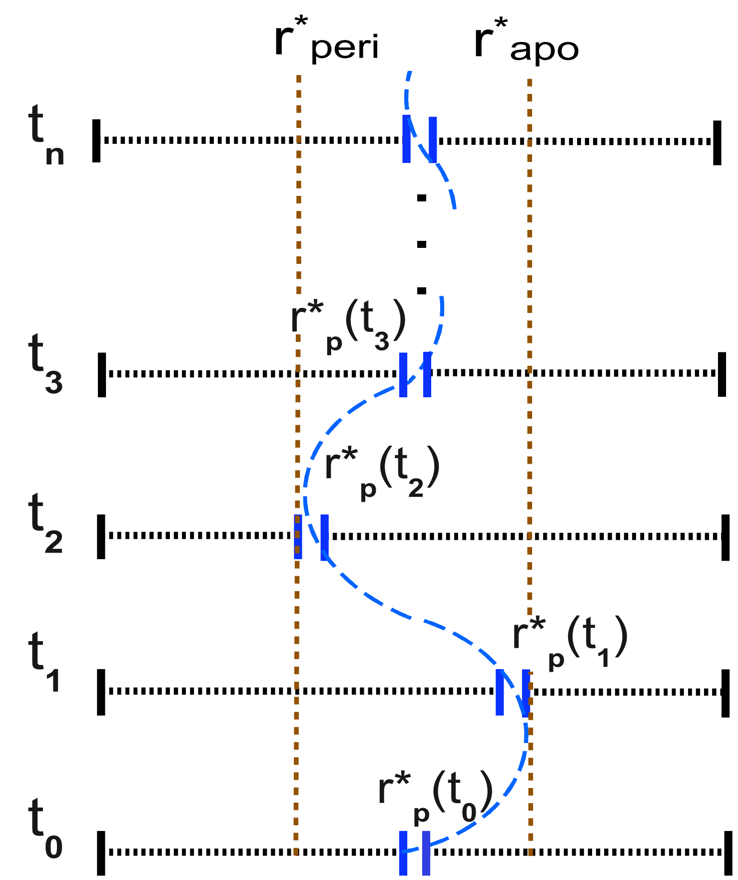
## From circular to eccentric orbits:

- The key point of the PwP method is to keep the particle at the interface between subdomains:




- For eccentric orbits we use a time dependent linear mapping between the physical and spectral domains.




$$r^*_p = r^*_p(t)$$

$$[\psi^{\ell m}]_p = 0\,,$$

$$\left[\partial_t \psi^{\ell m}\right]_p = -\frac{\dot{r}^*_p\, S^{\ell m}}{(1-\dot{r}^{*2}_p)f(r_p)}\,,$$

