Effective field theory

and

radiation reaction

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Overview

- The paradigm of Effective Field Theory (EFT)

- The classical mechanics of non-conservative systems

- Radiation reaction in EFT

Effective field theory paradigm

What's been done with EFT: A snapshot

Potentials for non-spinning binaries thru 3PN
(Goldberger, Rothstein, Gilmore, Ross, Chu, Foffa, Sturani)

Spin-orbit & spin-spin potentials thru 4PN & 3PN, resp.
(Porto, Rothstein, Levi, Perrodin)

PN radiation reaction thru 3.5PN
(CRG, Leibovich)

Gravitational waveform at LO
(CRG)

Radiative moments thru 3PN
(Ross, Goldberger, Porto, Rothstein)

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Tidal Love number for BH
(Smolkin, Kol)

First-order gravitational self-force
(CRG, Hu)

Third-order scalar self-force
(CRG)

Absorptive effects
(Rothstein, Goldberger, Porto)

Radiation reaction on extended charges
(CRG, Leibovich, Rothstein)

Caged black holes
(Kol, Smolkin, Chu, Goldberger, Rothstein)

Cosmological perturbation theory
(Baumann, Nicolis, Senatore, Zaldarriaga,...)

Inflation
(Senatore, Zaldarriaga,...)

Higher dimensional BHs
(Emparan, Harmark, Niarchos, Obers)

Hydrodynamics
(Nicolis, Dubovsky, Endlich, Hui, Son,...)

Condensed matter/Biophysics
(Yolcu, Rothstein, Deserno)

What is effective field theory?

- EFT is a way of parameterizing long-distance physics with effective degrees of freedom that account for the short-distance effects.

  - Symmetries are guiding principle

- EFT utilizes a separation of scales to describe perturbative corrections

  - e.g., lengths, masses, velocities

- Many features of EFT arise in more familiar contexts

  - Matched asymptotic expansions
  - Multipole moment expansions
  - Dimensional analysis

- EFT has bells & whistles to streamline perturbative computations

  - e.g., Feynman diagrams, renormalization group theory

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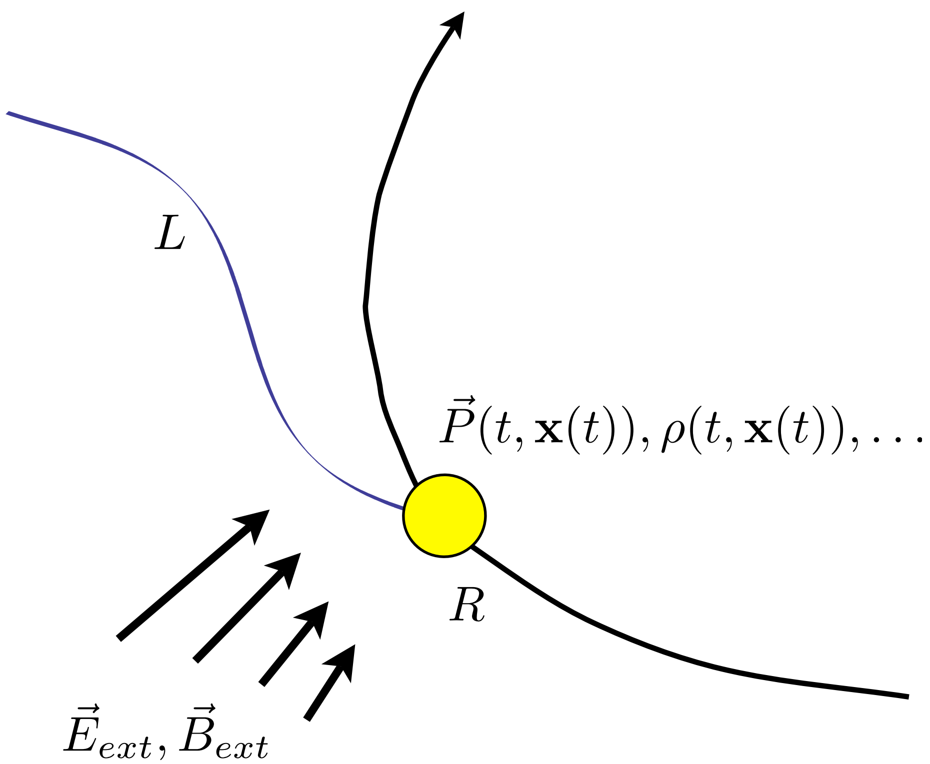
  Step 4: Computing stuff

Example: Motion of an extended charge

CRG, Leibovich & Rothstein, PRL (2010)

- Consider the motion of an extended (spherical) charge distribution



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Superconductor

Metamaterial

etc.

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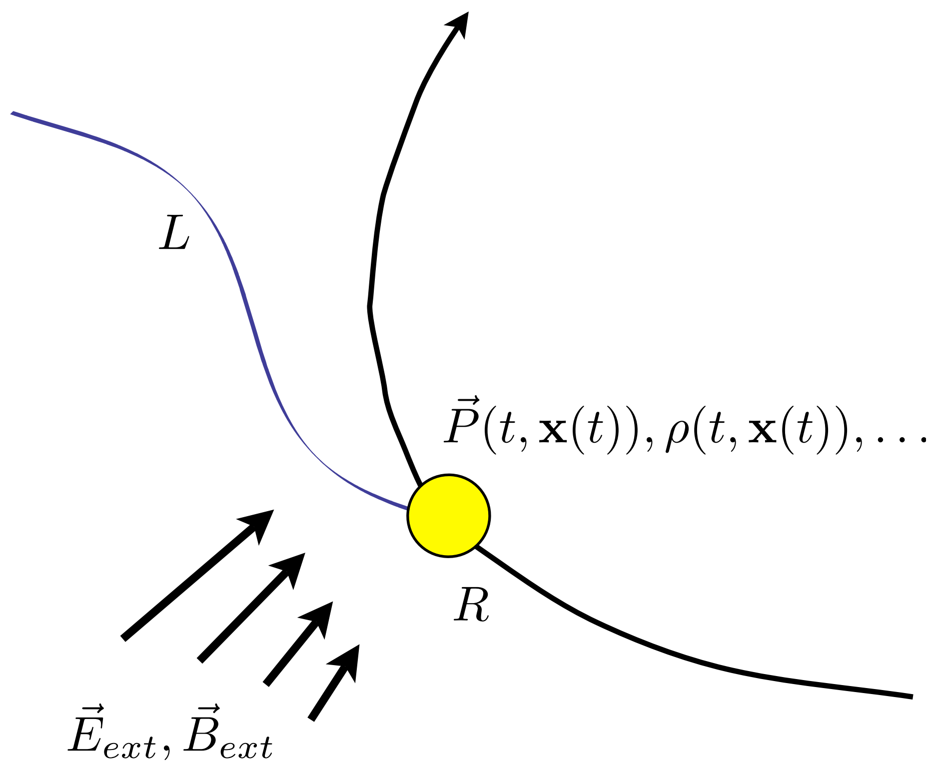

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- In many physical cases, *R* << *L* implying a scale separation

$\Longrightarrow$ An EFT description is admissable

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- Relevant degrees of freedom: $z^\mu(\lambda)\ ,\quad A_\mu(x^\alpha)$

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- Answer: Use symmetries as the guiding principle of what could possibly be

  Add extra worldline terms to the action that are consistent with the underlying symmetries of the full theory

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- The EFT action:

- Is model independent (matching coefficients C_d, C_e, C_m,... contain info about material)

- Yields results equivalent to the full theory when $R < L$ and the coefficients are known

- Extra interactions involve derivatives of radiation field and are perturbative corrections to a pure point particle description

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- The precise location of the apparent point-like charge is immaterial

$$\phi(\mathbf{x}) = q'\,\frac{1}{r'} + p'_i\,\frac{x'^i}{r'^3} + \frac{1}{2}Q'_{ij}\,\frac{x'^i x'^j}{r'^5} + \cdots \ , \quad r' = |\mathbf{x} + \delta\mathbf{x}| \ , \quad |\delta\mathbf{x}| \lesssim R$$

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$$C_d \int d\tau \, u^{[\alpha} a^{\beta]} F_{\alpha\beta}(z) = \frac{1}{2} \int d\tau \left[\vec{d}(\tau) \cdot \vec{E}(z) + \vec{m}(\tau) \cdot \vec{B}(z) \right]$$

$$d^i(\tau) = 2C_d(u^0 a^i - u^i a^0) \to 2C_d a^i$$

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$$C_e \int d\tau \, F^{\alpha\beta}(z) F_{\alpha\beta}(z) + C_m \int d\tau \, u^\alpha F_{\alpha\beta}(z) F^{\beta}{}_{\gamma}(z) u^\gamma$$

$$\vec{P} = 4\left(\frac{C_e}{2} - C_m\right) \vec{E}$$

$$\vec{M} = -4C_m \vec{B}$$

These terms describe susceptibilities

Matching

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  Power radiated in EFT:

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$$P_{\mathrm{full}} = \frac{e}{6\pi}\left[e\mathbf{a}^2 - \frac{eR^2}{5}\dot{\mathbf{a}}^2 + \cdots\right] \quad \Longrightarrow \quad C_d = \frac{1}{10}eR^2$$

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  Power radiated by harmonically oscillating spherical shell of charge

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Power counting

$$G = c = 1 \Longrightarrow [e] = [m] = \text{Length}$$

- EFT action has infinite number of terms, which are perturbative corrections in *R/L*:

Derivatives of radiation field scale according to wavelength, *L*

$$\partial_\alpha A_\mu \sim \frac{1}{L} A_\mu$$

Dynamical time-scale

$$\int d\tau \sim L \ , \quad a^\mu \sim \frac{1}{L}$$

Maxwell's equation gives scaling of field with *L* and *e*

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Scaling of extra interactions relative to point charge coupling

$$\frac{C_d \int d\tau \, u^{[\alpha} a^{\beta]} F_{\alpha\beta}(z)}{e \int d\tau \, u^\alpha A_\alpha(z)} \sim \frac{C_d}{e\lambda^2} \implies C_d \sim e R^2$$

$$\frac{C_e \int d\tau \, F^{\alpha\beta}(z) F_{\alpha\beta}(z)}{e \int d\tau \, u^\alpha A_\alpha(z)} \sim \frac{C_e}{\lambda^3} \implies C_e \sim R^3$$

The story of O(*R*)

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- Or so it was thought...

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$$\frac{eC/m}{C_d} \sim \frac{e^2}{mR} \ll 1 \quad \Longrightarrow \quad \frac{e^2}{R} \ll m$$

Thus, the a^2 term gives no measurable contribution

Feynman diagrams

- Wave equation $\Box A_\mu = e \int d\tau \, u_\mu \delta^4(x-z) - 2C_d \int d\tau \, u_{[\mu} a^{\alpha]} \partial_\alpha \delta^4(x-z) + \cdots$

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- Perturbative solution to wave equation can be represented diagrammatically

$$A_\mu(x) = \int d^4x'\, D^{\rm ret}_{\mu\nu}(x-x') \left[e \int d\tau\, u^\nu \delta^4(x'-z) \right]$$

$$+ \int d^4x'\, D^{\rm ret}_{\mu\nu}(x-x') \left[-2C_d \int d\tau\, u^{[\nu} a^{\alpha]} \partial_\alpha \delta^4(x'-z) \right]$$

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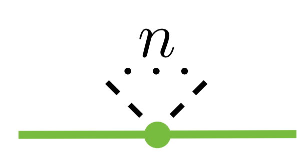
- Dictionary ("Feynman rules"):

x $\quad$ x' $\quad = D^{\text{ret}}_{\mu\nu}(x-x')$



$$= \left.\frac{\delta^n S}{\delta A_{\alpha_1}(x)\cdots \delta A_{\alpha_n}(x)}\right|_{A_\mu = 0}$$

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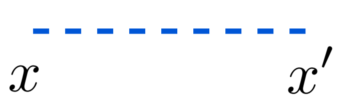
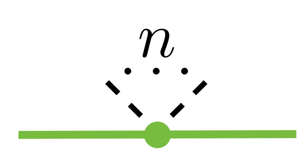


$= D^{\rm ret}_{\mu\nu}(x-x')$



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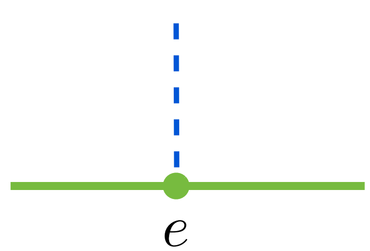
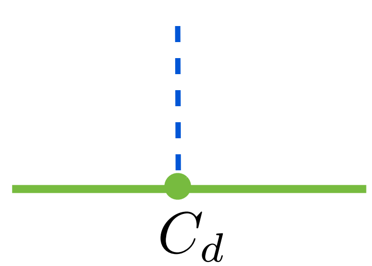
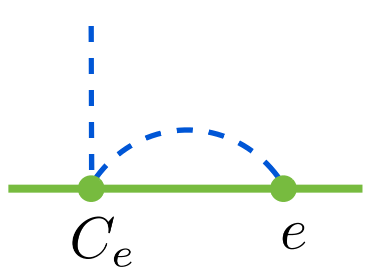
$A_\mu(x) =$ 

$+$ 

$+$ 

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- Naive application of EFT to radiating systems yields no radiation reaction

  - Radiation reaction in electrodynamics

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- ☐ The problem is not with EFT but with classical mechanics itself...

