Numerical calculation of Green functions in black hole spacetimes

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Motivation

The most general way of solving a wave propagation problem is by constructing its Green function. The Green function for the scalar wave operator $\Box$ satisfies

$$\Box G(x, x') = \delta^4(x - x') \qquad \Box G = \mathbf{1}$$

The inhomogeneous scalar wave equation

$$\Box \phi(x) = S(x), \qquad \Box \phi = S$$

can be solved via the Green function simply by

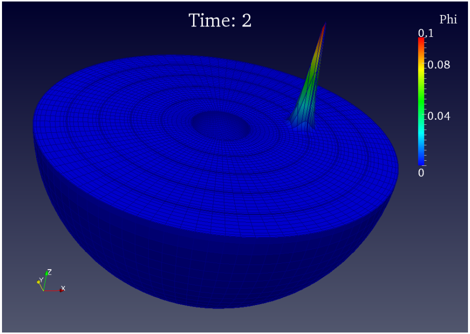
$$\phi(x) = \int G(x, x') S(x') d^4x', \qquad \phi = S/\Box = G\,S$$

A similar procedure applies to initial data.

Numerical Solution

An approximation to the Green function is obtained numerically for $\sigma = 0.2$

$$\Box \phi_{x'}(x) = \frac{1}{(\sqrt{2\pi}\sigma)^4} \exp\left[-\frac{(x-x')^2}{2\sigma^2}\right]$$



For the simulations we used SpEC with a hyperboloidal layer.

What happens then?

Focusing caused by the BH leads to formation of caustics.
Caustic formation is an everyday phenomenon.

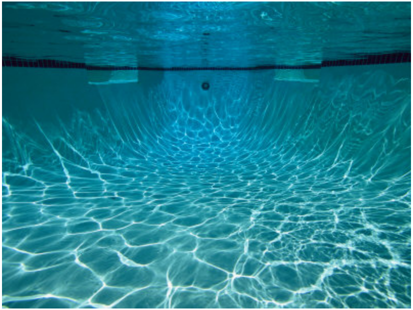


Image from www.shutterstock.com

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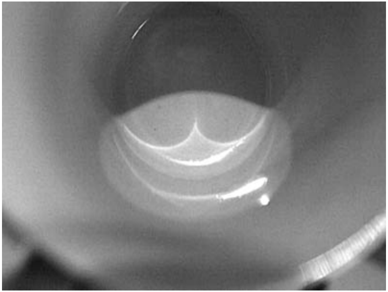
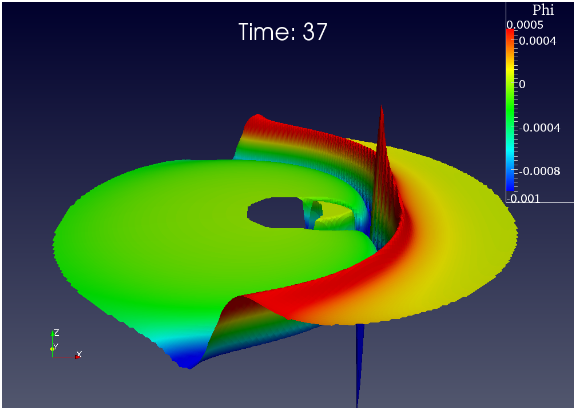


Images from Wikipedia

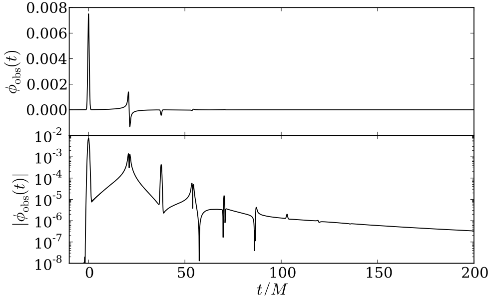
What happens then?



See Visualization

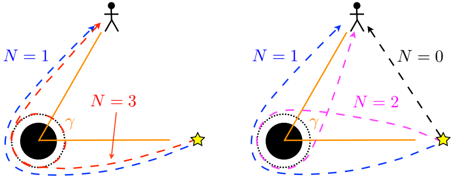
Observer's measurement

An observer at null infinity along the z-axis measures the signal below.



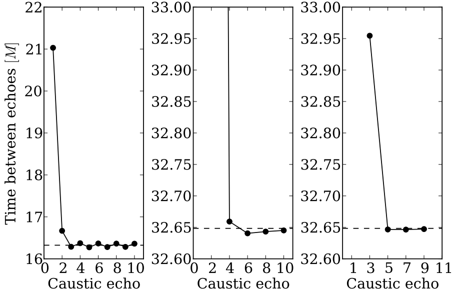
Arrival times

Infinitely many null geodesics connect the source and the observer because of the photon sphere.



The arrival times of the echoes agree with revolution around the photon sphere.

