Second-order gravitational self-force

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Outline

1. Introduction

2. $n$th-order outer expansion

3. Second-order outer expansion

4. Matching to inner expansion
Outline

1 Introduction

2 $n$th-order outer expansion

3 Second-order outer expansion

4 Matching to inner expansion
Why second order?

**EMRIs**
- inspiral takes radiation-reaction time $t_{rr} \sim M^2/m$
- to determine position accurately on that timescale, require second-order acceleration

**IMRIs and equal-mass binaries**
- second-order self-force could yield highly accurate model for IMRIs over short times
- would fix terms quadratic in mass in Effective One Body theory
Reminder: self-consistent expansion

- In buffer region, write metric in coordinates centered on $\gamma$
- Make body at “center” of coordinates, in that mass dipole terms vanish
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Outer expansion

Wave equations

- write \( g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} = g_{\mu\nu} + \sum \epsilon^n h^{(n)}_{\mu\nu} \)
- impose gauge condition \( \nabla_\nu \bar{h}^{\mu\nu} = 0 \) to relax EFE, and outside body, split EFE into sequence of wave equations

\[
E^{\mu\nu}[\bar{h}_{(n)}] \equiv \Box \bar{h}^{\mu\nu}_{(n)} + 2 R^\mu_\rho \rho^\nu_\sigma \bar{h}^{\rho\sigma}_{(n)} = S^{\mu\nu}_{(n)},
\]

where \( S^{\mu\nu}_{(n)} \) is \( n \)th-order term in nonlinear part of Einstein tensor

- could impose any condition \( \nabla_\nu \bar{h}^{\mu\nu} = Z^\mu_\rho_\sigma \bar{h}^{\rho\sigma} + C^{\mu\nu} + O(h^2) \)
Recall...

- given equation of motion in terms of regular field, singular field in buffer region $m \ll r \ll R$ is all that is required to numerically solve $n$th order EFE

Ansatz

- in local coordinates $(t, x^a)$ centered on $\gamma$, I assume a small-$r$ multipolar expansion

$$\bar{h}^\mu_\nu{(n)} = \sum_{p \geq -n} \sum_{\ell, q \geq 0} r^p (\ln r)^q \bar{h}^\mu_\nu^{L(npq\ell)}(t) \hat{n}_L$$

- $n^i = x^i / r$ is radial unit vector
- note $f_L \hat{n}^L = f_{i_1\ldots i_\ell} n^{i_1 \ldots i_\ell} \sim \sum_m f^{\ell m} Y^{\ell m}$
- logarithms arise due to deformation of light cones
Decomposing the solution

Homogeneous modes

- For small $r$, spatial derivatives dominate over time derivatives
  ⇒ solving wave equation consists of solving sequence of Poisson equations

- The homogeneous solutions are $\bar{h}_{(n,-\ell-1,0,\ell)}^{\mu\nu L} \hat{n}_L / r^{\ell+1}$ and $\bar{h}_{(n,\ell,0,\ell)}^{\mu\nu L} \hat{n}_L r^\ell$

- Call $\bar{h}_{(n,-\ell-1,0,\ell)}^{\mu\nu L}$ and $\bar{h}_{(n,\ell,0,\ell)}^{\mu\nu L}$ the homogeneous modes

- These modes fully characterize solutions to the wave equation

Meaning

- $\bar{h}_{(n,-\ell-1,0,\ell)}^{\mu\nu L} \equiv I_{(n)}^{\mu\nu L} \sim$ multipole moment of body or correction to it

- $\bar{h}_{(n,\ell,0,\ell)}^{\mu\nu L}$ defines a smooth homogeneous solution to wave equation
Singular and regular fields

**Defining a regular field**

- Define $\bar{h}_{n}^{R\mu\nu}$ to be solution to
  \[
  E^\mu\nu[\bar{h}_{n}^{R\mu\nu}] = S_{n}^{\mu\nu}[\bar{h}^{R}]
  \]
  containing only the $\bar{h}_{n,\ell,0,\ell}^{L} \hat{n}_{L} r^{\ell}$ modes
- the full regular field $\bar{h}_{n}^{R\mu\nu} = \sum e^{n} \bar{h}_{n}^{R\mu\nu}$ satisfies vacuum EFE to all orders