Classical mechanics

of non-conservative systems

CRG (2012)

An illustrative example (1)

- Coupled harmonic oscillators

$$S[q,\{Q_n\}] = \int_{t_i}^{t_f} dt \left\{ \frac{m}{2}\left(\dot{q}^2 - \omega^2 q^2\right) + q\sum_{n=1}^{N} \lambda_n Q_n + \sum_{n=1}^{N} \frac{M_n}{2}\left(\dot{Q}_n^2 - \Omega_n^2 Q_n^2\right) \right\}$$

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- "Integrate out" $\{Q_n(t)\}$ subject to initial conditions

$$\ddot{Q}_n + \Omega_n^2 Q_n = \frac{\lambda_n}{M} q \quad \Longrightarrow \quad Q_n(t) = Q_n^{(h)}(t) + \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt' \, G_{\text{ret}}^{(n)}(t-t') q(t')$$

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- Equation of motion for *q(t)* is (from Hamilton's principle)

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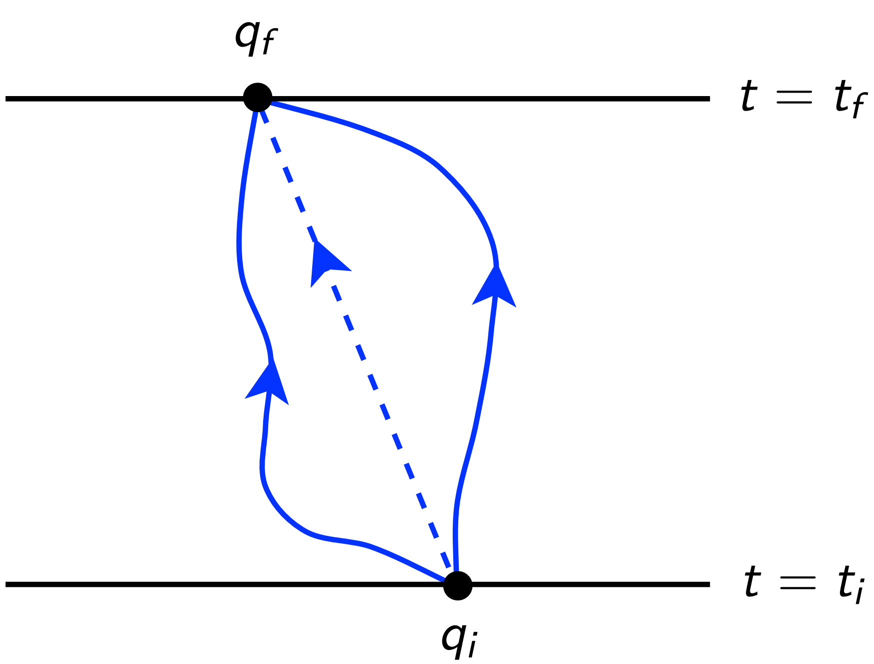
  -- *Kernel of the integral is symmetric in time, implies only conservative interactions*

  Does not account for dissipation (a time-asymmetric process)

Boundary conditions, Green functions & dynamics

- The reason for these unphysical features is the very formulation of Hamilton's principle itself



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Boundary conditions, Green functions & dynamics

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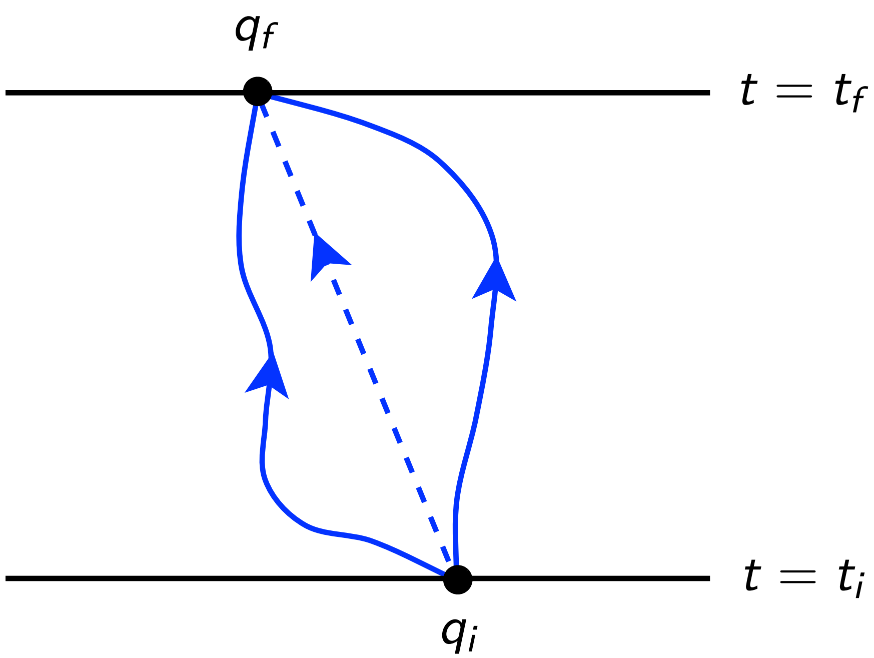

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in time

$\Updownarrow$ *Sturm-Liouville theory*

Time-symmetric
Green functions

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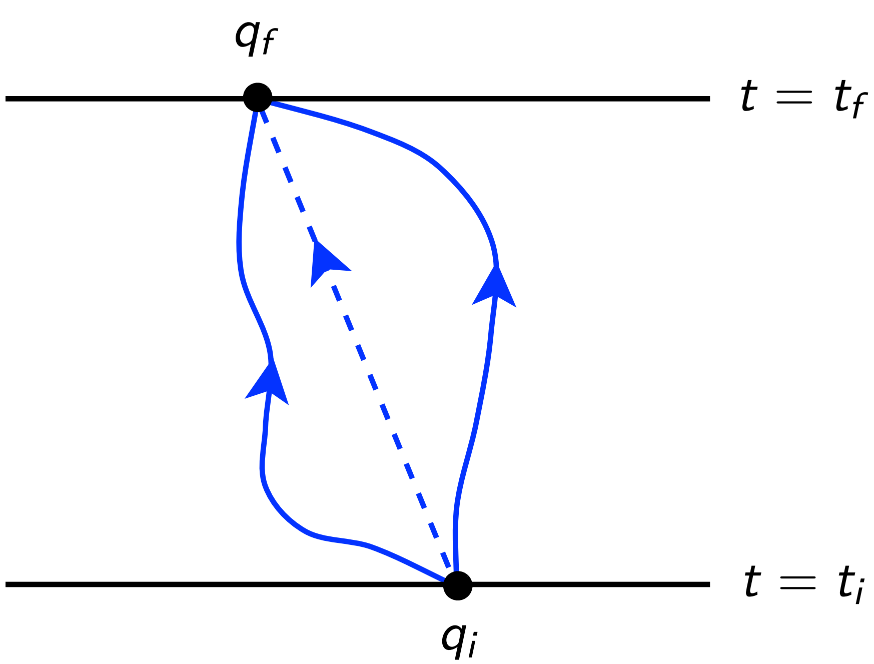

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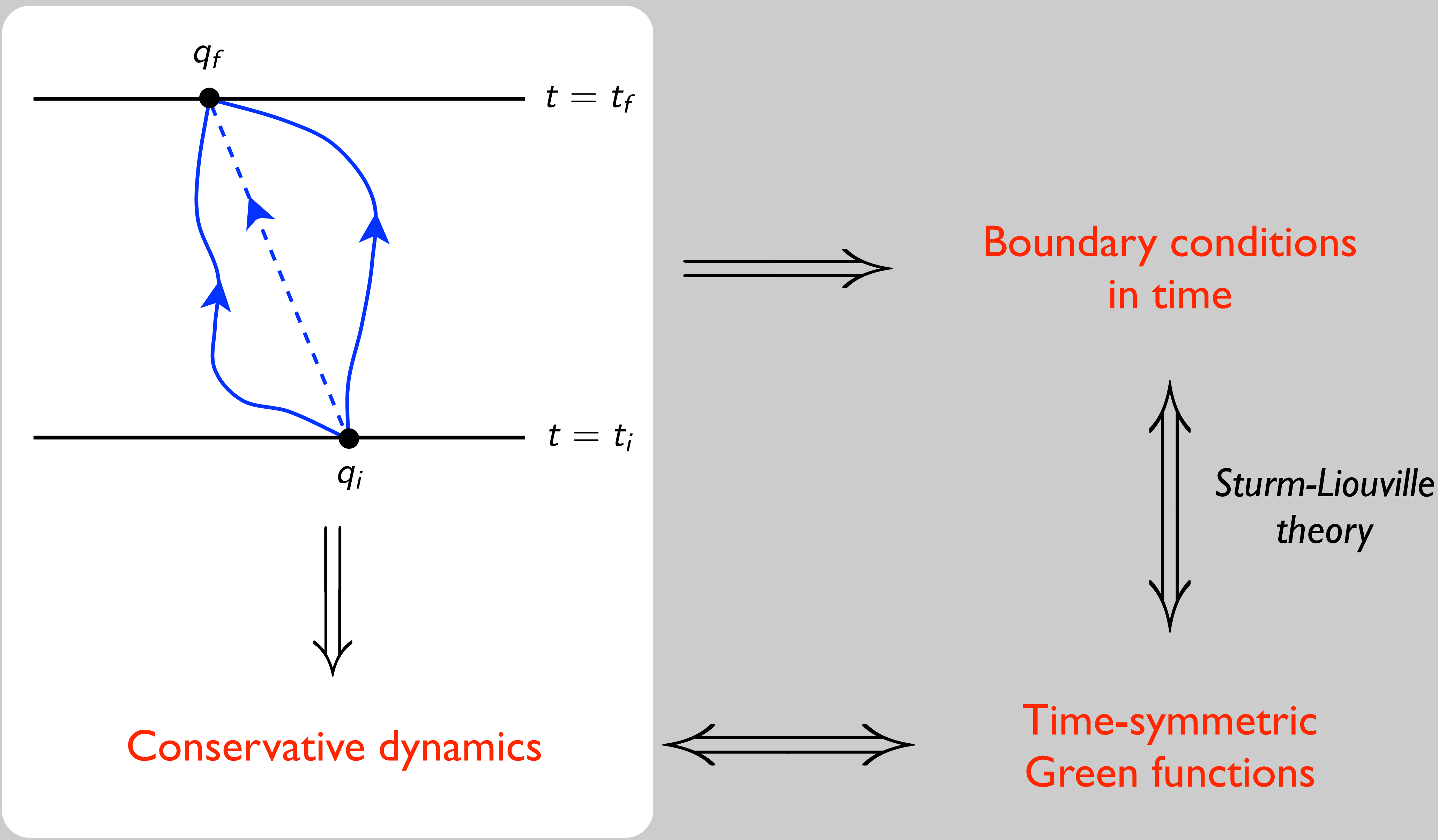