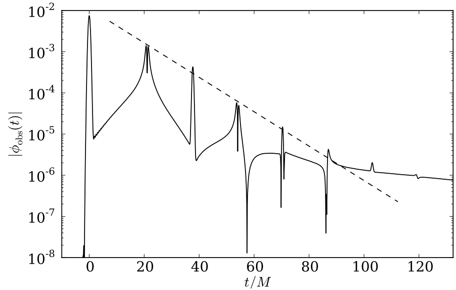
$$T_{\rm half} = \pi\sqrt{27}M \approx 16.32M.$$



Exponential decay

The amplitude decays with the Lyapunov exponent of the unstable null geodesics.

$$\lambda = \frac{1}{2\sqrt{27}M} \approx 0.096 M^{-1}.$$



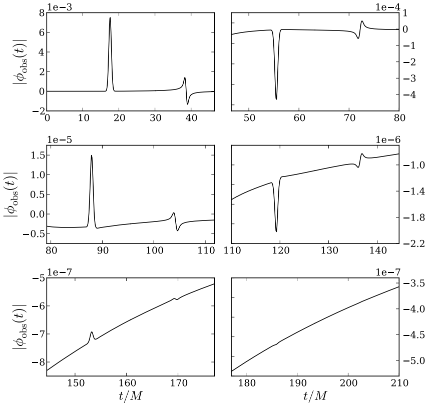
These two properties agree with the large ℓ limit of QNMs.

$$\omega \sim \frac{1}{2\sqrt{27}M}\left(1 - i\right).$$

Profiles and the four-fold structure

Each caustic passage induces a shift of $-\pi/2$ in the profile of the signal, known as Gouy phase shift (1890), known as a Hilbert transform in signal processing.

One can write down an analytic expression for the Green function in the geometrical optics approximation and explain the profiles of the caustic echoes.



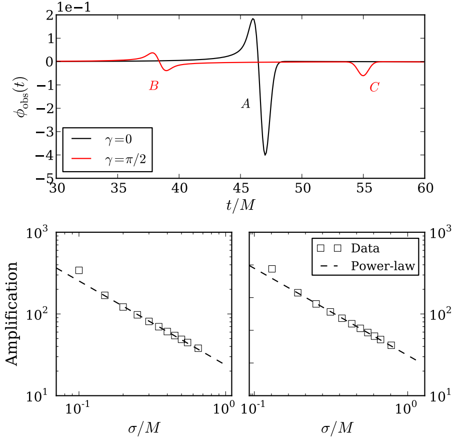
The two-fold cycle

The energy amplification at the caustics depends on the width of the Gaussian source. With

$$E = \frac{1}{2}\int_I dt\,\left|\dot{\phi}(t)\right|^2,$$

we show

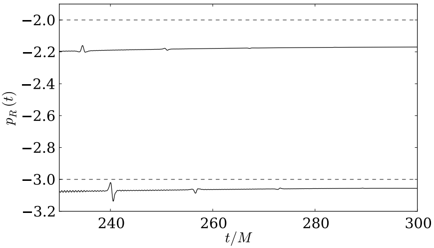
$$\mathrm{Amplification} \equiv \frac{E_A}{E_{B,C}} \sim \frac{1}{\sigma}.$$



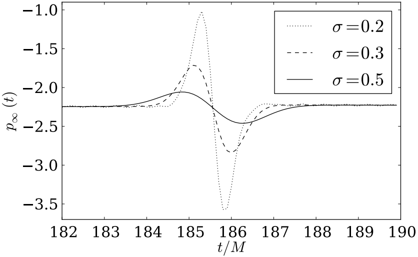
Finite number of "images"

Propagation within the null cone leads to power-law decay at late times.

$$p_R(t) = \frac{d \ln |\phi(t, R)|}{d \ln t}.$$

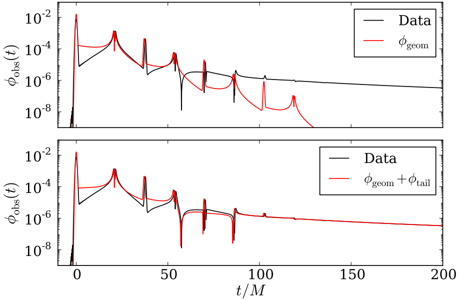


For any finite σ one can only see a finite number of echoes (or images).

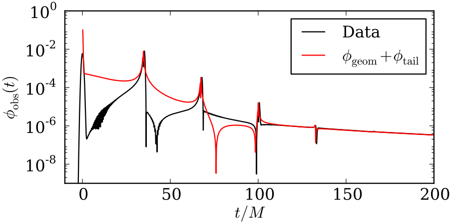


Heuristic description

The geometrical optics approximation to the Green function agrees well with the early part of the signal. Adding the tail contribution leads to good agreement everywhere, including the zero crossings.



The approximation breaks down along the caustic line.



Future directions

Analytical

- Use the numerical Green function in the computation of self-force.
- Analytic approximations to the Green function.
  - Large ℓ-limit of QNMs + tail contribution.
  - Geometrical optics + diffraction theory.
  - Gaussian approximation, Kichhoff's integral representation.
- Capture the base-point dependence of the Green function
- Understand the evolution in Kerr spacetime

Numerical

- Second order formulation of the scalar wave equation.
- Implicit-explicit time stepping.
- Adaptive Mesh Refinement (?).
- Frozen Gaussian beam approximation.
- Butterfly algorithm.

Inspired by Ori, Casals, Dolan, Ottewill, Wardell.

Thank You!