**Defining a singular field**

- $\bar{h}_{n}^{S\mu\nu}$ is the rest of the field. It satisfies
  \[
  E^\mu\nu[\bar{h}_{n}^{S\mu\nu}] = S_{n}^{\mu\nu}[\bar{h}] - S_{n}^{\mu\nu}[\bar{h}^{R}]
  \]
  and contains only the $\bar{h}_{n,-\ell-1,0,\ell}^{L} \hat{n}_{L} / r^{\ell+1}$ modes
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Pieces of second-order field

**Regular field**
- satisfies $E^{\mu\nu}[\bar{h}_R^{(2)}] = -2\delta^2 G^{\mu\nu}[\bar{h}_R^{(1)}]$
- contains all the homogeneous modes $\bar{h}^{\mu\nu L}_{(2,\ell,0,\ell)} \hat{n}_L r^\ell$

**Singular field**
- solution to $E^{\mu\nu}[\bar{h}_S^{(2)}] = -2\left(\delta^2 G^{\mu\nu}[\bar{h}_R^{(1)}] - \delta^2 G^{\mu\nu}[\bar{h}_L^{(1)}]\right)$
- split into three pieces: $\bar{h}_S^{\mu\nu}_{(2)} = \bar{h}^{\mu\nu}_{-2(2)} + \bar{h}^{\mu\nu}_{-1(2)} + \bar{h}^{\mu\nu}_{IH(2)}$

**$\bar{h}^{\mu\nu}_{IH(2)}$**
- inhomogeneous solution made up of terms quadratic in $\bar{h}^{\mu\nu}_{(1)}$
- i.e., particular solution containing no homogeneous modes
Intro outer expansion second order matching

Pieces of second-order field (continued)

\( \bar{h}^{\mu\nu}_{-2(2)} \)
- homogeneous solution (in buffer region; not at \( r = 0 \)) made up of terms linear in body’s dipole moments
- has form \( \sim n_i/r^2 + O(1/r) \)
- set mass dipole to zero, leaving only spin terms