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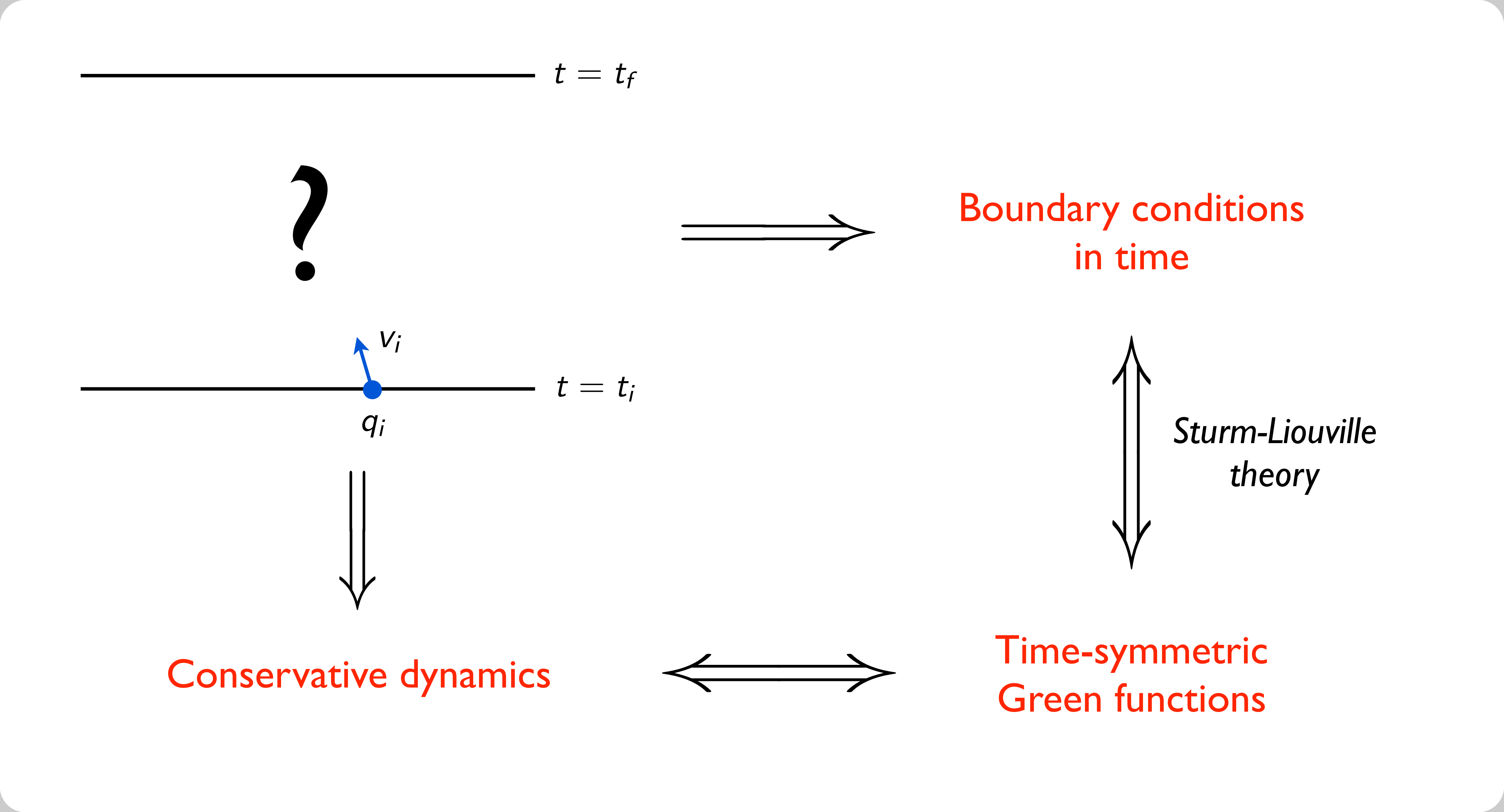

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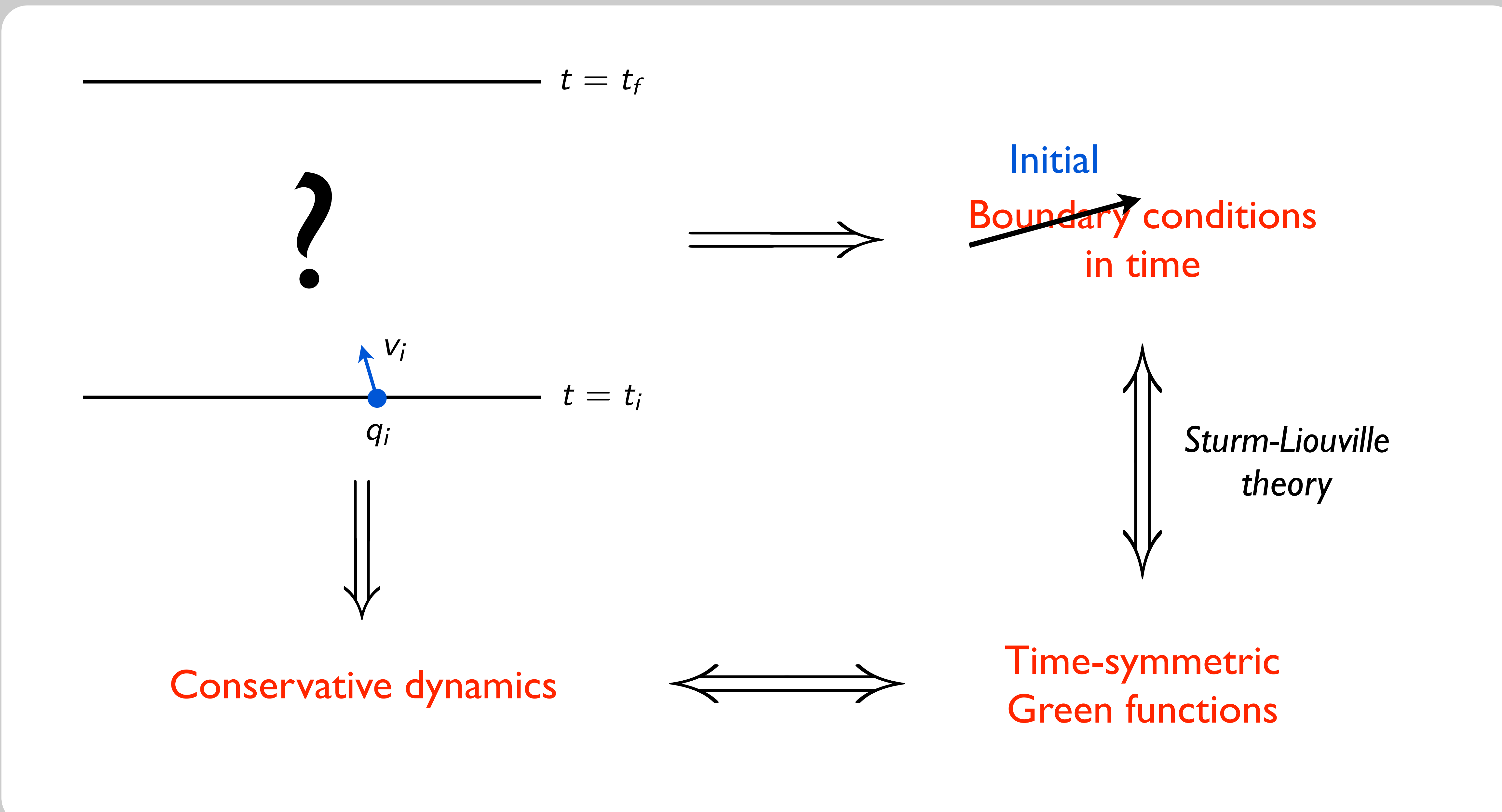

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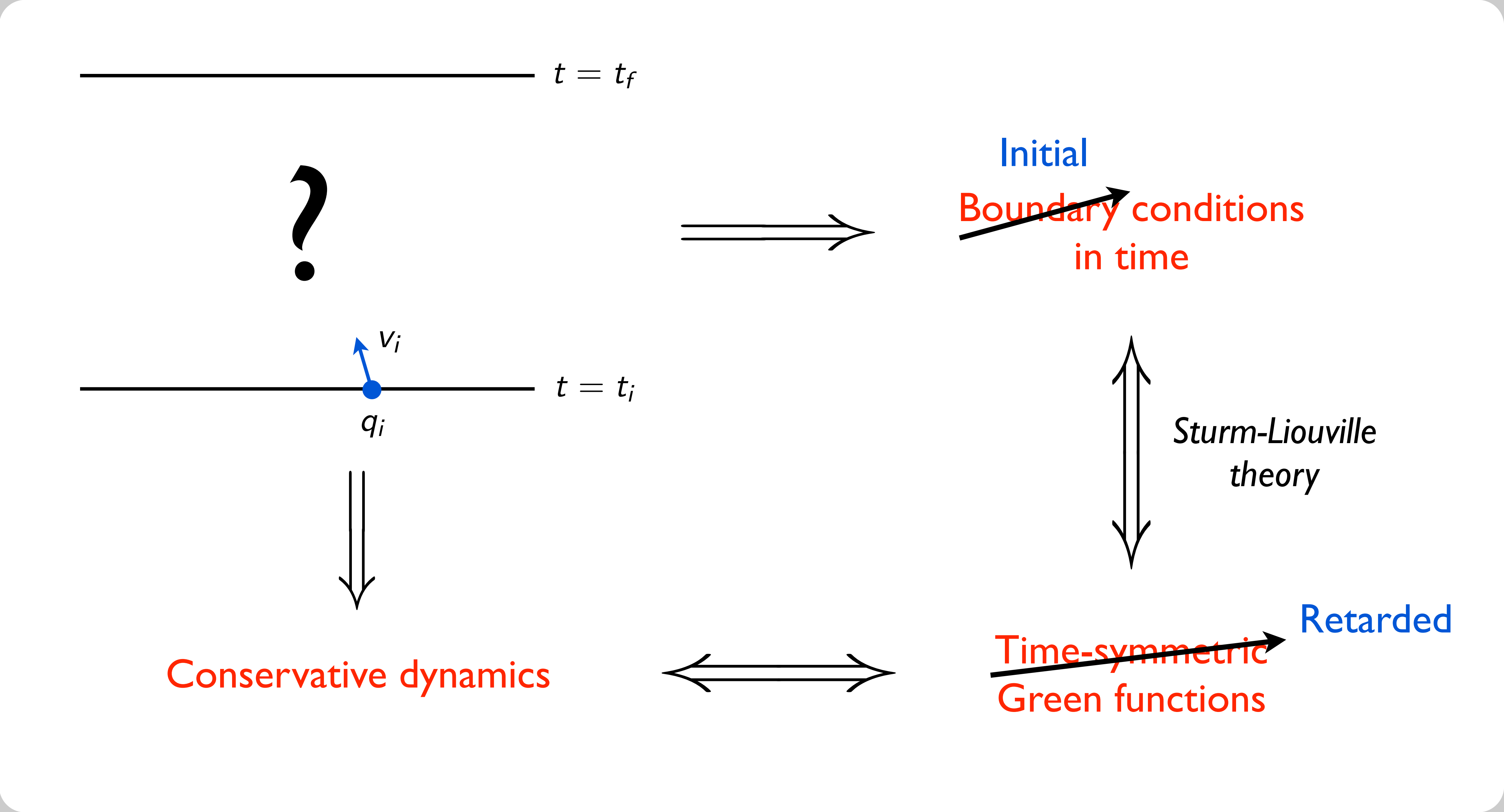

