Second-order gravitational self-force

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June 12, 2012

Outline

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Why second order?

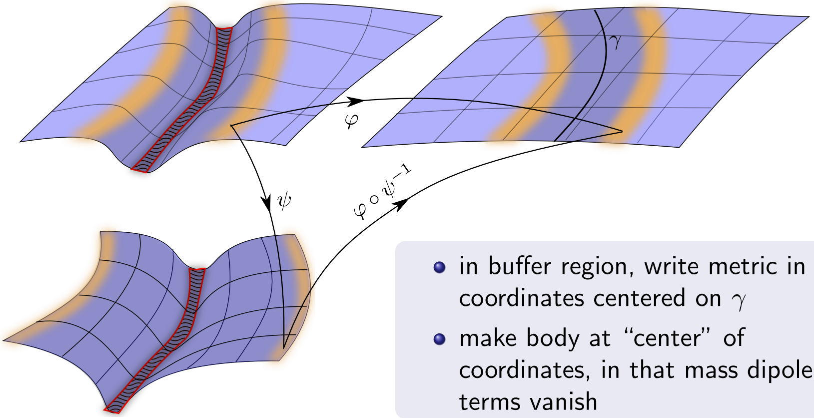
EMRIs

- inspiral takes radiation-reaction time $t_{rr} \sim M^2/m$
- to determine position accurately on that timescale, require second-order acceleration

IMRIs and equal-mass binaries

- second-order self-force could yield highly accurate model for IMRIs over short times
- would fix terms quadratic in mass in Effective One Body theory

Reminder: self-consistent expansion



- in buffer region, write metric in coordinates centered on γ
- make body at "center" of coordinates, in that mass dipole terms vanish

Outline

Outer expansion

Wave equations

- write $\mathsf{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} + \sum \epsilon^n h^{(n)}_{\mu\nu}$
- impose gauge condition $\nabla_\nu \bar{h}^{\mu\nu} = 0$ to relax EFE, and outside body, split EFE into sequence of wave equations

$$E^{\mu\nu}[\bar{h}_{(n)}] \equiv \Box \bar{h}^{\mu\nu}_{(n)} + 2R^{\mu}{}_{\rho}{}^{\nu}{}_{\sigma}\bar{h}^{\rho\sigma}_{(n)} = S^{\mu\nu}_{(n)},$$

  where $S^{\mu\nu}_{(n)}$ is nth-order term in nonlinear part of Einstein tensor
- could impose any condition $\nabla_\nu \bar{h}^{\mu\nu} = Z^{\mu}{}_{\rho\sigma}\bar{h}^{\rho\sigma} + C^{\mu\nu} + O(h^2)$

Expansion in buffer region

Recall...

- given equation of motion in terms of regular field, singular field in buffer region $m \ll r \ll \mathcal{R}$ is all that is required to numerically solve nth order EFE

Ansatz

- in local coordinates (t, x^a) centered on γ, I assume a small-r multipolar expansion

$$\bar{h}^{\mu\nu}_{(n)} = \sum_{p\geq -n} \sum_{\ell,q\geq 0} r^p (\ln r)^q \bar{h}^{\mu\nu L}_{(npq\ell)}(t)\hat{n}_L$$

- $n^i = x^i/r$ is radial unit vector
- note $f_L \hat{n}^L = f_{i_1\cdots i_\ell} n^{\langle i_1} \cdots n^{i_\ell\rangle} \sim \sum_m f^{\ell m} Y^{\ell m}$
- logarithms arise due to deformation of light cones

Decomposing the solution

Homogeneous modes

- for small r, spatial derivatives dominate over time derivatives $\Rightarrow$ solving wave equation consists of solving sequence of Poisson equations
- the homogeneous solutions are $\bar{h}^{\mu\nu L}_{(n,-\ell-1,0,\ell)}\hat{n}_L/r^{\ell+1}$ and $\bar{h}^{\mu\nu L}_{(n,\ell,0,\ell)}\hat{n}_L r^{\ell}$
- call $\bar{h}^{\mu\nu L}_{(n,-\ell-1,0,\ell)}$ and $\bar{h}^{\mu\nu L}_{(n,\ell,0,\ell)}$ the homogeneous modes
- these modes fully characterize solutions to the wave equation

Meaning

- $\bar{h}^{\mu\nu L}_{(n,-\ell-1,0,\ell)} \equiv I^{\mu\nu L}_{(n)}$ $\sim$multipole moment of body or correction to it
- $\bar{h}^{\mu\nu L}_{(n,\ell,0,\ell)}$ defines a smooth homogeneous solution to wave equation

