Journal of Computational Mathematics
Vol.42, No.2, 2024, 544–569.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2205-m2021-0234

A SCALAR AUXILIARY VARIABLE (SAV) FINITE ELEMENT NUMERICAL SCHEME FOR THE CAHN-HILLIARD-HELE-SHAW SYSTEM WITH DYNAMIC BOUNDARY CONDITIONS*

Changhui Yao[1) and Fengdan Zhang
School of Mathematics and Statistics, Zhengzhou University, Zhengzhou 450001, China
Email: chyao@lsec.cc.ac.cn, finezhang@gs.zzu.edu.cn
Cheng Wang
Department of Mathematics, University of Massachusetts, North Dartmouth, MA 02747, USA
Email: cwang1@umassd.edu

Abstract

In this paper, we consider the Cahn-Hilliard-Hele-Shaw (CHHS) system with the dynamic boundary conditions, in which both the bulk and surface energy parts play important roles. The scalar auxiliary variable approach is introduced for the physical system; the mass conservation and energy dissipation is proved for the CHHS system. Subsequently, a fully discrete SAV finite element scheme is proposed, with the mass conservation and energy dissipation laws established at a theoretical level. In addition, the convergence analysis and error estimate is provided for the proposed SAV numerical scheme.

Mathematics subject classification: 65N06, 65B99.
Key words: Cahn-Hilliard-Hele-Shaw system, Dynamic boundary conditions, Bulk energy and surface energy, Scalar auxiliary variable formulation, Energy stability, Convergence analysis.

1. Introduction

The Cahn-Hilliard-Hele-Shaw system (CHHS) has attracted more and more attentions in recent years, since this model describes two phase flows in a simple way. This system turns out to be the basic diffusion interface model for incompressible binary fluids confined in a Hele-Shaw cell [42, 43, 50], and it has been proposed to simplify the well-known Cahn-Hilliard-Navier-Stokes model, where the Navier-Stokes system is coupled with the convective Cahn-Hilliard equation [19, 38, 39, 59]. This model has also been used to describe spinodal decomposition of a binary fluid in a Hele-Shaw cell [33], tumor growth and cell sorting [25, 64], and two phase flows in porous media [17], etc.

The CHHS system with Neumann boundary conditions has been extensively studied in the existing literature [8, 9, 12, 31, 33, 49, 63]. On the other hand, the homogeneous Neumann boundary condition turns out to unsatisfactory in some cases, due to the fact that this simple boundary condition set-up ignores the effects of certain process on the boundary to the bulk dynamics; in other words, separate chemical reactions on the boundary are not taken into consideration. Nevertheless, in certain applications such as fluid dynamics and contact line problems, a more accurate description of the short-range interaction of the binary mixture with

* Received August 15, 2021 / Revised version received March 7, 2022 / Accepted May 9, 2022 / Published online March 21, 2023 /
1) Corresponding author

the solid wall of the vessel turns out to be necessary. At present, various dynamic boundary conditions have been derived and analyzed for the Cahn-Hilliard equation [5, 40, 41, 52], while the associated analysis for the CHHS system is very limited.

Let $\Omega \subset \mathbb{R}^d$ (where $d = 2, 3$) be a bounded domain with a boundary $\Gamma := \partial\Omega$. The unit outer normal vector on Γ will be denoted by $\mathbf{n} = \mathbf{n}(x)$. The standard CHHS system is formulated as

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) - \epsilon \Delta \mu = 0 \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\mu + \epsilon \Delta \phi - f(\phi) = 0 \quad \text{in } \Omega \times (0, T], \tag{1.2}$$

$$\mathbf{u} + \nabla p + \gamma \phi \nabla \mu = 0 \quad \text{in } \Omega \times (0, T], \tag{1.3}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T], \tag{1.4}$$

where $\gamma > 0$ is a dimensionless surface tension parameter, $\mathbf{u}$ is the advective velocity, and p is the pressure. To describe a mixture of two materials, the phase field variable ϕ stands for the difference of two local relative concentrations. In more details, $\phi(x)$ $(x \in \Omega)$ takes the distinct values, 1 and -1, in the respective pure phases of the materials, while $\{x \in \Omega : -1 < \phi(x) < 1\}$ matches with the diffuse interface between them, whose thickness is proportional to the very small positive constant ϵ. The variable μ stands for the chemical potential in the bulk, which can be derived from the Fréchet derivative [16] of the following Ginzburg-Landau free energy

$$E_{bulk}[\phi] = \int_\Omega \Big(\frac{\epsilon}{2} \big|\nabla \phi\big|^2 + F(\phi) \Big)\, dx,$$

where the functional F denotes the bulk potential and $f(\phi) = F'(\phi)$. Typically, F has a double well form, which reaches its global minima at $\phi = \pm 1$ and a local maximum at $\phi = 0$.