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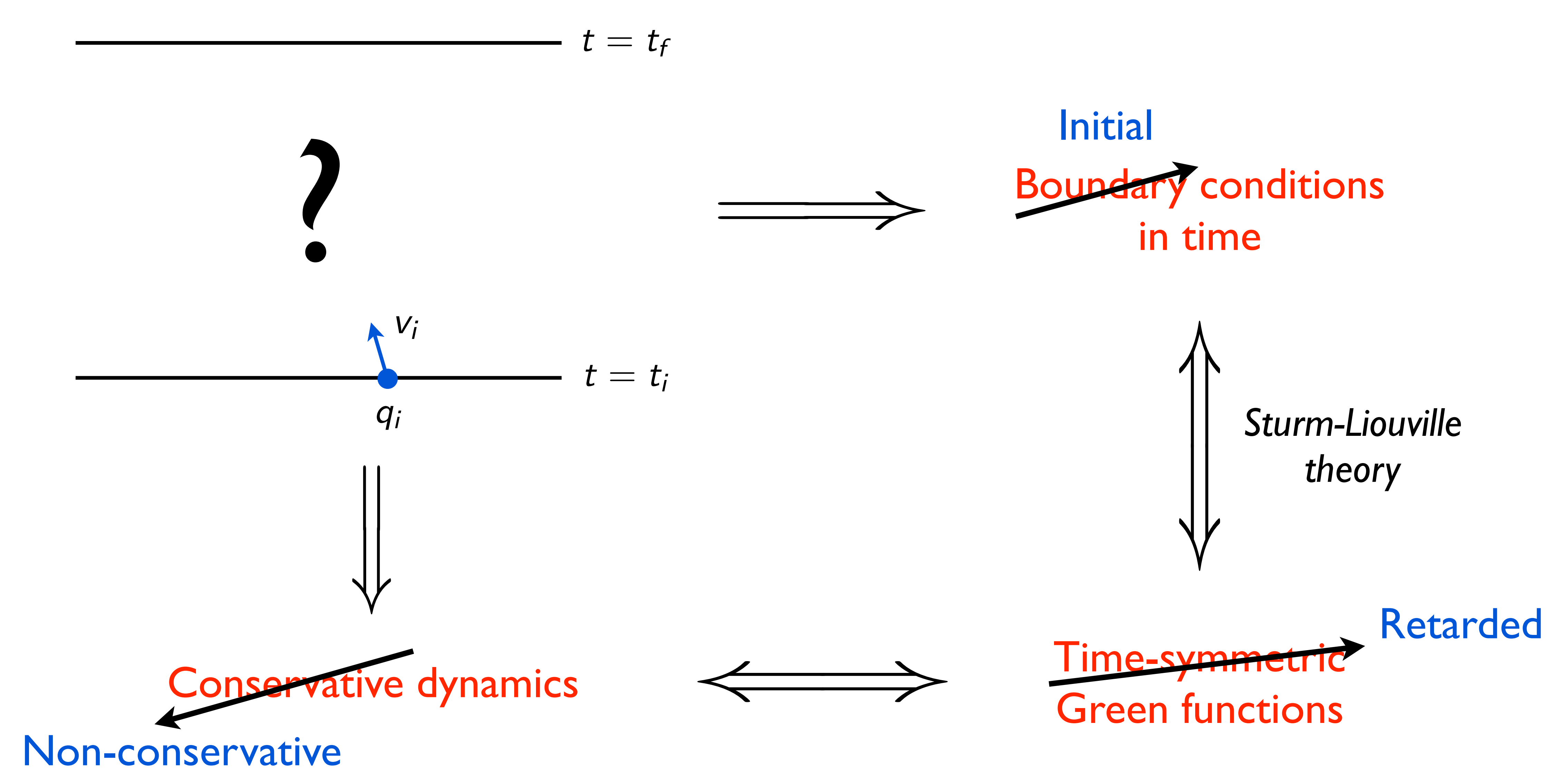

A hint...

- Advanced Green functions appears because $q(t)q(t')$ is symmetric in $t \leftrightarrow t'$

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- Hint:

Use two different sets of variables

$$\int_{t_i}^{t_f} dtdt'\, q(t) G_{\text{ret}}^{(n)}(t-t') q(t') \longrightarrow \int_{t_i}^{t_f} dtdt'\, q_1(t) G_{\text{ret}}^{(n)}(t-t') q_2(t')$$

Varying with respect to q_1 gives the correct force provided $q_2 = q_1$ after the variation

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- Interestingly, doubled variables appear as early as

- 1970 (Staruszkiewicz) for a Lagrangian, and

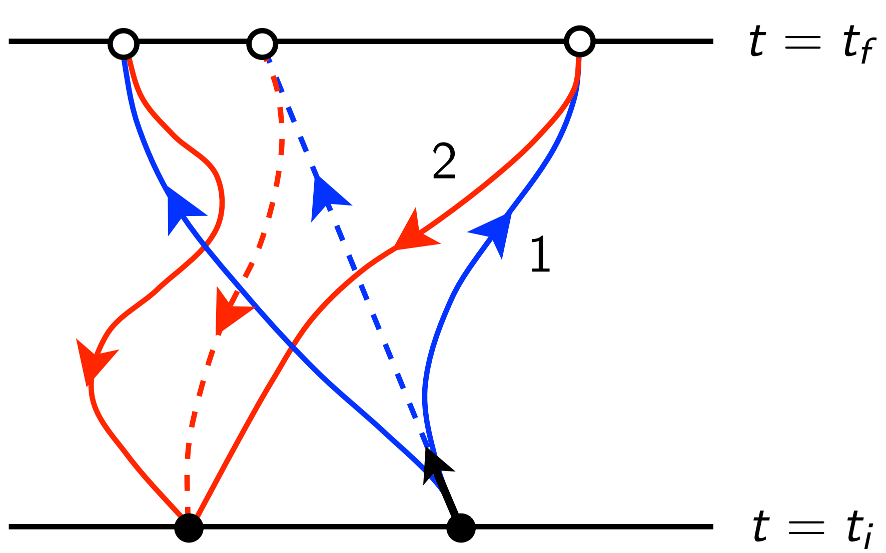
- 1997 (Schaefer, et al.) for a Hamiltonian

Hamilton's principle & initial conditions (I) *CRG (2012)*

- Introduce <u>two</u> paths such that:

  1) Both paths have vanishing displacements at the initial time

  2) The coordinates and velocities of both paths are equal at the final time (the *equality condition*)



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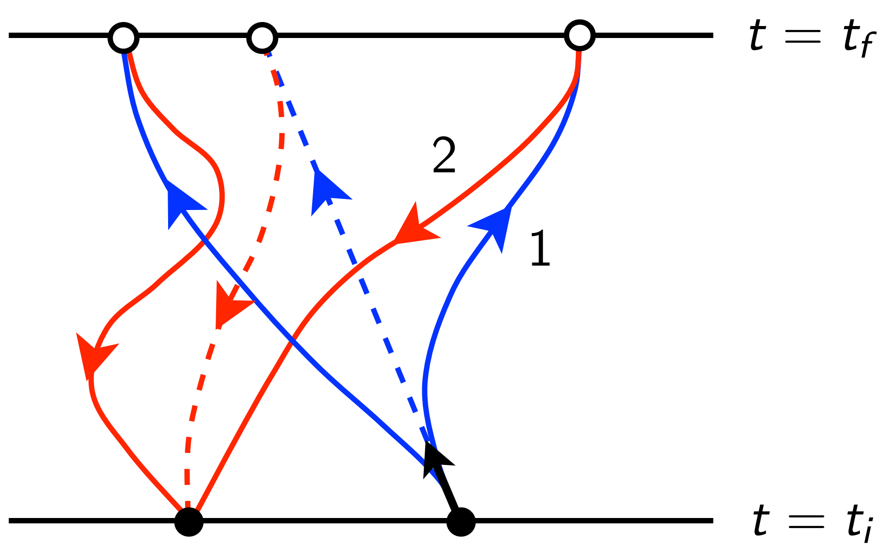
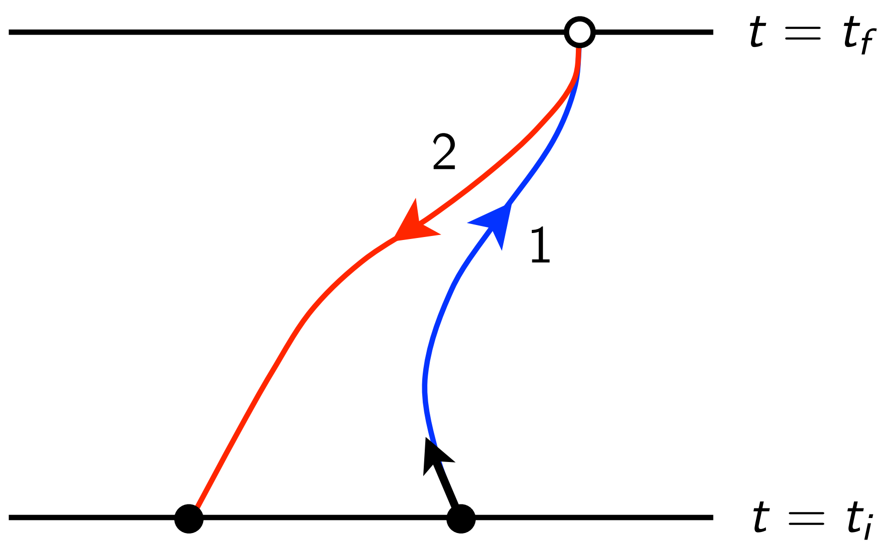


- After <u>all</u> variations are done, identify both paths with the physical one (the *physical limit*)

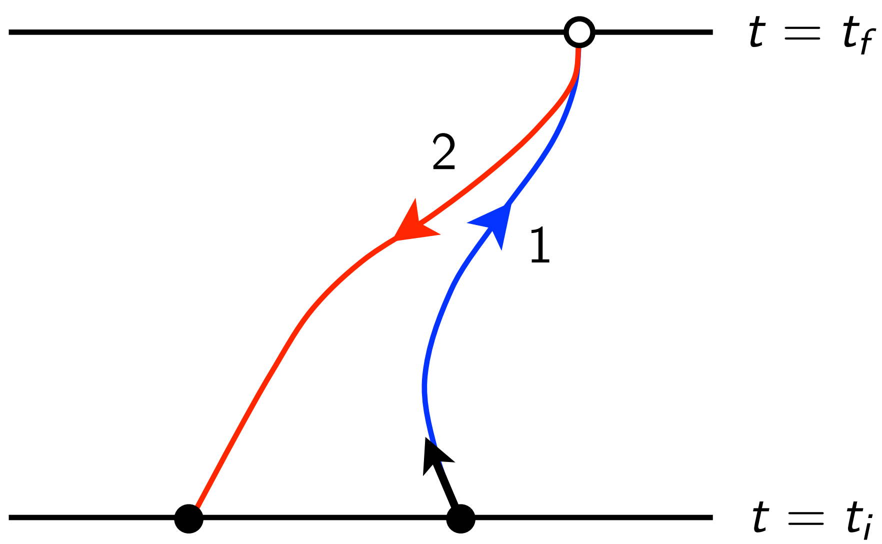
Hamilton's principle & initial conditions (2) *CRG (2012)*



- New action defined by the total line integral of the Lagrangian along both segments

$$S[q_1, q_2] \equiv \int_{t_i}^{t_f} dt\, L(q_1, \dot{q}_1) + \int_{t_f}^{t_i} dt\, L(q_2, \dot{q}_2) + \int_{t_i}^{t_f} dt\, K(q_1, q_2, \dot{q}_1, \dot{q}_2)$$

Hamilton's principle & initial conditions (2) *CRG (2012)*



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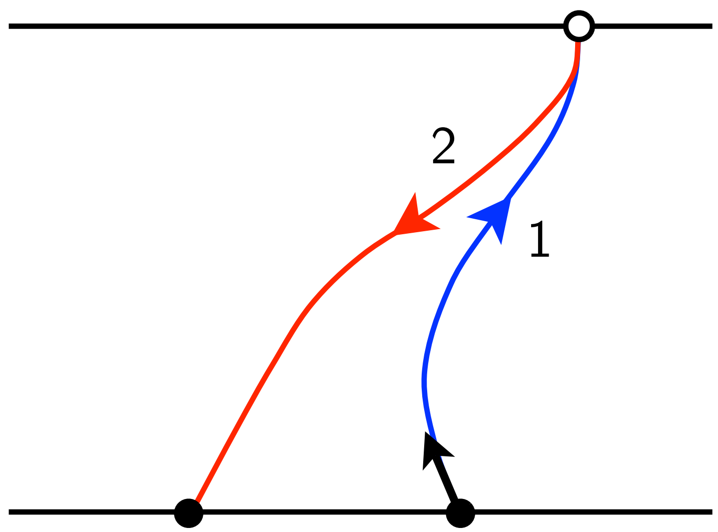