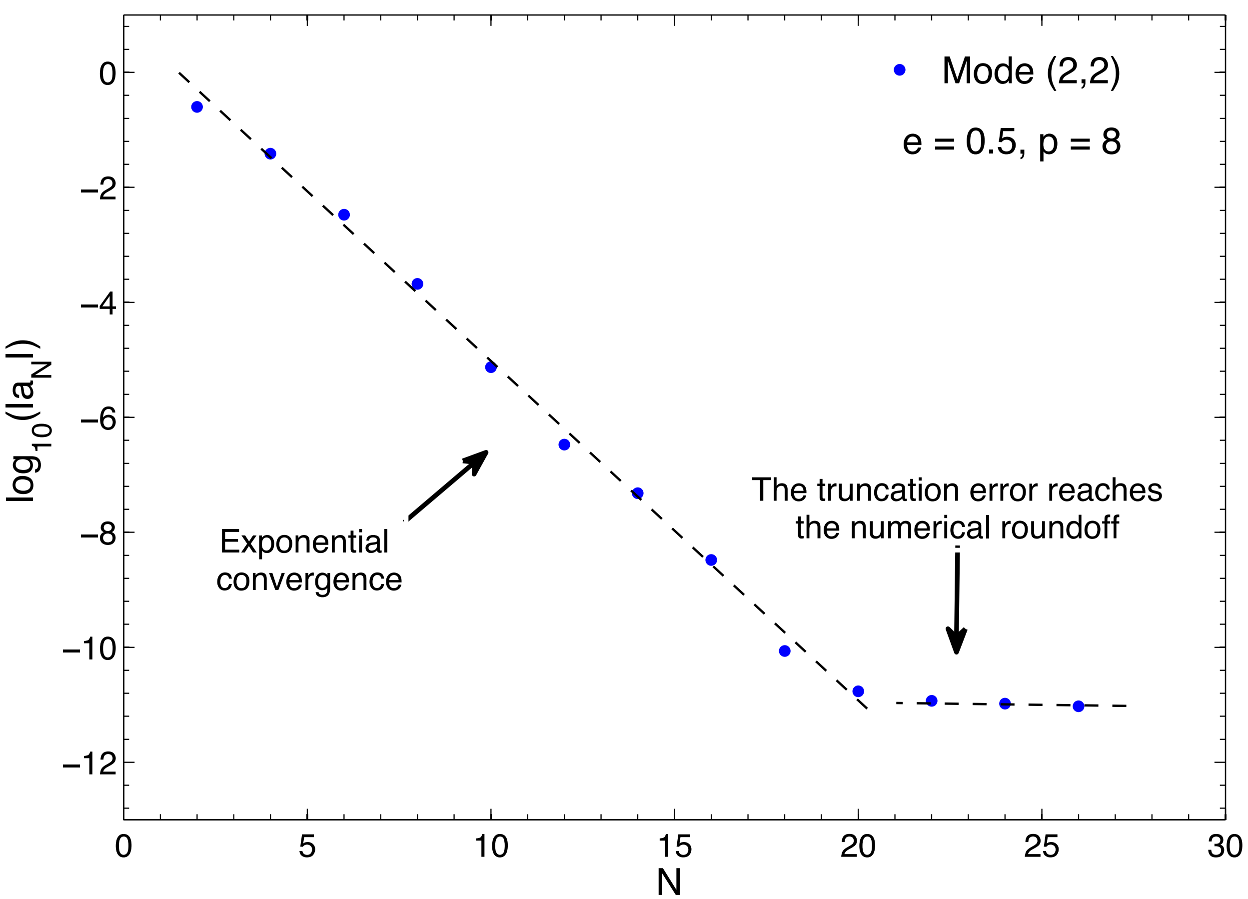
$$\left[\partial_{r^*} \psi^{\ell m}\right]_p = \frac{S^{\ell m}}{(1-\dot{r}^{*2}_p)f(r_p)}$$