\( \bar{h}^{\mu\nu}_{-1(2)} \)
- homogeneous solution (in buffer region; not at \( r = 0 \)) made up of terms linear in \( \delta m^{\mu\nu} \)
- has form \( \sim 1/r + O(r^0) \)
- gauge condition/Bianchi identity relates \( \delta m^{\mu\nu} \) to \( m\bar{h}^{R\mu\nu} \)
- could add solution with identical form as first order singular field (i.e., \( \delta^{\mu\nu} \text{const} \frac{m}{r} \)); I choose to incorporate it into \( m \)
\[
\begin{aligned}
\bar{h}^{ta}_{-2(2)} &= -\frac{2\epsilon^{aij} S_{j} n_{i}}{r^2} + \frac{3\epsilon^{ajd} S_{d} a^{i} \hat{n}_{ij}}{r} + \left( \frac{1}{3} S^{c} \mathcal{E}_{b} d \epsilon^{a}_{cd} - \frac{49}{45} S^{c} \mathcal{E}^{ad} \epsilon_{bcd} \right) n^{b} \\
&+ 3 a^{a} a^{b} S^{d} \epsilon_{bcd} n^{c} + \frac{1}{3} S^{b} \mathcal{E}^{cd} \epsilon_{bc} i \hat{n}_{di} - S^{b} \mathcal{E}^{cd} \epsilon_{b} a^{i} \hat{n}_{cdi} + \frac{15}{4} a^{b} a^{c} S^{d} \epsilon_{d} a^{i} \hat{n}_{bci} \\
&+ r \left( \frac{1}{4} S^{b} \mathcal{E}_{a} b \epsilon_{c} a_{d,tt} \hat{n}^{cd} + \frac{1}{3} S^{b} \mathcal{E}_{a} b c_{a,tt} \hat{n} - \frac{1}{18} S^{b} \mathcal{B}^{cd} \hat{n}_{bcd} - \frac{1}{63} S^{a} \mathcal{B}^{bc} \hat{n}_{bc} + \frac{34}{63} S^{b} \mathcal{B}^{c} b \hat{n}_{ac} + \frac{41}{63} S^{b} \mathcal{B}^{ac} \hat{n}_{bc} + \frac{16}{15} S^{b} \mathcal{B}^{a} b - \frac{1}{2} a^{b} S^{c} \mathcal{E}^{di} \epsilon_{cd} j \hat{n}_{a b i j} \\
&+ \frac{5}{2} a^{b} S^{c} \mathcal{E}^{di} \epsilon_{a c} j \hat{n}_{b d i j} + \frac{3}{7} a^{a} S^{b} \mathcal{E}^{cd} \epsilon_{b d i} \hat{n}_{c i} + \frac{2}{21} a^{b} S^{c} \mathcal{E}^{di} \epsilon_{b c i} \hat{n}_{a d} \\
&+ \frac{3}{7} a^{b} S^{c} \mathcal{E}_{b} d \epsilon_{c d i} \hat{n}^{a i} + \frac{13}{42} a^{b} S^{c} \mathcal{E}^{di} \epsilon_{a c i} \hat{n}_{bd} + \frac{25}{84} a^{b} S^{c} \mathcal{E}_{b} d \epsilon_{a c i} \hat{n}_{d i} - \frac{7}{3} a^{b} S^{c} \mathcal{E}^{ad} \epsilon_{bcd} \hat{n}_{i} \\
&+ \frac{2}{15} a^{b} S^{c} \mathcal{E}_{b} d \epsilon^{a}_{cd} + \frac{191}{45} a^{b} S^{c} \mathcal{E}^{ad} \epsilon_{bcd} + \frac{1}{6} S^{b} \epsilon_{b c} j \mathcal{E}^{cdi} \hat{n}^{a i d i j} \\
&+ \frac{1}{6} S^{b} \epsilon_{a b i} \mathcal{E}_{cd} i \hat{n}^{cd} + \frac{61}{42} S^{b} \mathcal{E}_{c} a i \epsilon_{b d i} \hat{n}^{cd} - \frac{35}{8} a^{b} a^{c} a^{d} S^{i} \epsilon_{a i j} \hat{n}_{b c d i j} \\
&+ \frac{15}{2} a^{a} a^{b} a^{c} S^{d} \epsilon_{cdi} \hat{n}_{b i} - \frac{1}{4} S^{b} \epsilon_{a b j} \mathcal{E}^{cdi} \hat{n}_{cdi j} \right) + O(r^2),
\end{aligned}
\]
Obtaining a global solution

Puncture scheme

- can use and desired effective-source/puncture method
- in region effectively covering body, define $h^{P(n)}_{\mu\nu}$ as small-$r$ expansion of $h^{S(n)}_{\mu\nu}$ truncated at highest order available. Solve

$$E^{\mu\nu}[\bar{h}^{R(1)}_{\mu\nu}] = T^{(1)}_{\mu\nu} - E^{\mu\nu}[\bar{h}^{P(1)}_{\mu\nu}]$$

$$E^{\mu\nu}[\bar{h}^{R(2)}_{\mu\nu}] = 2\delta^2 G^{\mu\nu}[h^{(1)}_{\mu\nu}] + T^{(2)}_{\mu\nu} - E^{\mu\nu}[\bar{h}^{P(2)}_{\mu\nu}]$$

- transition to full fields $\bar{h}^{R(1)}_{\mu\nu}$ using worldtube method, window function method, etc.

Reminder: stress-energy tensor

- can show the homogeneous solutions defined by the multipole moments are identical to those sourced by

$$T^{\mu\nu}_{(n)}[\gamma] = \sum_{\ell} \int_{\gamma} I^{\mu\nu\alpha_1\ldots\alpha_\ell}_{(n)} \nabla_{\alpha_1} \cdots \nabla_{\alpha_\ell} \frac{\delta^4(x^\rho - z^\rho(\tau))}{\sqrt{-g}} d\tau$$
Motion at $n$th order

- Gauge condition determines relationship between acceleration of worldline, multipole moments, and $\bar{h}_{\mu\nu}^R$
- Need to relate the multipole moments to the body and model their evolution for realistic bodies
- Need to impose some reasonable “mass-centeredness” condition to make this into an ODE that can be solved for the trajectory — best choice not obvious, since gravitational “energy” might not be centered at same place as unperturbed mass
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Intro

outer expansion

second order

matching

Inner expansion

Change of gears

- rather than deriving second-order equation of motion from gauge condition on $\bar{h}_{(3)}^\mu\nu$, use method along lines of original Mino, Sasaki, Tanaka derivation
- assume inner expansion is tidally perturbed Schwarzschild, find unique gauge transformation between inner and outer