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3) K measures the "openness" of a system

Closed if $K = 0$ Open if $K \neq 0$

Hamilton's principle & initial conditions (3) *CRG (2012)*

- Hamilton's principle: Extremize the new action $S[q_1, q_2]$

  - Convenient to make a change of variables:

$$q_+ = \frac{q_1 + q_2}{2} \qquad\qquad q_- = q_1 - q_2$$

Physical limit: $q_-(t) \to 0 \ , \quad q_+(t) \to q(t)$

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  - Virtual paths are given by:

$$q_+(t,\epsilon) = q_+(t,0) + \epsilon\eta_+(t) \qquad\qquad q_-(t,\epsilon) = q_-(t,0) + \epsilon\eta_-(t)$$

Hamilton's principle & initial conditions (3) *CRG (2012)*

- Hamilton's principle: Extremize the new action $S[q_1, q_2]$

  - Convenient to make a change of variables:

$$q_+ = \frac{q_1 + q_2}{2} \qquad\qquad q_- = q_1 - q_2$$

Physical limit: $q_-(t) \to 0 \ , \quad q_+(t) \to q(t)$

  - Virtual paths are given by:

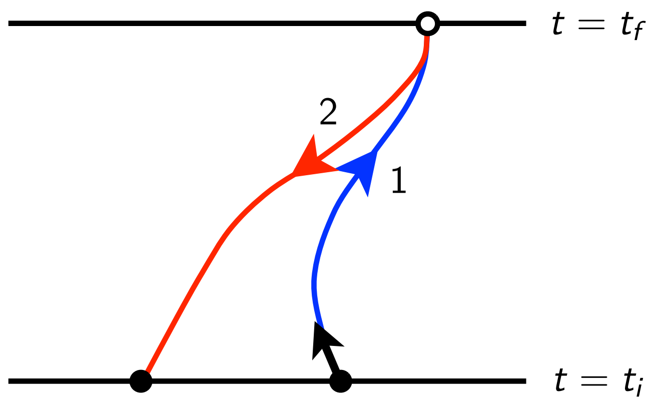
$$q_+(t,\epsilon) = q_+(t,0) + \epsilon\eta_+(t) \qquad\qquad q_-(t,\epsilon) = q_-(t,0) + \epsilon\eta_-(t)$$

  - Variation of the new action:

$$\left.\frac{dS[q_+, q_-]}{d\epsilon}\right|_{\epsilon=0} = \int_{t_i}^{t_f} dt \left\{ \eta_+(t)\left(\frac{\partial\Lambda}{\partial q_+} - \frac{d}{dt}\frac{\partial\Lambda}{\partial \dot{q}_+}\right)_0 + \eta_-(t)\left(\frac{\partial\Lambda}{\partial q_-} - \frac{d}{dt}\frac{\partial\Lambda}{\partial \dot{q}_-}\right)_0 \right\}$$

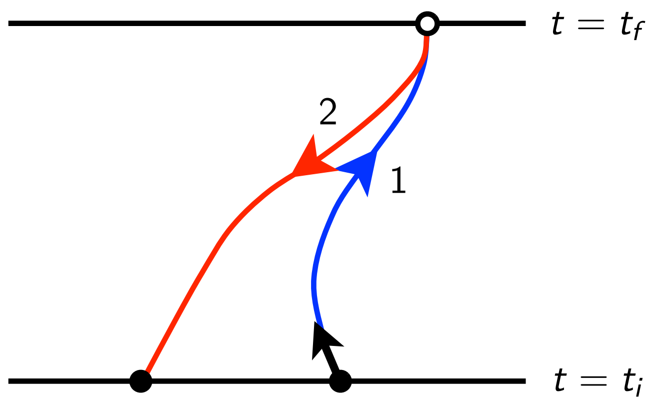
$$+ \Big[\eta_+(t)p_-(t) + \eta_-(t)p_+(t)\Big]_{t=t_i}^{t_f}$$

Hamilton's principle & initial conditions (4) *CRG (2012)*



- Conditions at the time boundaries

Hamilton's principle & initial conditions (4) *CRG (2012)*



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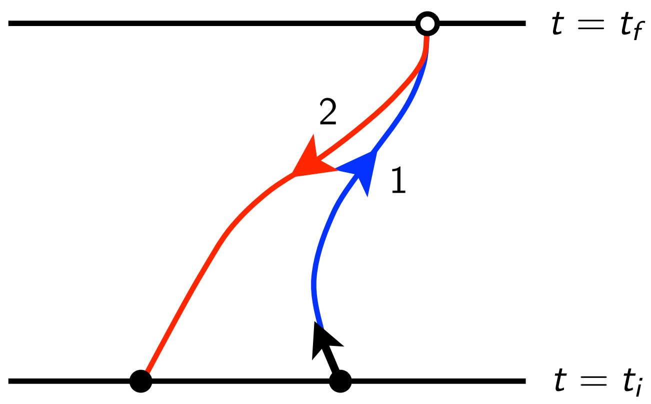

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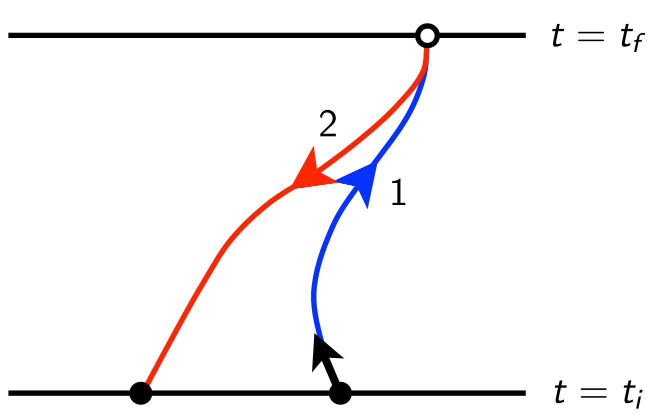

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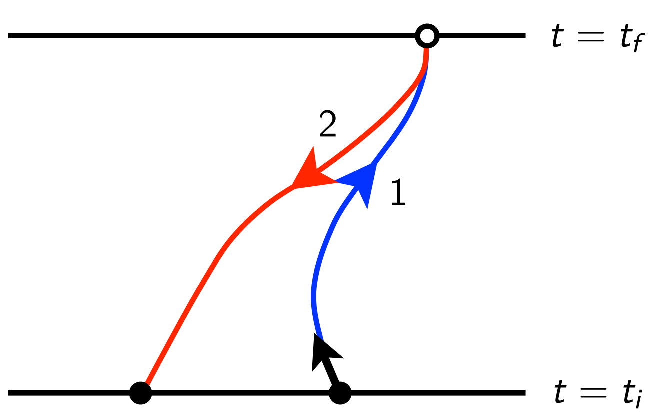
Hamilton's principle & initial conditions (4) *CRG (2012)*



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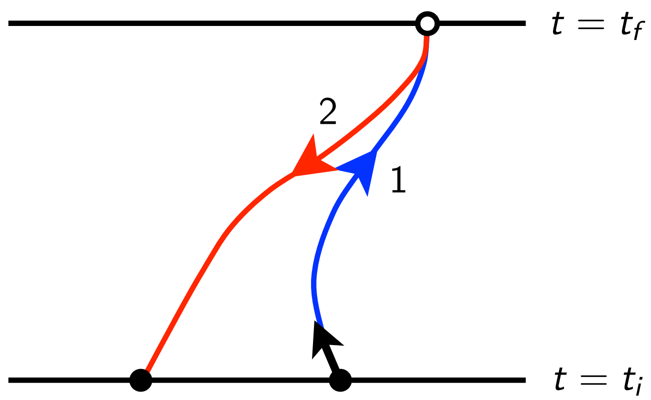
- Boundary contributions to action

$$\Big[\eta_+(t)p_-(t) + \eta_-(t)p_+(t)\Big]_{t=t_i}^{t_f} = 0$$

Hamilton's principle & initial conditions (5) *CRG (2012)*

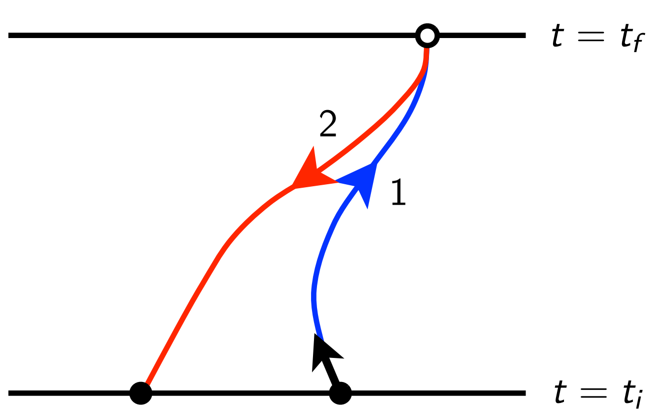


- Equations of motion

  - With the boundary term eliminated:

$$\left.\frac{dS[q_+, q_-]}{d\epsilon}\right|_{\epsilon=0} = \int_{t_i}^{t_f} dt \left\{ \eta_+(t) \left(\frac{\partial \Lambda}{\partial q_+} - \frac{d}{dt} \frac{\partial \Lambda}{\partial \dot{q}_+} \right)_0 + \eta_-(t) \left(\frac{\partial \Lambda}{\partial q_-} - \frac{d}{dt} \frac{\partial \Lambda}{\partial \dot{q}_-} \right)_0 \right\}$$

Hamilton's principle & initial conditions (5) *CRG (2012)*



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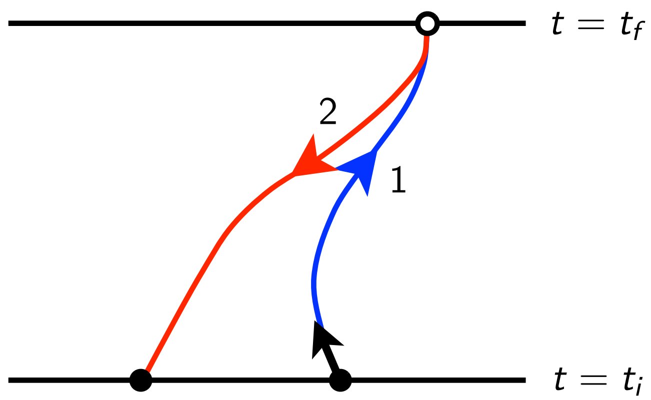

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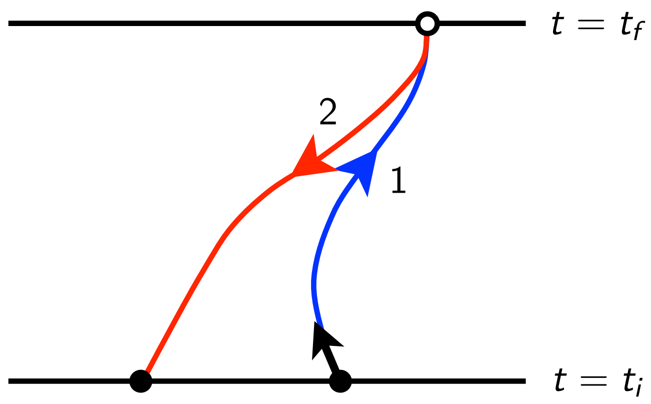

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- Lastly, identify both paths as the physical one, *q(t)* -- the "physical limit"

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = -\left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_-} - \frac{\partial K}{\partial q_-}\right]_{\text{p.l.}}} \quad \Longrightarrow \quad \boxed{0 = \left[\frac{\delta S[q_+, q_-]}{\delta q_-(t)}\right]_{\text{p.l.}}}$$

The energy function

- Total time derivative of the Lagrangian

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt}$$

- Use the new Lagrange's equations to substitute in for $\partial L/\partial q$

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_-} - \frac{\partial K}{\partial q_-}\right]_{\text{p.l.}}$$

- And define the energy function as the value of the Hamiltonian

$$h(q, \dot{q}) \equiv \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

$$\Longrightarrow \boxed{\frac{dh}{dt} = -\frac{\partial L}{\partial t} - \dot{q}\left[\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_-} - \frac{\partial K}{\partial q_-}\right]_{\text{p.l.}}}$$

Extra goodies

- A Hamiltonian for non-conservative systems can be constructed

$$A(q_1, q_2, p_1, p_2) \equiv H(q_1, p_1) - H(q_2, p_2) - K(q_1, q_2, p_1, p_2)$$

- Poisson brackets

$$\{\{f, g\}\} \equiv \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}$$

- New Hamilton's equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} - \left[\frac{\partial K}{\partial p_-}\right]_{\text{p.l.}} \qquad \dot{p} = -\frac{\partial H}{\partial q} + \left[\frac{\partial K}{\partial q_-}\right]_{\text{p.l.}}$$

- New action (with *K=0*) is classical limit of Schwinger's initial value formulation of quantum theory *Schwinger (1961)*

- etc...