The homogeneous Neumann boundary conditions corresponding to the system (1.1)-(1.4) are given by

$$\partial_n \phi = 0 \quad \text{on } \Gamma \times (0, T], \tag{1.5}$$

$$\partial_n \mu = 0 \quad \text{on } \Gamma \times (0, T], \tag{1.6}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma \times (0, T]. \tag{1.7}$$

However, especially for certain materials in the bounded region, boundary condition (1.5) is not well-pleasing, since certain additional effects of the boundary to the bulk dynamics are ignored. Meanwhile, several dynamic boundary conditions have been proposed in the existing literatures [18, 46, 47, 53, 65], to replace the homogeneous Neumann condition. In order to improve this phenomenon and to better describe the whole system, physicists put forward a surface free energy

$$E_{surf}[\phi_\Gamma] = \int_\Gamma \Big(\frac{\kappa \epsilon}{2} \big|\nabla_\Gamma \phi_\Gamma\big|^2 + G(\phi_\Gamma) \Big)\, dS,$$

where ∇_Γ denotes the surface gradient operator on Γ and G is a surface potential. Furthermore, $\kappa > 0$ is related to the effects of surface diffusion. Some numerical works [2, 3, 52] have been reported as well.

The total free energy corresponding to the dynamic boundary conditions becomes

$$E = E_{bulk}[\phi] + E_{surf}[\phi_\Gamma]. \tag{1.8}$$

In this paper, we consider the CHHS system with dynamic boundary conditions

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) - \epsilon\Delta\mu = 0 \quad \text{in } \Omega\times(0,T], \tag{1.9}$$

$$\mu + \epsilon\Delta\phi - f(\phi) = 0 \quad \text{in } \Omega\times(0,T], \tag{1.10}$$

$$\mathbf{u} + \nabla p + \gamma\phi\nabla\mu = 0 \quad \text{in } \Omega\times(0,T], \tag{1.11}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T], \tag{1.12}$$

$$\frac{\partial\phi_\Gamma}{\partial t} + \nabla_\Gamma\cdot(\phi_\Gamma\mathbf{u}_\Gamma) - \epsilon\Delta_\Gamma\mu_\Gamma = 0 \quad \text{on } \Gamma\times(0,T], \tag{1.13}$$

$$\mu_\Gamma + \kappa\epsilon\Delta_\Gamma\phi_\Gamma + \epsilon\partial_n\phi - g(\phi_\Gamma) = 0 \quad \text{on } \Gamma\times(0,T], \tag{1.14}$$

$$\mathbf{u}_\Gamma + \nabla_\Gamma p_\Gamma + \gamma\phi_\Gamma\nabla_\Gamma\mu_\Gamma = 0 \quad \text{on } \Gamma\times(0,T], \tag{1.15}$$

$$\nabla_\Gamma\cdot\mathbf{u}_\Gamma = 0 \quad \text{on } \Gamma\times(0,T], \tag{1.16}$$

where Δ_Γ denote the surface Laplace-Beltrami operator on Γ and $g(\phi_\Gamma) = G'(\phi_\Gamma)$. In addition, this system is endowed with initial conditions

$$\phi(0,x) = \phi_0(x), \tag{1.17}$$

$$\phi_\Gamma(0,x) = \phi_{\Gamma,0}(x). \tag{1.18}$$

The boundary conditions are presented as follows

$$\partial_n\mu = 0, \quad \partial_n p = 0 \quad \text{on } \Gamma\times(0,T], \tag{1.19}$$

$$\partial_{n_\Gamma}\mu_\Gamma = 0, \quad \partial_{n_\Gamma} p_\Gamma = 0 \quad \text{on } \partial\Gamma\times(0,T], \tag{1.20}$$

where n_Γ is the unit outer normal vector on Γ, and $\phi_\Gamma(\cdot,t)$ is Γ-periodic. For the above system, ϕ_Γ and μ_Γ does not necessarily consistent with the trace of ϕ and μ on Γ, respectively, but can be understood as independent variables.

Similar to the compactness arguments in [26], since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, the properties of subspaces and other basic definitions, theorems and properties in functional analysis, the existence and uniqueness of the weak solution of this system can be established. The more detailed procedure for the proof is left to interested readers.

Due to the second law of thermodynamics, dissipative physical systems are everywhere. Maintaining the energy law allows the numerical solution of the physical model to fit the dynamics correctly for a long time. Therefore, it is vital and necessary to design numerical methods to preserve discrete energy dissipation laws. Many efforts have been devoted to the development of numerical methods for energy stability in this active research field, which include, but are not limited to, the convex splitting method [4,23,28–30,33,45,54,56], the average vector field method [6,55], exponential time differencing (ETD) method [10,11,13,22,36,37,44,51] and the invariant energy quadratization (IEQ) method [34,66–68,70]. In addition, the scalar auxiliary variable (SAV) method [1,14,15,27,57,58,62,69] has been successfully developed, inspired by a similar idea of the IEQ method. Meanwhile, it has overcome many shortcomings, while maintaining the basic advantages of the IEQ approach. The SAV method is not limited to a specific form of the nonlinear part or the free energy, while it only requires the adoption of scalar variables independent of the spatial variables to obtain a linear decoupled system with constant coefficients.