Singular and regular fields

Defining a regular field

- Define $\bar{h}^{\mathrm{R}\mu\nu}_{(n)}$ to be solution to

$$E^{\mu\nu}[\bar{h}^{\mathrm{R}}_{(n)}] = S^{\mu\nu}_{(n)}[\bar{h}^{\mathrm{R}}]$$

  containing only the $\bar{h}^{\mu\nu L}_{(n,\ell,0,\ell)}\hat{n}_L r^\ell$ modes
- the full regular field $\bar{h}^{\mathrm{R}\mu\nu} = \sum \epsilon^n \bar{h}^{\mathrm{R}\mu\nu}_{(n)}$ satisfies vacuum EFE to all orders

Defining a singular field

- $\bar{h}^{\mathrm{S}\mu\nu}_{(n)}$ is the rest of the field. It satisfies

$$E^{\mu\nu}[\bar{h}^{\mathrm{S}\mu\nu}_{(n)}] = S^{\mu\nu}_{(n)}[\bar{h}] - S^{\mu\nu}_{(n)}[\bar{h}^{\mathrm{R}}]$$

  and contains only the $\bar{h}^{\mu\nu L}_{(n,-\ell-1,0,\ell)}\hat{n}_L/r^{\ell+1}$ modes

Outline

Pieces of second-order field

Regular field

- satisfies $E^{\mu\nu}[\bar{h}^{\rm R}_{(2)}] = -2\delta^2 G^{\mu\nu}[\bar{h}^{\rm R}_{(1)}]$
- contains all the homogeneous modes $\bar{h}^{\mu\nu L}_{(2,\ell,0,\ell)}\hat{n}_L r^\ell$

Singular field

- solution to $E^{\mu\nu}[\bar{h}^{\rm S}_{(2)}] = -2\left(\delta^2 G^{\mu\nu}[\bar{h}_{(1)}] - \delta^2 G^{\mu\nu}[\bar{h}^{\rm R}_{(1)}]\right)$
- split into three pieces: $\bar{h}^{{\rm S}\mu\nu}_{(2)} = \bar{h}^{\mu\nu}_{-2(2)} + \bar{h}^{\mu\nu}_{-1(2)} + \bar{h}^{{\rm IH}\mu\nu}_{(2)}$

$\bar{h}^{{\rm IH}\mu\nu}_{(2)}$

- inhomogeneous solution made up of terms quadratic in $\bar{h}^{\mu\nu}_{(1)}$
- i.e., particular solution containing no homogeneous modes

Pieces of second-order field (continued)

$\bar{h}^{\mu\nu}_{-2(2)}$

- homogeneous solution (in buffer region; not at $r = 0$) made up of terms linear in body's dipole moments
- has form $\sim n_i/r^2 + O(1/r)$
- set mass dipole to zero, leaving only spin terms

$\bar{h}^{\mu\nu}_{-1(2)}$

- homogeneous solution (in buffer region; not at $r = 0$) made up of terms linear in $\delta m^{\mu\nu}$
- has form $\sim 1/r + O(r^0)$
- gauge condition/Bianchi identity relates $\delta m^{\mu\nu}$ to $m\bar{h}^{\mathrm{R}\mu\nu}_{(1)}$
- could add solution with identical form as first order singular field (i.e., $\frac{\delta_{\mu\nu} const}{r}$); I choose to incorporate it into m