Coupled oscillators again

- Formally double the variables

$$(q, \{Q_n\}) \rightarrow (q_1, q_2, \{Q_{n1}\}, \{Q_{n2}\})$$

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- Integrate out the Q's

$$Q_{n+}(t) = Q_{n+}^{(h)}(t) + \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt' \; G_{\text{ret}}^{(n)}(t - t') q_+(t')$$

$$Q_{n-}(t) = \frac{\lambda_n}{M} \int_{t_i}^{t_f} dt' \; G_{\text{adv}}^{(n)}(t - t') q_-(t')$$

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- Effective action and equations of motion

$$S_{\text{eff}}[q_1, q_2] = \int_{t_i}^{t_f} dt \left\{ m(\dot{q}_- \dot{q}_+ - \omega^2 q_- q_+) + q_- \sum_{n=1}^{N} \lambda_n Q_n^{(h)} + \sum_{n=1}^{N} \frac{\lambda_n^2}{M_n} \int_{t_i}^{t_f} dt'\, q_-(t) G_{\text{ret}}^{(n)}(t - t') q_+(t') \right\}$$

$$\Longrightarrow \quad m\ddot{q} + m\omega^2 q = \sum_{n=1}^{N} \lambda_n Q_n^{(h)}(t) + \sum_{n=1}^{N} \frac{\lambda_n^2}{M_n} \int_{t_i}^{t_f} dt'\, G_{\text{ret}}^{(n)}(t - t') q(t')$$

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$$\Downarrow$$

Lagrangian and Hamiltonian formulations for general non-conservative systems

Radiation reaction in EFT

Radiation reaction on extended charge (1)

- Effective action

$$S_{\rm eff}[z_1^\mu, z_2^\mu] = -m\int (d\tau_1 - d\tau_2) + \frac{e^2}{6\pi}\int d\tau_+\, z_{-\alpha}\big(\dot{a}_+^\alpha + u_+^\alpha u_+^\beta \dot{a}_{+\beta}\big) + \cdots$$

- Compare to effective action from usual Hamilton's Principle

$$S_{\rm eff}[z^\mu] = -m\int d\tau - \frac{e^2}{8\pi}\int d\tau\, z^\alpha \dot{a}_\alpha$$

Radiation reaction on extended charge (I)

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$$S_{\text{eff}}[z^\mu] = -m \int d\tau - \frac{e^2}{8\pi} \int d\tau \, z^\alpha \dot{a}_\alpha$$

- Self-force

$$0 = \frac{\delta S}{\delta z_-^\mu(\tau)}\bigg|_{z_-=0, z_+=z} \qquad \Longrightarrow \qquad F_{\text{ALD}}^\alpha(\tau) = \frac{e^2}{6\pi}(\dot{a}^\alpha + u^\alpha u^\beta \dot{a}_\beta)$$

- Thus, radiation reaction (a dissipative force) is derived from a Lagrangian

Radiation reaction on extended charge (2)

- Effective action in non-relativistic limit

$$\Lambda_{\text{eff}}[\vec{z}_\pm] = \frac{m}{2}(\vec{v}_1^{\,2} - \vec{v}_2^{\,2}) + \overbrace{\frac{e^2}{6\pi} z_{-i} \dot{a}_+^i}^{K} + \cdots$$

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- Change in mechanical energy

$$h = \vec{v} \cdot \frac{\partial L}{\partial \vec{v}} - L$$

$$\frac{dh}{dt} = -\vec{v} \cdot \left[\frac{d}{dt} \frac{\partial K}{\partial \vec{v}_-} - \frac{\partial K}{\partial \vec{z}_-} \right]_{\text{p.l.}}$$

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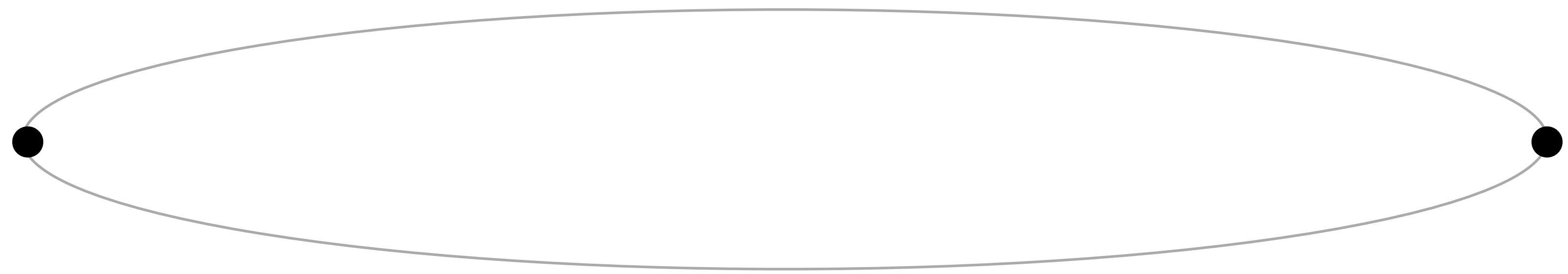
- Move time derivative onto the velocity

$$\frac{d}{dt}\left(h - \frac{e^2}{6\pi}\vec{v}\cdot\vec{a}\right) = -\frac{e^2}{6\pi}\vec{a}^{\,2}$$

Radiation reaction in compact binaries (1)

- The EFT for this system:



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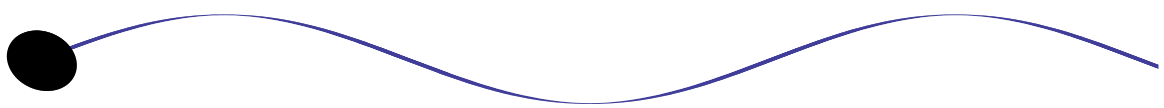

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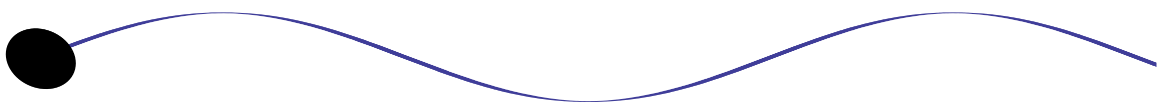

Identify the relevant degrees of freedom and their symmetries

$$\{X^\mu(\tau), h_{\mu\nu}(x^\alpha)\}$$

General coordinate invariance

Reparameterization invariance of worldline

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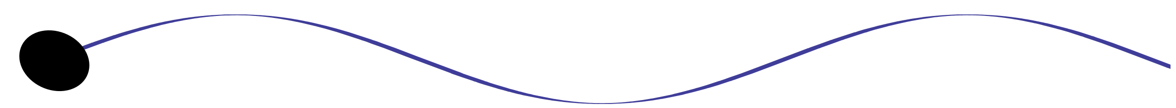

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General coordinate invariance

Reparameterization invariance of worldline

Write most general action consistent with symmetries (i.e., derivative expansion)

$$S[X^\mu, h_{\mu\nu}] = \frac{1}{16\pi G}\int_x g^{1/2}R - M\int d\tau + \int d\tau\, Q_{IJ}(\tau)\mathcal{E}^{IJ}(X) + \cdots$$

Radiation reaction in compact binaries (2)

Matching to a specific theory, model, or data

$$M = m_1 + m_2 + \cdots \qquad\qquad Q^{ij}(t) = \left[\sum_{K=1}^{2} m_K x_K^i(t) x_K^j(t)\right]_{\mathrm{STF}} + \cdots$$

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Compute stuff: Leading order radiation reaction force (2.5PN = v^5)

CRG & Tiglio, (2009)
CRG & Leibovich, (2012)

$$S_{\mathrm{eff}}[\vec{x}_{1\pm}, \vec{x}_{2\pm}] = \frac{m}{2} \int dt\, (\vec{v}_1^{\,2} - \vec{v}_2^{\,2}) + (\text{3PN potentials}) - \frac{G}{5} \int dt\, Q_-^{ij}(t) \frac{d^5 Q_{+ij}}{dt^5} + O(v^7)$$

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- Resulting force is precisely the 2.5PN radiation reaction of Burke & Thorne

$$0 = \left.\frac{\delta S}{\delta x^i_{K-}(t)}\right|_{x_{K-}=0,\, x_{K+}=x_K} \qquad \Longrightarrow \qquad F^i_{\rm Burke-Thorne}(t) = -\frac{2 m_K G}{5}\frac{d^5 Q^i{}_j(t)}{dt^5} x^j_K(t)$$

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Radiation reaction in compact binaries (2)

Matching to a specific theory, model, or data

$$M = m_1 + m_2 + \cdots \qquad\qquad Q^{ij}(t) = \left[\sum_{K=1}^{2} m_K x_K^i(t) x_K^j(t) \right]_{\rm STF} + \cdots$$

Compute stuff: Leading order radiation reaction force (2.5PN = v^5) *CRG & Tiglio, (2009) CRG & Leibovich, (2012)*

$$S_{\rm eff}[\vec{x}_{1\pm}, \vec{x}_{2\pm}] = \frac{m}{2} \int dt\, (\vec{v}_1^{\,2} - \vec{v}_2^{\,2}) + (\text{3PN potentials}) - \frac{G}{5} \int dt\, Q_-^{ij}(t) \frac{d^5 Q_{+ij}}{dt^5} + O(v^7)$$

- Resulting force is precisely the 2.5PN radiation reaction of Burke & Thorne

$$0 = \frac{\delta S}{\delta x^i_{K-}(t)} \bigg|_{x_{K-}=0,\, x_{K+}=x_K} \qquad \Longrightarrow \qquad F^i_{\rm Burke-Thorne}(t) = -\frac{2 m_K G}{5} \frac{d^5 Q^i{}_j(t)}{dt^5} x^j_K(t)$$

- Compare to effective action from usual Hamilton's Principle

$$S_{\rm eff}[\vec{x}_1, \vec{x}_2] = \frac{m}{2} \int dt\, \vec{v}^{\,2} + (\text{3PN potentials}) - \frac{G}{5} \int dt\, Q^{ij}(t) \frac{d^5 Q_{ij}}{dt^5} + O(v^7)$$

- Recently, the 3.5PN radiation reaction was computed in EFT and agreement found with published results *CRG & Leibovich, (2012)*

Radiation reaction in compact binaries (3)