[ Canizares, Sopuerta & Jaramillo (2010 )].

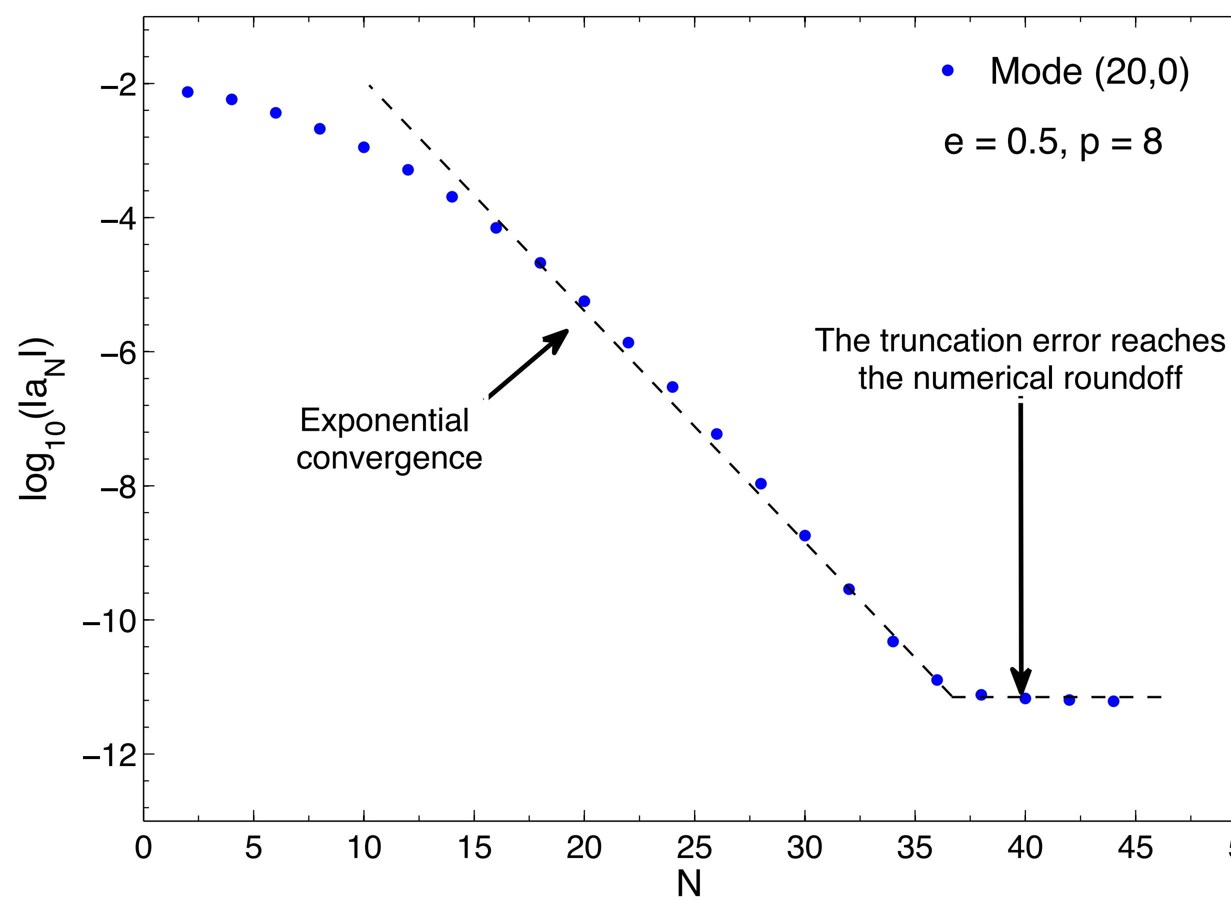
# Convergence Test (eccentric case)