Tidally perturbed black hole

- Schwarzschild black hole plus tidal fields due to external spacetime (at orders of interest here, describes tidally perturbed spacetime of any spherical body)
- tidal perturbations begin at quadrupole order; monopole and dipoles are not included
Matching

Metric in schematic form

- in inner gauge:
  \[ g_{tt} \sim -1 + \frac{m^2}{r^2} + \frac{m}{r} + r^2 e_1 (m/r) \tilde{E}^q + r^3 [e_2 (m/r) \dot{\tilde{E}}^q + e_3 (m/r) \tilde{E}^o] + \ldots \]

- in outer gauge:
  \[ g_{tt} \sim -1 + \frac{m^2}{r^2} + \frac{m + \delta m}{r} + h^R + r (a_i + \partial h^R) + r^2 (E^q + \partial \partial h^R) \]
  \[ + r^3 (\dot{E}^q + E^o) + \ldots \]

Matching condition

- both gauges are mass-centered
- demand a unique gauge transformation that makes them agree 
  *without translating worldline*
  (i.e., no spatial translation terms at \( r = 0 \) allowed)
Results of matching

Tidal moments

- \( \tilde{\mathcal{E}}_{ab} = \mathcal{E}_{ab} + \delta \mathcal{E}_{ab} + \ldots \), where

\[
\delta \mathcal{E}_{ab} = -\frac{1}{2} h^{(1)R}_{tt,\langle ab \rangle} + h^{(1)R}_{tt} \mathcal{E}_{ab} + \frac{8}{3} m \dot{\mathcal{E}}_{ab} - \text{STF} \mathcal{E}_{a} h^{(1)R}_{\langle bi \rangle} \\
- \mathcal{E}_{ab} \delta^{ij} h^{(1)R}_{ij} + 2 \text{STF} \mathcal{E}_{a} \int h_{t[b,i]}^{(1)R} dt + h_{t\langle a,b \rangle t}^{(1)R} \\
+ \frac{1}{2} \dot{\mathcal{E}}_{ab} \int h_{tt}^{(1)R} dt - \frac{1}{2} h^{(1)R}_{\langle ab \rangle,tt}.
\]

- note \( m \dot{\mathcal{E}}_{ab} \) term; indicates contribution from singular field

\( \Rightarrow \tilde{\mathcal{E}}_{ab} \) is not made from Riemann tensor of \( g_{\mu\nu} + h_{\mu\nu}^{R} \)

- also note potentially growing terms, which arise due to exclusion of monopole and dipole terms in inner expansion

- analogous results for \( \delta \mathcal{B}_{ab} \)
Intro outer expansion second order matching

Results of matching (continued)

Geodesic motion in an effective geometry

\[
a^\mu = \frac{1}{2} \left( g^{\mu\nu} + u^\mu u^\nu \right) \left( g^\nu_\rho - h^R_\nu^\rho \right) \left( h^R_\sigma^\lambda_\rho - 2 h^R_\rho^\sigma_\lambda \right) u^\sigma u^\lambda + O(\epsilon^3)
\]

- here \( a^\mu = a^\mu_0 + \epsilon a^\mu_1 + \epsilon^2 a^\mu_2 + \ldots \)
- this is geodesic equation in \( g_{\mu\nu} + h^R_{\mu\nu} \), which is a smooth solution to the vacuum EFE
Conclusion

Summary

- in principle, no major obstacle to going to $n$th order
- at second order, singular field known to sufficient accuracy to implement puncture scheme
- motion through second order (for spherical body) is geodesic in an effective metric that would be calculated in puncture scheme

Future work

- find closed-form expression for $h_{\mu\nu}^{R(2)}$ and $h_{\mu\nu}^{S(2)}$ (or some other fields that agree with my definitions through order $r$), analogous to Detweiler-Whiting fields at first order
- find equation of motion for spinning body with quadrupole moments