Moreover, it is realized that the finite element spatial discretization is more advantageous than the collocation approach when dealing with problems concerning dynamic boundary conditions. In this paper, we adopt the combination of SAV method and finite element spatial approximation for the CHHS system with dynamic boundary conditions.

In the process of the numerical design, we adopt the SAV approach for the bulk free energy and the surface free energy respectively to linearize the nonlinear term. Meanwhile, semi-implicit treatment is applied to the convective and stress terms. The discrete format of the combined finite element and SAV approach maintains the modified energy dissipation law, which is theoretically justified in the paper. In addition, an error analysis is performed for the fully discrete numerical scheme, with dynamic boundary conditions.

Throughout this paper, for $s \in Z_+$ and $1 \le q \le \infty$, let $W^{s,p}(\Omega)$ and $W^{s,p}(\Gamma)$ stand for the standard Sobolev spaces of Ω and Γ, respectively, with corresponding norms $\|\cdot\|_{W^{s,p}(\Omega)}$ and $\|\cdot\|_{W^{s,p}(\Gamma)}$. For any $1 \le p \le \infty$, the Lebesgue spaces on Ω and Γ are denoted as $L^p(\Omega)$ and $L^p(\Gamma)$, respectively, associated with the norms $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(\Gamma)}$. Moreover, $W^{0,p}$ can be identified with L^p. An alternative notation of Sobolev spaces for $p = 2$ becomes $H^s(\Omega)$ and $H^s(\Gamma)$, equipped with the norms $\|\cdot\|_{H^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$, respectively. Let $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^d$, $\mathbf{H}^s(\Gamma) = [H^s(\Gamma)]^d$, $\mathbf{L}^p(\Omega) = [L^p(\Omega)]^d$ and $\mathbf{L}^p(\Gamma) = [L^p(\Gamma)]^d$ with bold faced letters for Sobolev spaces or Lebesgue spaces of the vector-valued functions with d components. For a fixed time $T > 0$, the space $L^p(0, T; X)$ represents the L^p space on the interval $(0, T)$ with values in the Banana space X. If X is a Hilbert space, $L^2(\Omega)-$inner product on X is denoted by $(\cdot,\cdot)$, and $L^2(\Gamma)-$inner product on X is denoted by $\langle\cdot,\cdot\rangle$. In addition, we set $L^2_0(\Omega) = \{v \in L^2(\Omega)|\ (v, 1) = 0\}$, and $L^2_0(\Gamma) = \{v_\Gamma \in L^2(\Gamma)|\langle v_\Gamma, 1\rangle = 0\}$.

The structure of this paper is organized as follows. In Section 2, an equivalent physical system based on the SAV formulation is introduced, and the corresponding weak form and energy decreasing law are derived as well. In Section 3, the fully discrete numerical scheme with the SAV formulation is constructed and a modified energy stability is proved. Subsequently, a convergence analysis and error estimate in provided in Section 4, with the help of the regularity assumption for the exact solution, Ritz projection and interpolation estimates, as well as the stability analysis for the numerical error functions. The convergent order is obtained as $\mathcal{O}(h^q + \Delta t)$. Finally, some concluding remarks are given in Section 5.

2. Equivalent Physical System in the SAV Formulation and Its Energy Dissipation Law

The SAV approach is an efficient way to solve a gradient flow, while the energy stability is maintained [7,27,35]. The key point is an introduction of the scalar auxiliary variable. More precisely, it is necessary to separately introduce the auxiliary variables of the bulk and surface parts in this system. First, we set E_1 and $E_{\Gamma,1}$ in the following form:

$$E_1[\phi] = \int_\Omega F(\phi)\ dx, \quad E_{\Gamma,1}[\phi_\Gamma] = \int_\Gamma G(\phi_\Gamma)\ dS, \tag{2.1}$$

under assumption that $E_1[\phi] > -c_1$ and $E_{\Gamma,1}[\phi_\Gamma] > -c_2$, and let $C_1 > c_1, C_2 > c_2$ so that $E_1[\phi] + C_1 > 0$ and $E_{\Gamma,1}[\phi_\Gamma] + C_2 > 0$. For simplicity of presentation, we replace E_1 by $E_1 + C_1$ without changing the gradient flow. In this setting, $E_1[\phi]$ always has a positive lower bound $C_1 - c_1$ for any ϕ, which we still denote as C_1. Similarly, we substitute $E_{\Gamma,1}$ by $E_{\Gamma,1} + C_2$, and apparently $E_{\Gamma,1}[\phi_\Gamma]$ is always bounded by a positive lower bound $C_2 - c_2$ for any ϕ_Γ, which we still denote as C_2. Subsequently, the auxiliary variables of this system take the form of

$$r(t) = \sqrt{E_1[\phi]}, \quad r_\Gamma(t) = \sqrt{E_{\Gamma,1}[\phi_\Gamma]}. \tag{2.2}$$