*The dependence of the truncation error ($\sim |a_N|$) with respect increasing numbers of collocation points, N, give us an estimation of the exponential convergence of the code:* $e^{-N}$

$(\ell, m) = (2, 2)$

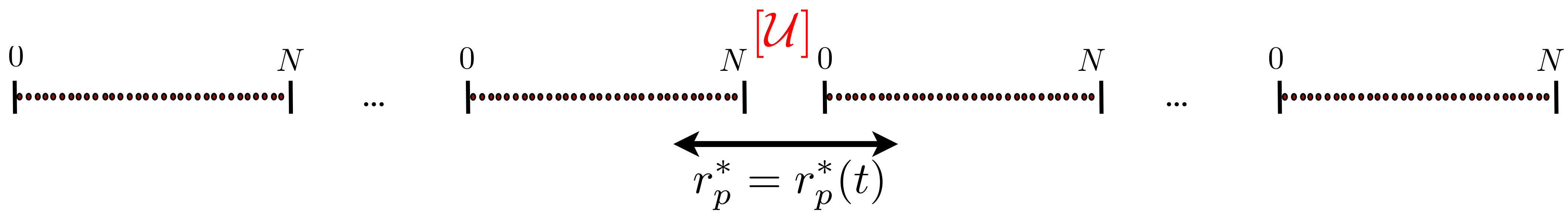


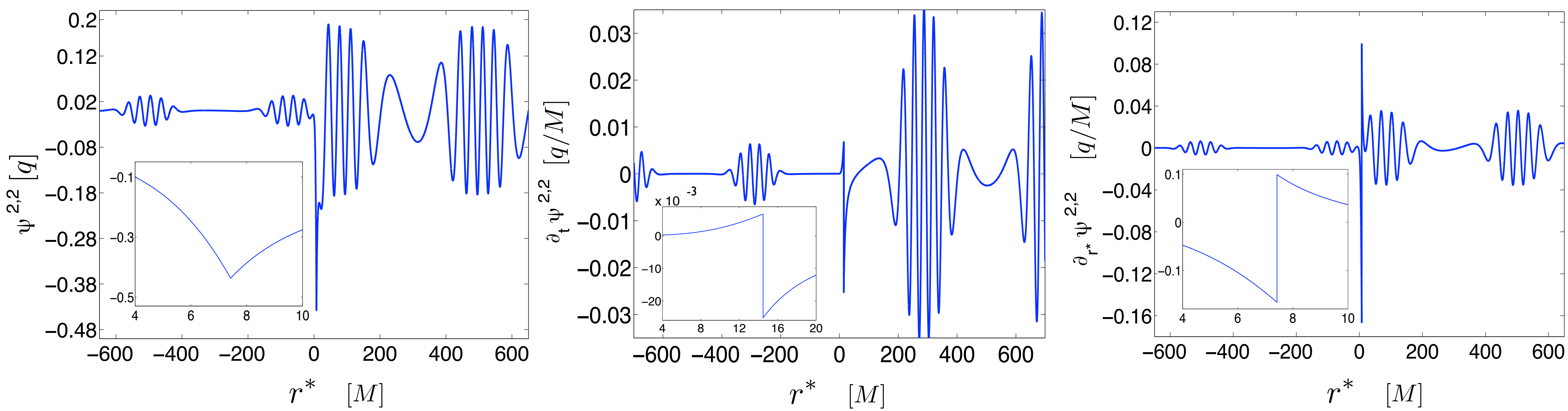

$(\ell, m) = (20, 0)$




*[ Canizares & Sopuerta (2011)].*

## *Snapshots from the Eccentric (e=0.5, p= 7.1) case (D=10, N= 100)* $(\ell, m) = (2,2)$




$$[\psi^{\ell m}]_p = 0\,, \qquad [\partial_t \psi^{\ell m}]_p = -\frac{\dot{r}_p^*\, A^{\ell m}}{(1-\dot{r}_p^{*2})}\,, \quad [\partial_{r^*} \psi^{\ell m}]_p = \frac{A^{\ell m}}{(1-\dot{r}_p^{*2})}$$

*[ Canizares, Sopuerta & Jaramillo (2010 )].*

## Results for the self-force components:

- *We have employed* $\ell_{max} = 17$ , $D = 10$ , $N = 100$ *and* $\Delta r^* = 2 - 5M_\bullet$

  *The Self-force values have been obtained near the pericenter.* *[ Canizares, Sopuerta & Jaramillo (2010 )].*

- *We compare our results with posterior ones obtained in the frequency-domain*

  *[Warburton & Barack (2010)]*

| $(e, p)$ | $\mathcal{F}_\alpha$ | PwP | Frequency-Domain | Relative Difference |
|---|---|---|---|---|
| $(0.1, 6.3)$ | $\frac{M_\bullet^2}{q}\,\Phi_t^{\mathrm{R}}$ | $4.517\,196 \cdot 10^{-4}$ | $4.517\,994 \cdot 10^{-4}$ | $0.01\%$ |
| | $\frac{M_\bullet^2}{q}\,\Phi_r^{\mathrm{R}}$ | $2.125\,049 \cdot 10^{-4}$ | $2.125\,7 \cdot 10^{-4}$ | $0.03\%$ |
| | $\frac{M_\bullet}{q}\,\Phi_\phi^{\mathrm{R}}$ | $-6.204\,083 \cdot 10^{-3}$ | $-6.20\,401 \cdot 10^{-3}$ | $3 \cdot 10^{-5}\%$ |
| $(0.3, 6.7)$ | $\frac{M_\bullet^2}{q}\,\Phi_t^{\mathrm{R}}$ | $7.698\,048 \cdot 10^{-4}$ | $7.177\,3 \cdot 10^{-4}$ | $0.25\%$ |
| | $\frac{M_\bullet^2}{q}\,\Phi_r^{\mathrm{R}}$ | $3.63\,3926 \cdot 10^{-4}$ | $3.632\,2 \cdot 10^{-4}$ | $0.04\%$ |
| | $\frac{M_\bullet}{q}\,\Phi_\phi^{\mathrm{R}}$ | $-9.040\,222 \cdot 10^{-3}$ | $-9.0402\,1 \cdot 10^{-3}$ | $1.5 \cdot 10^{-5}\%$ |
| $(0.5, 7.1)$ | $\frac{M_\bullet^2}{q}\,\Phi_t^{\mathrm{R}}$ | $1.233\,071 \cdot 10^{-3}$ | $1.233\,1 \cdot 10^{-3}$ | $0.015\%$ |
| | $\frac{M_\bullet^2}{q}\,\Phi_r^{\mathrm{R}}$ | $5.612\,209 \cdot 10^{-4}$ | $5.617\,9 \cdot 10^{-4}$ | $0.1\%$ |
| | $\frac{M_\bullet}{q}\,\Phi_\phi^{\mathrm{R}}$ | $-1.268\,560 \cdot 10^{-2}$ | $-1.2685\,7 \cdot 10^{-2}$ | $6.1 \cdot 10^{-4}\%$ |

## Summary

- *The PwP scheme provides accurate and efficient self-force computations in (1+1)*
- *It is a robust method suitable to deal with generic EMRI orbits.*
- *We are working to extend the PwP scheme to 2+1 computations*