- Effective action

$$\Lambda_{\text{eff}}[\vec{x}_{1\pm},\vec{x}_{2\pm}] = L_{3\text{PN}}[\vec{x}_{K1}] - L_{3\text{PN}}[\vec{x}_{K2}] \underbrace{- \frac{G}{5}\, I_{-}^{ij}(t)\, I_{+ij}^{(5)}(t)}_{K} + O(v^7)$$

$$I_{-}^{ij}(t) = \sum_{K=1}^{2} m_K \left[x_{K-}^{i} x_{K+}^{j}\right]_{\text{STF}} \qquad I_{+}^{ij}(t) = \sum_{K=1}^{2} m_K \left[x_{K+}^{i} x_{K+}^{j}\right]_{\text{TF}} + O(x_{K-}^{2})$$

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- Change in mechanical energy

$$h = \sum_{K=1}^{2} \vec{v}_K \cdot \frac{\partial L}{\partial \vec{v}_K} - L$$

$$\frac{dh}{dt} = -\sum_{K=1}^{2} \vec{v}_K \cdot \left[\frac{d}{dt}\frac{\partial K}{\partial \vec{v}_{K-}} - \frac{\partial K}{\partial \vec{x}_{K-}}\right]_{\rm p.l.}$$

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$$\left\langle \frac{dh}{dt} \right\rangle = -\frac{G}{5} \left\langle \dddot{I}^{ij}(t)\, \dddot{I}_{ij}(t) \right\rangle = -\frac{dE_{\rm GW}}{dt}$$

Take-home points

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  - A variational calculus for initial value problems

  - Naturally yields Lagrangian & Hamiltonian formulations for non-conservative systems

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- Hamilton's Principle compatible with initial data + EFT is a powerful framework

Conclusion

- New Hamilton's principle compatible with initial data is successfully applied in EFT:

  1) Finite size corrections to radiation reaction force on an extended charge
  CRG, Leibovich & Rothstein, PRL (2010)

  2) 2.5 & 3.5 PN radiation reaction forces
  CRG, & Tiglio, PRD (2009); CRG & Leibovich (arXiv: 1205.3842)

  3) 4 PN tail contribution to potential
  Foffa & Sturani (arXiv:1111.5488)

  4) Gravitational self-force and waveform at 1st order in mass ratio
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  5) Scalar self-force and waveforms through 3rd order in mass ratio (see Wed talk)
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- Some possible applications of the new Hamilton's Principle:

  - Engineering/Economics: Optimal control/Pontryagin's Minimum Principle

  - Numerical computing: Variational/Symplectic integrators

  - Mathematics: Variational calculus for initial value problems

  - Physics: Lots! -- Statistical mechanics, fluid mechanics, kinetic theory, nonlinear dynamics,...

Extra slides

A brief history of Effective Field Theory (EFT)

- Long history...

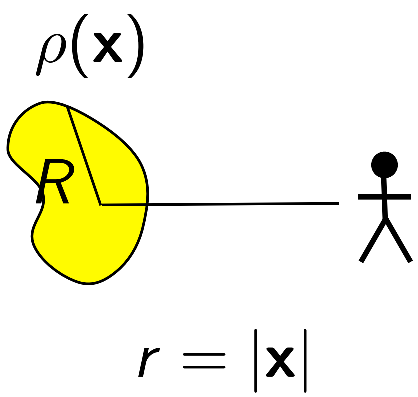
- Initial motivations come from Ken Wilson's ideas on renormalization group

  (talk about coarse-graiing, how that goes into redefining the interactions, etc.?)

- Application of EFT to classical physics by Goldberger & Rothstein

Common ground with EFT (1)

- Aspects of EFT appear in very familiar examples (e.g., extended charge)



$$\phi(\mathbf{x}) = \int_R d^3x' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

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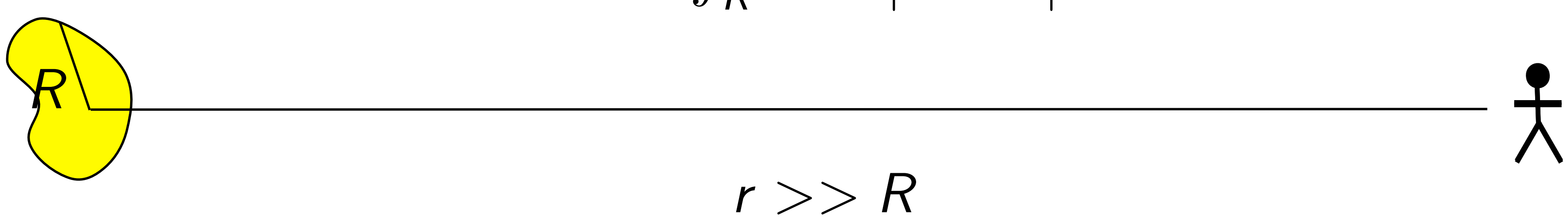

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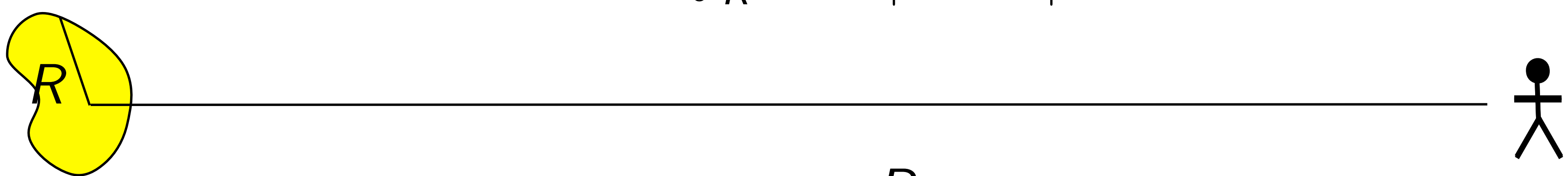

$r >> R$

- Far away, the charge looks like a point carrying multipole moments

$$\phi(x) = q\frac{1}{r} + p_i \frac{x^i}{r^3} + \frac{1}{2} I_{ij} \frac{x^i x^j}{r^5} + \cdots$$

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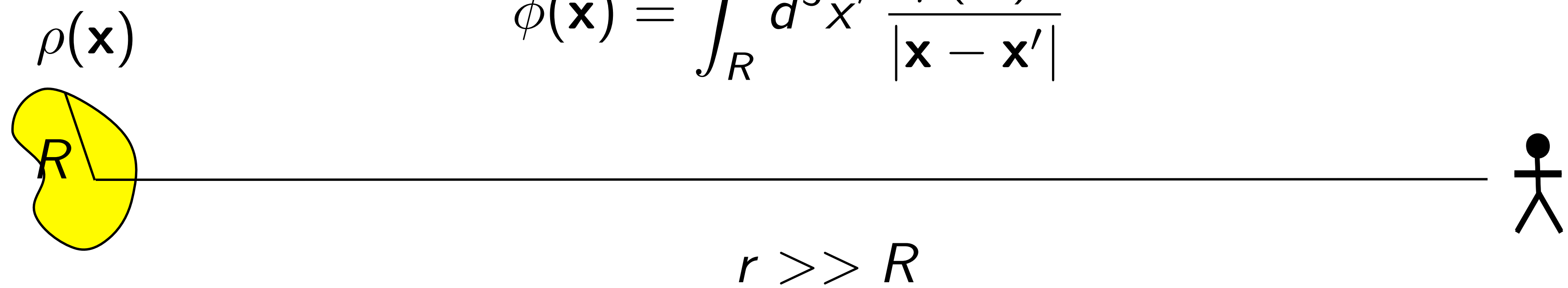

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- The precise location of the apparent point-like charge is immaterial

$$\phi(x) = q' \frac{1}{r'} + p'_i \frac{x'^i}{r'^3} + \frac{1}{2} I'_{ij} \frac{x'^i x'^j}{r'^5} + \cdots \ , \quad r' = |\mathbf{x} + \delta\mathbf{x}| \ , \quad |\delta\mathbf{x}| \lesssim R$$

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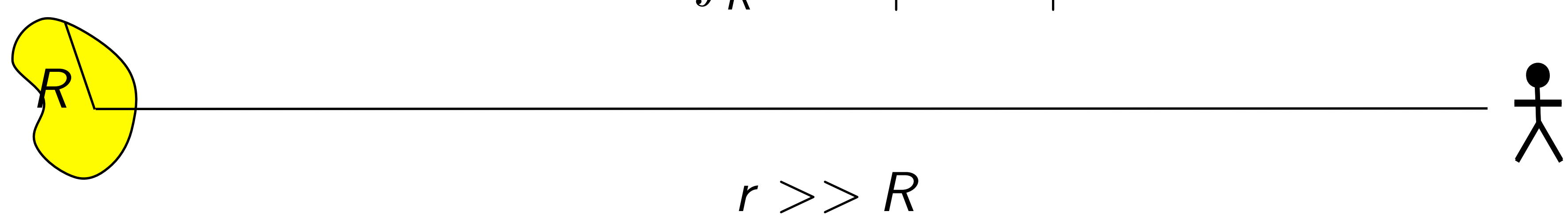

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- The short-distance physics factorizes from the long-distance physics

Common ground with EFT (2)

- Field on worldline diverges, simply because the expansion is invalid there

Use highly successful renormalization group

- Long-wavelength perturbations interact locally in space and time

- The long-distance field is parameterized by STF moments

$$\phi(x) = \frac{q}{r} + \frac{p_i x^i}{r^3} + \frac{1}{2}\frac{I_{ij} x^i x^j}{r^5} + \cdots$$

Can determine the moments through experiment

or

matching to the predictions of a theory

Electrostatics in a Lagrangian

- Lagrangian for electric potential

$$L[\phi] = -\frac{1}{2}\int_{\mathbf{x}} \partial_i\phi(\mathbf{x})\partial^i\phi(\mathbf{x}) + \int_{\mathbf{x}} \rho(\mathbf{x})\phi(\mathbf{x})$$

- Expand field about arbitrary point in charge distribution

$$\phi(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} x^N \partial_N \phi(t, \mathbf{x}_0)$$

$$L[\phi] = -\frac{1}{2}\int_{\mathbf{x}} \partial_i\phi(\mathbf{x})\partial^i\phi(\mathbf{x}) + \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{I}^N_{\mathsf{STF}}\, \partial_N\phi(\mathbf{x}_0)$$

Electro*dynamics* in a Lagrangian

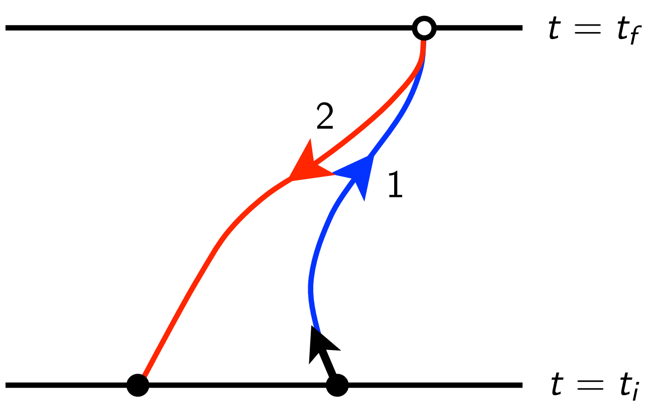
- In an inertial frame, the action is

- In terms of field strength tensor (gauge invariant)

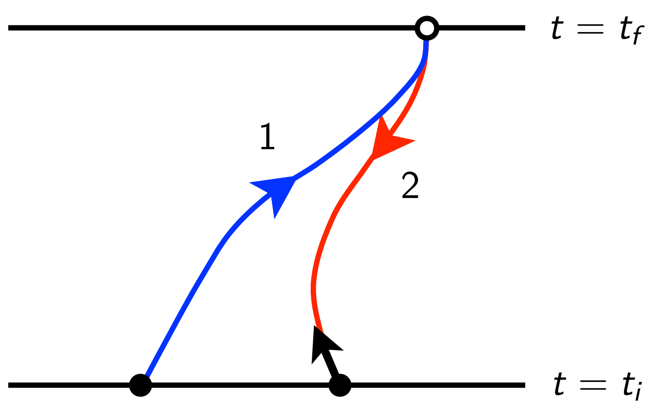
$$L[\phi] = -\frac{1}{4}\int_{\mathbf{x}} F_{i0}(\mathbf{x})F^{i0}(\mathbf{x}) + \sum_{n=0}^{\infty}\frac{1}{n!}\,\mathcal{I}_{\mathsf{STF}}^{N-1\,i}\,\partial_{N-1}F_{i0}(\mathbf{x}_0)$$