In turn, by applying (2.2), the equations can be equivalently rewritten as

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) - \epsilon \Delta \mu = 0 \qquad \text{in } \Omega \times (0,T], \tag{2.3}$$

$$\mu + \epsilon \Delta \phi - \frac{r}{\sqrt{E_1[\phi]}} f(\phi) = 0 \qquad \text{in } \Omega \times (0,T], \tag{2.4}$$

$$\mathbf{u} + \nabla p + \gamma \phi \nabla \mu = 0 \qquad \text{in } \Omega \times (0,T], \tag{2.5}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \times (0,T], \tag{2.6}$$

$$\frac{\partial \phi_\Gamma}{\partial t} + \nabla_\Gamma \cdot (\phi_\Gamma \mathbf{u}_\Gamma) - \epsilon \Delta_\Gamma \mu_\Gamma = 0 \qquad \text{on } \Gamma \times (0,T], \tag{2.7}$$

$$\mu_\Gamma + \kappa \epsilon \Delta_\Gamma \phi_\Gamma + \epsilon \partial_n \phi - \frac{r_\Gamma}{\sqrt{E_{\Gamma,1}[\phi_\Gamma]}} g(\phi_\Gamma) = 0 \qquad \text{on } \Gamma \times (0,T], \tag{2.8}$$

$$\mathbf{u}_\Gamma + \nabla_\Gamma p_\Gamma + \gamma \phi_\Gamma \nabla_\Gamma \mu_\Gamma = 0 \qquad \text{on } \Gamma \times (0,T], \tag{2.9}$$

$$\nabla_\Gamma \cdot \mathbf{u}_\Gamma = 0 \qquad \text{on } \Gamma \times (0,T], \tag{2.10}$$

$$\frac{dr}{\partial t} = \frac{1}{2\sqrt{E_1[\phi]}} \int_\Omega f(\phi) \frac{\partial \phi}{\partial t} dx, \qquad t \in (0,T], \tag{2.11}$$

$$\frac{dr_\Gamma}{\partial t} = \frac{1}{2\sqrt{E_{\Gamma,1}[\phi_\Gamma]}} \int_\Gamma g(\phi_\Gamma) \frac{\partial \phi_\Gamma}{\partial t} dS, \qquad t \in (0,T]. \tag{2.12}$$

It is noticed that the transformed SAV system is exactly identical to the original system (1.9)-(1.16), since (2.2) can be obtained by integrating (2.11) and (2.12) with respect to time, which does not involve spatial derivative, so that the initial conditions (1.17), (1.18) and the boundary conditions (1.19), (1.20) are still valid. In addition, the initial conditions of r and r_Γ turn out to be

$$r(0) = \sqrt{E_1[\phi_0(x)]}, \quad r_\Gamma(0) = \sqrt{E_{\Gamma,1}[\phi_{\Gamma,0}(x)]}.$$

Inserting (2.6) into (2.5), (2.10) into (2.9), exploiting the boundary conditions (1.19), (1.20), and using integration by parts, a weak formulation of the system (2.3)-(2.12) can be expressed as follows: For any $t \in (0,T]$, find $(\phi, \mu, p, \phi_\Gamma, \mu_\Gamma, p_\Gamma, r, r_\Gamma)$,

$$\phi \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^4\big(0,T;L^\infty(\Omega)\big), \tag{2.13}$$

$$\partial_t \phi \in L^2\big(0,T;H^{-1}(\Omega)\big), \tag{2.14}$$

$$\mu \in L^2\big(0,T;H^1(\Omega)\big), \tag{2.15}$$

$$p \in L^2\big(0,T;H^1(\Omega) \cap L_0^2(\Omega)\big), \tag{2.16}$$

$$\phi_\Gamma \in L^\infty\big(0,T;H^1(\Gamma)\big) \cap L^4\big(0,T;L^\infty(\Gamma)\big), \tag{2.17}$$

$$\partial_t \phi_\Gamma \in L^2\big(0,T;H^{-1}(\Gamma)\big), \tag{2.18}$$

$$\mu_\Gamma \in L^2\big(0,T;H^1(\Gamma)\big), \tag{2.19}$$

$$p_\Gamma \in L^2\big(0,T;H^1(\Gamma) \cap L_0^2(\Gamma)\big), \tag{2.20}$$

such that

$$\left(\frac{\partial \phi}{\partial t}, \chi\right) - (\phi \mathbf{u}, \nabla \chi) + \epsilon(\nabla \mu, \nabla \chi) = 0, \qquad \forall \chi \in H^1(\Omega), \tag{2.21}$$

$$(\mu, \zeta) = \epsilon(\nabla \phi, \nabla \zeta) - \epsilon \langle \partial_n \phi, \zeta \rangle + \frac{r}{\sqrt{E_1[\phi]}} \big(f(\phi), \zeta\big), \quad \forall \zeta \in H^1(\Omega), \tag{2.22}$$

$$(\nabla p, \nabla q) + \gamma(\phi \nabla \mu, \nabla q) = 0, \qquad \forall q \in H^1(\Omega), \tag{2.23}$$