Singular field

$$
\begin{aligned}
\bar{h}^{ta}_{-2(2)} = {} & -\frac{2\epsilon^{aij}S_j n_i}{r^2} + \frac{3\epsilon^{ajd}S_d a^i \hat{n}_{ij}}{r} + \left(\tfrac{1}{3}S^c\mathcal{E}_b{}^d\epsilon^a{}_{cd} - \tfrac{49}{15}S^c\mathcal{E}^{ad}\epsilon_{bcd}\right)n^b \\
& + 3a^a a^b S^d \epsilon_{bcd} n^c + \tfrac{1}{3} S^b \mathcal{E}^{cd}\epsilon_{bc}{}^i \hat{n}^a{}_{di} - S^b\mathcal{E}^{cd}\epsilon^a{}_b{}^i \hat{n}_{cdi} + \tfrac{15}{4} a^b a^c S^d \epsilon^a{}_d{}^i \hat{n}_{bci} \\
& + r\Big(\tfrac{1}{4} S^b\epsilon^a{}_{bc} a_{d,tt}\hat{n}^{cd} + \tfrac{1}{3} S^b \epsilon^a{}_b{}^c a_{c,tt} - \tfrac{1}{18} S^b \dot{\mathcal{B}}^{cd}\hat{n}^a{}_{bcd} - \tfrac{11}{63} S^a \dot{\mathcal{B}}^{bc}\hat{n}_{bc} \\
& + \tfrac{34}{63} S^b\dot{\mathcal{B}}_b{}^c \hat{n}^a{}_c + \tfrac{41}{63} S^b \dot{\mathcal{B}}^{ac}\hat{n}_{bc} + \tfrac{16}{15} S^b\dot{\mathcal{B}}^a{}_b - \tfrac{1}{2} a^b S^c \mathcal{E}^{di}\epsilon_{cd}{}^j \hat{n}^a{}_{bij} \\
& + \tfrac{5}{2} a^b S^c \mathcal{E}^{di}\epsilon^a{}_c{}^j \hat{n}_{bdij} + \tfrac{3}{7} a^a S^b \mathcal{E}^{cd}\epsilon_{bdi}\hat{n}_c{}^i + \tfrac{2}{21} a^b S^c \mathcal{E}^{di}\epsilon_{bci}\hat{n}^a{}_d \\
& + \tfrac{3}{7} a^b S^c \mathcal{E}_b{}^d \epsilon_{cdi}\hat{n}^{ai} + \tfrac{13}{42} a^b S^c \mathcal{E}^{di}\epsilon^a{}_{ci}\hat{n}_{bd} + \tfrac{69}{14} a^b S^c \mathcal{E}^{ad}\epsilon_{cdi}\hat{n}_b{}^i \\
& + \tfrac{17}{21} a^b S^c \mathcal{E}^{di}\epsilon^a{}_{bc}\hat{n}_{di} + \tfrac{25}{84} a^b S^c \mathcal{E}_b{}^d \epsilon^a{}_{ci}\hat{n}_d{}^i - \tfrac{7}{3} a^b S^c\mathcal{E}^{ad}\epsilon_{bci}\hat{n}_d{}^i \\
& + \tfrac{2}{15} a^b S^c \mathcal{E}_b{}^d \epsilon^a{}_{cd} + \tfrac{191}{45} a^b S^c \mathcal{E}^{ad}\epsilon_{bcd} + \tfrac{1}{6} S^b \epsilon_{bc}{}^j \mathcal{E}^{cdi}\hat{n}^a{}_{dij} \\
& + \tfrac{1}{6} S^b \epsilon^a{}_{bi}\mathcal{E}_{cd}{}^i \hat{n}^{cd} + \tfrac{61}{42} S^b \mathcal{E}^a{}_c{}^i \epsilon_{bdi}\hat{n}^{cd} - \tfrac{35}{8} a^b a^c a^d S^i \epsilon^a{}_i{}^j \hat{n}_{bcdj} \\
& + \tfrac{15}{2} a^a a^b a^c S^d \epsilon_{cdi}\hat{n}_b{}^i - \tfrac{1}{4} S^b \epsilon^a{}_b{}^j \mathcal{E}^{cdi}\hat{n}_{cdij}\Big) + O(r^2),
\end{aligned}
$$

Obtaining a global solution

Puncture scheme

- can use and desired effective-source/puncture method
- in region effectively covering body, define $h^{P(n)}_{\mu\nu}$ as small-r expansion of $h^{S(n)}_{\mu\nu}$ truncated at highest order available. Solve

$$E^{\mu\nu}[\bar{h}^{R(1)}] = T^{\mu\nu}_{(1)} - E^{\mu\nu}[\bar{h}^{P(1)}]$$

$$E^{\mu\nu}[\bar{h}^{R(2)}] = 2\delta^2 G^{\mu\nu}[h^{(1)}] + T^{\mu\nu}_{(2)} - E^{\mu\nu}[\bar{h}^{P(2)}]$$

- transition to full fields $\bar{h}^{R(1)}_{\mu\nu}$ using worldtube method, window function method, etc.

Reminder: stress-energy tensor

- can show the homogeneous solutions defined by the multipole moments are identical to those sourced by $T^{\mu\nu}_{(n)}[\gamma] = \sum_\ell \int_\gamma I^{\mu\nu\alpha_1\ldots\alpha_\ell}_{(n)} \nabla_{\alpha_1} \cdots \nabla_{\alpha_1} \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$

Motion at nth order

- gauge condition determines relationship between acceleration of worldline, multipole moments, and $\bar{h}^{\mathrm{R}}_{\mu\nu}$
- need to relate the multipole moments to the body and model their evolution for realistic bodies
- need to impose some reasonable "mass-centeredness" condition to make this into an ODE that can be solved for the trajectory
  —best choice not obvious, since gravitational "energy" might not be centered at same place as unperturbed mass