Identifying the paths: The "physical limit" (1)

- An ambiguity in associating physical initial data for a physical path with two unphysical paths



$q_1(t_i) = q_i$



$q_2(t_i) = q_i$

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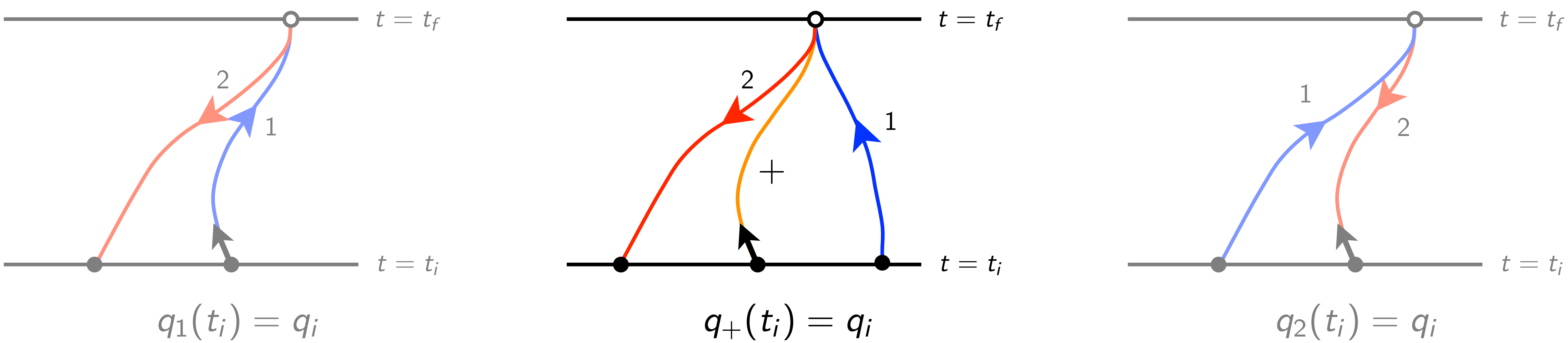

- Natural to identify physical initial data with $q_+(t_i)$

$$\text{Physical limit} \implies q_-(t) \to 0,\; q_+(t) \to q(t)$$

- Make this a convention

Connection to quantum theory

- The new action is the one that arises in the classical limit of the corresponding "in-in" or "closed-time-path (CTP)" quantum theory

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  - Generating functional

$$Z_{\text{in}-\text{in}}[J_1, J_2] \equiv \int \mathcal{D}q_1(t)\mathcal{D}q_2(t) \, \exp\left\{ \frac{i}{\hbar}\big(S[q_1] - S[q_2]\big) + \frac{i}{\hbar}\int dt (J_1 q_1 - J_2 q_2) \right\}$$

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  - Approach the classical limit

$$\begin{aligned} S_{\text{classical}} &= S[q_1] - S[q_2] + \int dt(J_1 q_1 - J_2 q_2) + O(\hbar) \\ &= S[q_1, q_2] + O(\hbar) \end{aligned}$$

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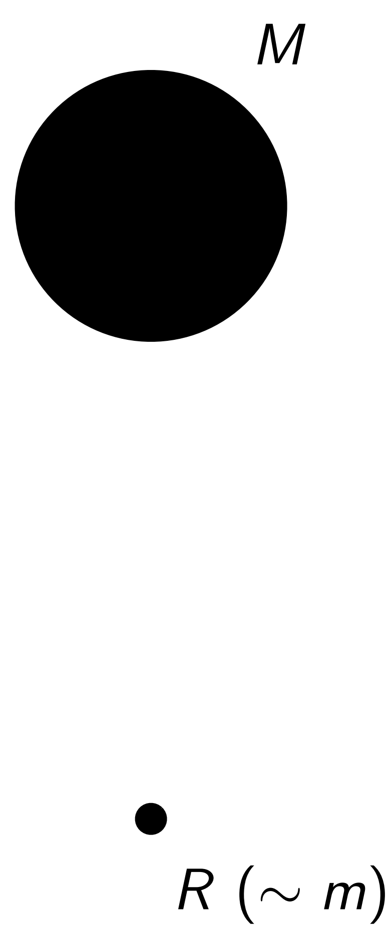
  - On the other hand, path integral quantization based on the new action

$$Z[J_1, J_2] \equiv \int \mathcal{D}q_1(t)\mathcal{D}q_2(t) \, \exp\left\{ \frac{i}{\hbar} S[q_1, q_2] + \frac{i}{\hbar}\int dt(J_1 q_1 - J_2 q_2) \right\} = Z_{\text{in-in}}[J_1, J_2]$$

Self-force in extreme mass ratio binaries (1)



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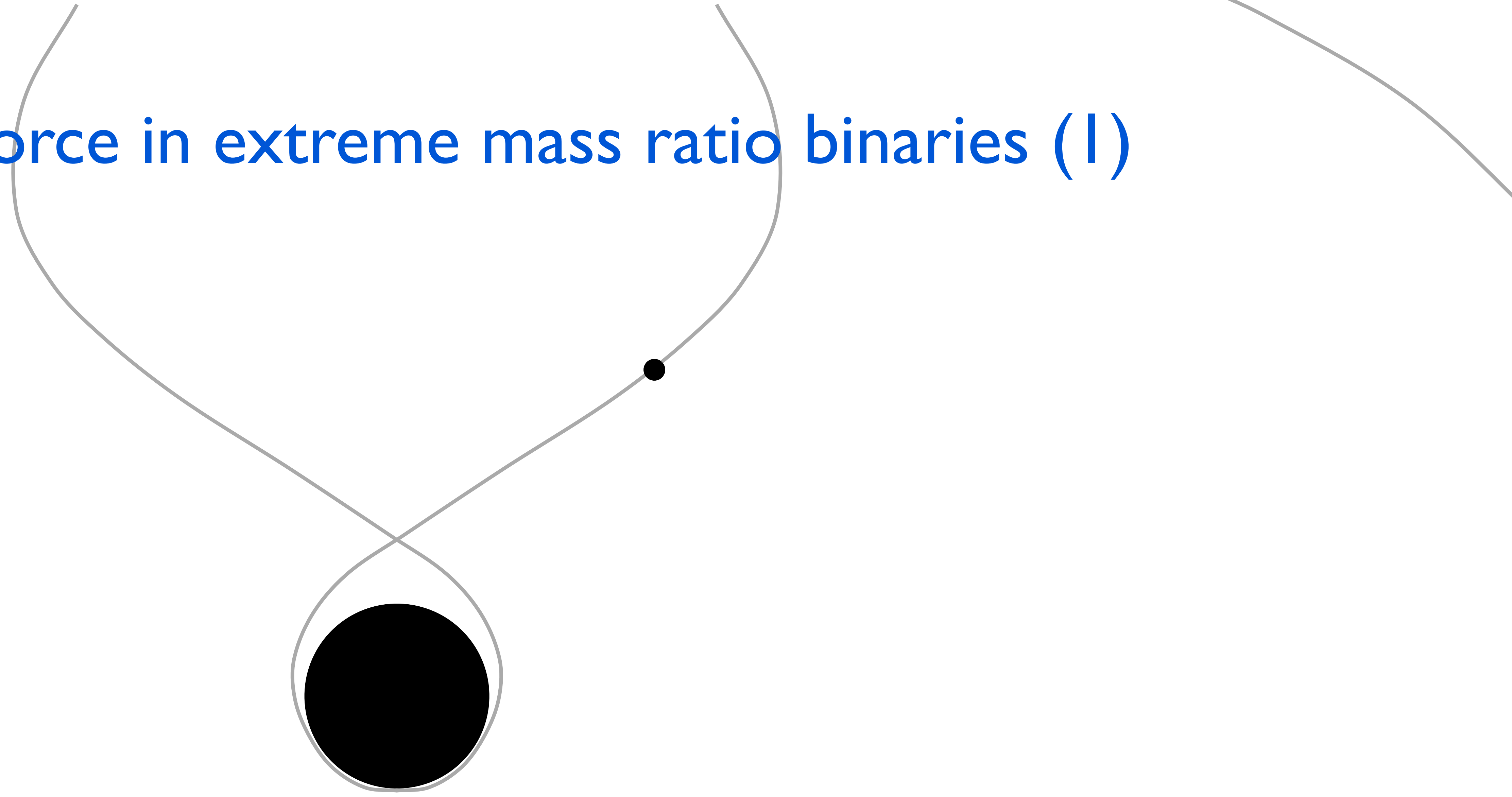


Self-force in extreme mass ratio binaries (II)

- The EFT for this system:

Identify the relevant degrees of freedom and their symmetries

$$\{z^\mu(\tau), h_{\mu\nu}(x^\alpha)\}$$

General coordinate invariance

Reparameterization invariance of worldline

Write most general action consistent with symmetries (i.e., derivative expansion)

$$S[z^\mu, h_{\mu\nu}] = \frac{1}{16\pi}\int_x g^{1/2} R - m \int d\tau + C_e \int d\tau \, \mathcal{E}^{\alpha\beta}(z)\mathcal{E}_{\alpha\beta}(z) + \cdots$$

- Power counting

$$C_e \sim R^5 \sim m^5 \qquad \Longrightarrow$$

Geodesic deviation from finite-size is *$O(R^4)$*

CRG & Hu (2009)

Matching to a specific theory, model, or data

- C_e vanishes for a black hole

Damour & Nagar (2009), Binnington & Poisson (2009), Kol & Smolkin (2011)

Self-force in extreme mass ratio binaries (III)

Compute stuff: Self-force through first order in mass ratio

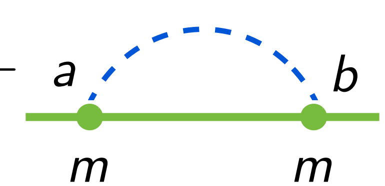
- New action

$$S[z^\mu_{1,2}, h^{\mu\nu}_{1,2}] \equiv S[z^\mu_1, h^{\mu\nu}_1] - S[z^\mu_2, h^{\mu\nu}_2]$$

$$= -\frac{1}{64\pi} \int_x \left(h_a^{\alpha\beta;\mu} h^a_{\alpha\beta;\mu} - \frac{1}{2} h_a^{;\mu} h^a_{;\mu} \right) - m \int (d\tau_1 - d\tau_2) + \frac{1}{2} \int_x h^a_{\alpha\beta} T_a^{\alpha\beta} + \cdots$$

$$T^{\alpha\beta}_{1,2} \equiv m \int d\lambda \, \frac{\delta^4(x - z_{1,2})}{g^{1/2}} \frac{u^\alpha_{1,2} u^\beta_{1,2}}{(-u^2_{1,2})^{1/2}}$$

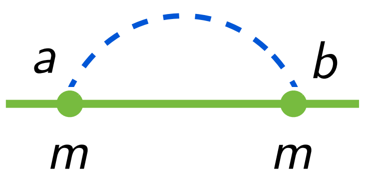
- Integrate out metric perturbations using Feynman diagrams and rules to get eff action

$$S_{\text{eff}}[z^\mu_{1,2}] = -m \int (d\tau_1 - d\tau_2) +$$ 

$$+ O\left((m/M)^2\right)$$

Self-force in extreme mass ratio binaries (IV)

- Diagram for leading order self-force



$$0 = \frac{\delta S}{\delta z_-^\mu(\tau)}\bigg|_{z_-=0, z_+=z}$$

$$\Downarrow$$

$$F^\mu_{\text{MiSaTaQuWa}}(\tau) = 16\pi m^2 P^{\mu\alpha\beta\nu} \lim_{\epsilon to 0} \int_{-\infty}^{\tau-\epsilon} d\tau' \, \nabla_\nu D^{\text{ret}}_{\alpha\beta\gamma'\delta'}(z^\mu, z^{\mu'}) u^{\gamma'} u^{\delta'}$$