$$\left\langle \frac{\partial \phi_\Gamma}{\partial t}, \psi \right\rangle - \langle \phi_\Gamma \mathbf{u}_\Gamma, \nabla_\Gamma \psi \rangle + \epsilon \langle \nabla_\Gamma \mu_\Gamma, \nabla_\Gamma \psi \rangle = 0, \qquad \forall \psi \in H^1(\Gamma), \tag{2.24}$$

$$\langle \mu_\Gamma, \nu \rangle = \kappa\epsilon \langle \nabla_\Gamma \phi_\Gamma, \nabla_\Gamma \nu \rangle + \epsilon \langle \partial_n \phi, \nu \rangle + \frac{r_\Gamma}{\sqrt{E_{\Gamma,1}[\phi_\Gamma]}} \langle g(\phi_\Gamma), \nu \rangle, \quad \forall \nu \in H^1(\Gamma), \tag{2.25}$$

$$\langle \nabla_\Gamma p_\Gamma, \nabla_\Gamma q_\Gamma \rangle + \gamma \langle \phi_\Gamma \nabla_\Gamma \mu_\Gamma, \nabla_\Gamma q_\Gamma \rangle = 0, \qquad \forall q_\Gamma \in H^1(\Gamma), \tag{2.26}$$

$$\frac{dr}{\partial t} = \frac{1}{2\sqrt{E_1[\phi]}} \left(f(\phi), \frac{\partial \phi}{\partial t} \right), \tag{2.27}$$

$$\frac{dr_\Gamma}{\partial t} = \frac{1}{2\sqrt{E_{\Gamma,1}[\phi_\Gamma]}} \left\langle g(\phi_\Gamma), \frac{\partial \phi_\Gamma}{\partial t} \right\rangle. \tag{2.28}$$

After solving the above system, $\mathbf{u}$ and $\mathbf{u}_\Gamma$ can be defined by the following form:

$$(\mathbf{u}, \lambda) = -(\nabla p, \lambda) - \gamma(\phi \nabla \mu, \lambda) = 0, \qquad \forall \lambda \in \mathbf{L}^2(\Omega), \tag{2.29}$$

$$\langle \mathbf{u}_\Gamma, \lambda_\Gamma \rangle = -\langle \nabla_\Gamma p_\Gamma, \lambda_\Gamma \rangle - \gamma \langle \phi_\Gamma \nabla_\Gamma \mu_\Gamma, \lambda_\Gamma \rangle = 0, \quad \forall \lambda_\Gamma \in \mathbf{L}^2(\Gamma). \tag{2.30}$$

The above weak formulation (2.21)-(2.30) still preserves two significant features, mass conservation and energy dissipation.

Theorem 2.1. *Let $(\phi, \mu, p, \phi_\Gamma, \mu_\Gamma, p_\Gamma, r, r_\Gamma)$ be the smooth solution of the weak formulation* (2.21)-(2.28)*. Then the solution satisfies the mass conservation identity*

$$\int_\Omega \phi(t,x)\ dx = \int_\Omega \phi_0(x) dx, \quad \int_\Gamma \phi_\Gamma(t,x)\ dS = \int_\Gamma \phi_{\Gamma,0}(x) dS, \tag{2.31}$$

and the energy dissipation law

$$\frac{d}{dt} E = -\int_\Omega \left(\epsilon |\nabla \mu|^2 + \frac{1}{\gamma} |\mathbf{u}|^2 \right) dx + \int_\Gamma \left(\epsilon |\nabla_\Gamma \mu_\Gamma|^2 + \frac{1}{\gamma} |\mathbf{u}_\Gamma|^2 \right) dS \le 0. \tag{2.32}$$

Proof. By choosing $\chi = 1$ in (2.21) and $\psi = 1$ in (2.24), respectively, we can obtain

$$\frac{d}{dt} \int_\Omega \phi(t,x)\ dx = 0, \quad \frac{d}{dt} \int_\Gamma \phi_\Gamma(t,x)\ dS = 0. \tag{2.33}$$

In turn, a combination of (1.17) and (1.18) gives (2.31). Taking $\chi = \mu$ in (2.21), $\zeta = -\phi_t$ in (2.22), $\lambda = \mathbf{u}/\gamma$ in (2.29), $\psi = \mu_\Gamma$ in (2.24), $\nu = -(\phi_\Gamma)_t$ in (2.25), $\lambda_\Gamma = \mathbf{u}_\Gamma/\gamma$ in (2.30), multiplying (2.27), (2.28) by $2r$ and $2r_\Gamma$, respectively, and summarizing (2.21)-(2.30) except for (2.23) and (2.26), we arrive at

$$\begin{aligned} &\frac{d}{dt} \left(\int_\Omega \frac{\epsilon}{2} |\nabla \phi|^2 dx + \int_\Gamma \frac{\kappa\epsilon}{2} |\nabla_\Gamma \phi_\Gamma|^2 dS + r^2 + r_\Gamma^2 \right) \\ &\quad + \int_\Omega \left(\epsilon |\nabla \mu|^2 + \frac{1}{\gamma} |\mathbf{u}|^2 \right) dx + \int_\Gamma \left(\epsilon |\nabla_\Gamma \mu_\Gamma|^2 + \frac{1}{\gamma} |\mathbf{u}_\Gamma|^2 \right) dS = 0. \end{aligned} \tag{2.34}$$