Outline

Inner expansion

Change of gears

- rather than deriving second-order equation of motion from gauge condition on $\bar{h}^{\mu\nu}_{(3)}$, use method along lines of original Mino, Sasaki, Tanaka derivation
- assume inner expansion is tidally perturbed Schwarzschild, find unique gauge transformation between inner and outer

Tidally perturbed black hole

- Schwarzschild black hole plus tidal fields due to external spacetime (at orders of interest here, describes tidally perturbed spacetime of any spherical body)
- tidal perturbations begin at quadrupole order; monopole and dipoles are not included

Matching

Metric in schematic form

- in inner gauge:

$$g_{tt} \sim -1+\frac{m^2}{r^2}+\frac{m}{r}+r^2 e_1(m/r)\tilde{\mathcal{E}}^{\mathsf{q}}+r^3[e_2(m/r)\dot{\tilde{\mathcal{E}}}^{\mathsf{q}}+e_3(m/r)\tilde{\mathcal{E}}^{\mathsf{o}}]+\ldots$$

- in outer gauge:

$$g_{tt} \sim -1+\frac{m^2}{r^2}+\frac{m+\delta m}{r}+h^{\mathrm{R}}+r(a_i+\partial h^{\mathrm{R}})+r^2(\mathcal{E}^{\mathsf{q}}+\partial\partial h^{\mathrm{R}}) + r^3(\dot{\mathcal{E}}^{\mathsf{q}}+\mathcal{E}^{\mathsf{o}})+\ldots$$

Matching condition

- both gauges are mass-centered
- demand a unique gauge transformation that makes them agree *without translating worldline* (i.e., no spatial translation terms at $r=0$ allowed)

Results of matching

Tidal moments

- $\tilde{\mathcal{E}}_{ab} = \mathcal{E}_{ab} + \delta\mathcal{E}_{ab} + \ldots$, where

$$\delta\mathcal{E}_{ab} = -\frac{1}{2}h^{(1)\mathrm{R}}_{tt,\langle ab\rangle} + h^{(1)\mathrm{R}}_{tt}\mathcal{E}_{ab} + \frac{8}{3}m\dot{\mathcal{E}}_{ab} - \underset{ab}{\mathrm{STF}}\,\mathcal{E}^i_a h^{(1)\mathrm{R}}_{\langle bi\rangle} - \mathcal{E}_{ab}\delta^{ij}h^{(1)\mathrm{R}}_{ij} + 2\,\underset{ab}{\mathrm{STF}}\,\mathcal{E}^i_a\int h^{(1)\mathrm{R}}_{t[b,i]}dt + h^{(1)\mathrm{R}}_{t\langle a,b\rangle t} + \frac{1}{2}\dot{\mathcal{E}}_{ab}\int h^{(1)\mathrm{R}}_{tt}dt - \frac{1}{2}h^{(1)\mathrm{R}}_{\langle ab\rangle,tt}.$$

- note $m\dot{\mathcal{E}}_{ab}$ term; indicates contribution from singular field $\Rightarrow \tilde{\mathcal{E}}_{ab}$ is not made from Riemann tensor of $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$
- also note potentially growing terms, which arise due to exclusion of monopole and dipole terms in inner expansion
- analogous results for $\delta\mathcal{B}_{ab}$

Results of matching (continued)

Geodesic motion in an effective geometry

$$a^\mu = \frac{1}{2}\left(g^{\mu\nu} + u^\mu u^\nu\right)\left(g_\nu{}^\rho - h_\nu^{\mathrm{R}\rho}\right)\left(h^{\mathrm{R}}_{\sigma\lambda;\rho} - 2h^{\mathrm{R}}_{\rho\sigma;\lambda}\right)u^\sigma u^\lambda + O(\epsilon^3)$$

- here $a^\mu = a^\mu_{(0)} + \epsilon a^\mu_{(1)} + \epsilon^2 a^\mu_{(2)} + \dots$
- this is geodesic equation in $g_{\mu\nu} + h^{\mathrm{R}}_{\mu\nu}$, which is a smooth solution to the vacuum EFE

Conclusion

Summary

- in principle, no major obstacle to going to nth order
- at second order, singular field known to sufficient accuracy to implement puncture scheme
- motion through second order (for spherical body) is geodesic in an effective metric that would be calculated in puncture scheme

Future work

- find closed-form expression for $h^{\mathrm{R}(2)}_{\mu\nu}$ and $h^{\mathrm{S}(2)}_{\mu\nu}$ (or some other fields that agree with my definitions through order r), analogous to Detweiler-Whiting fields at first order
- find equation of motion for spinning body with quadrupole moments