This is exactly (2.32), by recalling the notations in (2.1) and (2.2). □

3. The Fully Discrete SAV Numerical Scheme and the Energy Stability Analysis

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0,T]$, i.e., $t_i = i\Delta t, \Delta t = T/N, i = 0,1,\ldots,N$, where N is a positive integer. Let $T_h = \{K\}$ be a conforming, shape-regular, globally quasi-uniform family of triangulations or tetrahedrons of Ω. For any positive integer $q \geq 1$, we introduce the finite element space

$$M_h = \big\{v_h \in C^0(\Omega) \;\big|\; v_h|_K \in P_q, \forall K \in T_h\big\} \subset H^1(\Omega),$$
$$S_h = \big\{v_h \in C^0(\Gamma) \;\big|\; v_h|_K \in P_q, \forall K \in T_h\big\} \subset H^1(\Gamma),$$

where P_q is the space of polynomials of degree not exceeding q. Furthermore, we define the subspace $\mathring{M}_h := M_h \cap L_0^2(\Omega), \mathring{S}_h := S_h \cap L_0^2(\Gamma)$.

The first order accurate (in time), finite element SAV scheme for the CHHS system with dynamic boundary conditions is proposed as follows. For any $0 \leq n \leq N$, find

$$\left(\phi_h^{n+1}, \mu_h^{n+1}, p_h^{n+1}, \phi_{\Gamma,h}^{n+1}, \mu_{\Gamma,h}^{n+1}, p_{\Gamma,h}^{n+1}, r_h^{n+1}, r_{\Gamma,h}^{n+1}\right) \in [M_h]^3 \times [S_h]^3 \times [R]^2$$

such that

$$\left(\frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}, \chi_h\right) + \big(\phi_h^n(\nabla p_h^n + \gamma\phi_h^n\nabla\mu_h^{n+1}), \nabla\chi_h\big) + \epsilon(\nabla\mu_h^{n+1}, \nabla\chi_h) = 0, \quad \forall \chi_h \in M_h, \tag{3.1}$$

$$\big(\mu_h^{n+1}, \zeta_h\big) = \epsilon\left(\nabla\phi_h^{n+1}, \nabla\zeta_h\right) - \epsilon\left\langle\partial_n\phi_h^{n+1}, \zeta_h\right\rangle + \frac{r_h^{n+1}}{\sqrt{E_1[\phi_h^n]}}\big(f(\phi_h^n), \zeta_h\big), \quad \forall \zeta_h \in M_h, \tag{3.2}$$

$$\big(\nabla p_h^{n+1}, \nabla q_h\big) + \gamma\left(\phi_h^n\nabla\mu_h^{n+1}, \nabla q_h\right) = 0, \quad \forall q_h \in M_h, \tag{3.3}$$

$$\begin{aligned}\left\langle\frac{\phi_{\Gamma,h}^{n+1}-\phi_{\Gamma,h}^n}{\Delta t}, \psi_h\right\rangle &+ \big\langle\phi_{\Gamma,h}^n(\nabla_\Gamma p_{\Gamma,h}^n + \gamma\phi_{\Gamma,h}^n\nabla_\Gamma\mu_{\Gamma,h}^{n+1}), \nabla_\Gamma\psi_h\big\rangle \\ &- \epsilon\big\langle\nabla_\Gamma\mu_{\Gamma,h}^{n+1}, \nabla_\Gamma\psi_h\big\rangle = 0, \quad \forall \psi_h \in S_h,\end{aligned} \tag{3.4}$$

$$\begin{aligned}\big\langle\mu_{\Gamma,h}^{n+1}, \nu_h\big\rangle &= \kappa\epsilon\big\langle\nabla_\Gamma\phi_{\Gamma,h}^{n+1}, \nabla_\Gamma\nu_h\big\rangle + \epsilon\left\langle\partial_n\phi_h^{n+1}, \nu_h\right\rangle \\ &+ \frac{r_{\Gamma,h}^{n+1}}{\sqrt{E_{\Gamma,1}[\phi_{\Gamma,h}^n]}}\big\langle g(\phi_{\Gamma,h}^n), \nu_h\big\rangle, \quad \forall \nu_h \in S_h,\end{aligned} \tag{3.5}$$

$$\big\langle\nabla_\Gamma p_{\Gamma,h}^{n+1}, \nabla_\Gamma q_{\Gamma,h}\big\rangle + \epsilon\big\langle\phi_{\Gamma,h}^n\nabla_\Gamma\mu_{\Gamma,h}^{n+1}, \nabla_\Gamma q_{\Gamma,h}\big\rangle = 0, \quad \forall q_{\Gamma,h} \in S_h, \tag{3.6}$$

$$\frac{r_h^{n+1}-r_h^n}{\Delta t} = \frac{1}{2\sqrt{E_1[\phi_h^n]}}\left(f(\phi_h^n), \frac{\phi_h^{n+1}-\phi_h^n}{\Delta t}\right), \tag{3.7}$$

$$\frac{r_{\Gamma,h}^{n+1}-r_{\Gamma,h}^n}{\Delta t} = \frac{1}{2\sqrt{E_{\Gamma,1}[\phi_{\Gamma,h}^n]}}\left\langle g(\phi_{\Gamma,h}^n), \frac{\phi_{\Gamma,h}^{n+1}-\phi_{\Gamma,h}^n}{\Delta t}\right\rangle, \tag{3.8}$$

and the initial data are set as $\phi_h^0 = R_h\phi_0$ and $\phi_{\Gamma,h}^0 = R_h\phi_{\Gamma,0}$. The following identity is standard for the Ritz projection operator $R_h : \psi \in X^i \to Y_h^i$ $(i = 1,2)$ (with $X^1 = H^1(\Omega), Y_h^1 = M_h$ while $X^2 = H^1(\Gamma), Y_h^2 = S_h$):

$$\big(\nabla(R_h\psi - v), \nabla v\big) = 0, \quad \forall v \in Y_h^i, \quad (R_h\psi - \psi, 1) = 0. \tag{3.9}$$

In fact, for $\psi \in X^1$, the following estimates are valid for the Ritz projection [19,32]:

$$\big\|R_h\psi\big\|_{W^{1,p}(\Omega)} \leq C\big\|\psi\big\|_{W^{1,p}(\Omega)}, \quad \forall p \in [2,\infty), \tag{3.10}$$

$$\big\|\psi - R_h\psi\big\|_{L^p(\Omega)} + h\big\|\psi - R_h\psi\big\|_{W^{1,p}(\Omega)} \leq Ch^{q+1}\big\|\psi\big\|_{W^{q+1,p}(\Omega)}, \quad \forall p \in [2,\infty). \tag{3.11}$$

Similarly, for $\psi_\Gamma \in X^2$, the following estimates could be derived:

$$\big\|R_h\psi_\Gamma\big\|_{W^{1,p}(\Gamma)} \le C\big\|\psi_\Gamma\big\|_{W^{1,p}(\Gamma)}, \qquad \forall p \in [2,\infty), \tag{3.12}$$

$$\big\|\psi_\Gamma - R_h\psi_\Gamma\big\|_{L^p(\Gamma)} + h\big\|\psi_\Gamma - R_h\psi_\Gamma\big\|_{W^{1,p}(\Gamma)} \le Ch^{q+1}\big\|\psi_\Gamma\big\|_{W^{q+1,p}(\Gamma)}, \quad \forall p \in [2,\infty). \tag{3.13}$$

To facilitate the nonlinear analysis, we introduce the following negative norms:

$$\|v\|_{H^{-s}(\Omega)} = \sup\left\{\frac{(v,\zeta)}{\|\zeta\|_{H^s(\Omega)}}; \zeta \in H^s(\Omega)\right\} \quad \text{for} \quad s \ge 0 \text{ integer},$$

and the corresponding norms over the boundary are defined as

$$\|v_\Gamma\|_{H^{-s}(\Gamma)} = \sup\left\{\frac{(v_\Gamma,\zeta_\Gamma)}{\|\zeta_\Gamma\|_{H^s(\Gamma)}}; \zeta_\Gamma \in H^s(\Gamma)\right\} \quad \text{for} \quad s \ge 0 \text{ integer}.$$

Lemma 3.1. *Based on the above definitions, we have*

$$\|v\|_{H^{-1}(\Omega)} \le \|v\|_{L^2(\Omega)}, \quad \|v_\Gamma\|_{H^{-1}(\Gamma)} \le \|v_\Gamma\|_{L^2(\Gamma)}. \tag{3.14}$$

Lemma 3.2 ([61]). *Suppose $R_h\psi$ is the Ritz projection of ψ, then it holds that*

$$\|\psi - R_h\psi\|_{H^{-s}(\Omega)} \le Ch^{s+r}\|\psi\|_{H^r(\Omega)} \quad \textit{for} \quad 0 \le s \le q-1, \quad 1 \le r \le q+1. \tag{3.15}$$

For the sake of further analysis, we need the following preliminary estimates, which have been derived in the existing works [20, 21, 49], etc.

The projection operator $\mathcal{P} : \mathbf{L}^2(\Omega) \to \mathbf{W}$ is defined by

$$\mathcal{P}(\mathbf{w}) = \nabla p + \mathbf{w}, \tag{3.16}$$

and $p \in \mathring{H}^1(\Omega) := \{v \in H^1(\Omega) |\ (v,1) = 0\}$ is the unique solution of

$$(\nabla p + \mathbf{w}, \nabla q) = 0, \quad \forall q \in H^1(\Omega),$$

where $\mathbf{W} := \{\mathbf{u} \in \mathbf{L}^2(\Omega) |\ (\mathbf{u}, \nabla q) = 0, \forall q \in H^1(\Omega)\}$.

Lemma 3.3 ([49]). *The projection $\mathcal{P}$ is linear, and for any given $\mathbf{w} \in L^2(\Omega)$, we have*

$$\big(\mathcal{P}(\mathbf{w}) - \mathbf{w}, \mathbf{v}\big) = 0, \quad \forall \mathbf{v} \in \mathbf{W}. \tag{3.17}$$

Moreover, since $\mathcal{P}(\mathbf{w}) \in \mathbf{W}$, by applying Cauchy-Schwarz inequality, we obtain

$$\big\|\mathcal{P}(\mathbf{w})\big\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \tag{3.18}$$

In a similar manner, we define the projection operator $\widetilde{\mathcal{P}} : \mathbf{L}^2(\Gamma) \to \widetilde{\mathbf{W}}$ via

$$\widetilde{\mathcal{P}}(\mathbf{w}_\Gamma) = \nabla_\Gamma p_\Gamma + \mathbf{w}_\Gamma, \tag{3.19}$$

and $p_\Gamma \in \mathring{H}^1(\Gamma) := \{v_\Gamma \in H^1(\Gamma) |\ \langle v, 1\rangle = 0\}$ is the unique solution of

$$\langle \nabla_\Gamma p_\Gamma + \mathbf{w}_\Gamma, \nabla_\Gamma q_\Gamma \rangle = 0, \quad \forall q_\Gamma \in H^1(\Gamma),$$

where $\widetilde{\mathbf{W}} := \{\mathbf{u}_\Gamma \in \mathbf{L}^2(\Gamma) |\ \langle \mathbf{u}_\Gamma, \nabla_\Gamma q_\Gamma \rangle = 0, \forall q \in H^1(\Gamma)\}$.

Similar to Lemma 3.3, the following estimates are available.

Lemma 3.4. *The projection $\widetilde{\mathcal{P}}$ is linear, and for any given $\mathbf{w}_\Gamma \in L^2(\Gamma)$, we have*

$$\big\langle \widetilde{\mathcal{P}}(\mathbf{w}_\Gamma) - \mathbf{w}_\Gamma, \mathbf{v}_\Gamma \big\rangle = 0, \quad \forall \mathbf{v}_\Gamma \in \widetilde{\mathbf{W}}. \tag{3.20}$$

It is clear that $\widetilde{\mathcal{P}}(\mathbf{w}_\Gamma) \in \widetilde{\mathbf{W}}$, and by using Cauchy-Schwarz inequality, we obtain

$$\big\|\widetilde{\mathcal{P}}(\mathbf{w}_\Gamma)\big\|_{\mathbf{L}^2(\Gamma)} \le \|\mathbf{w}_\Gamma\|_{\mathbf{L}^2(\Gamma)}. \tag{3.21}$$

A discrete version of $\mathbf{W}$ is the space $\mathbf{W}_h := \{\mathbf{u}_h \in \mathbf{L}^2(\Omega)|\ (\mathbf{u}_h, \nabla q_h) = 0, \forall q_h \in M_h\}$. The corresponding discrete projection operator $\mathcal{P}_h : \mathbf{w} \in \mathbf{L}^2(\Omega) \to \mathbf{W}_h$ is defined as follows:

$$\mathcal{P}_h(\mathbf{w}) = \nabla p_h + \mathbf{w}, \tag{3.22}$$

where $p_h \in \mathring{M}_h$ is the unique solution of

$$(\nabla p_h + \mathbf{w}, \nabla q_h) = 0, \quad \forall q_h \in M_h.$$

The projection $\mathcal{P}_h$ satisfies the following properties.

Lemma 3.5 ([49]). *The projection $\mathcal{P}_h$ is linear, and for any given $\mathbf{w} \in \mathbf{L}^2(\Omega)$, we have*

$$\big(\mathcal{P}_h(\mathbf{w}) - \mathbf{w}, \mathbf{v}_h\big) = 0, \quad \forall \mathbf{v}_h \in \mathbf{W}_h. \tag{3.23}$$

It is easy to find that $\mathcal{P}_h(\mathbf{w}) \in \mathbf{W}_h$, and consequently, we get

$$\big\|\mathcal{P}_h(\mathbf{w})\big\|_{\mathbf{L}^2(\Omega)} \le \|\mathbf{w}\|_{\mathbf{L}^2(\Omega)}. \tag{3.24}$$

Lemma 3.6 ([49]). *Suppose that $\mathbf{w} \in \mathbf{H}^q(\Omega)$ satisfies the compatible boundary condition $\mathbf{w}\cdot\mathbf{n} = 0$ on Γ and $p \in H^{q+1}(\Omega)$, then we have*

$$\big\|\mathcal{P}_h(\mathbf{w}) - \mathcal{P}(\mathbf{w})\big\|_{\mathbf{L}^2(\Omega)} = \big\|\nabla(p - p_h)\big\|_{\mathbf{L}^2(\Omega)} \le Ch^q|p|_{H^{q+1}(\Omega)}. \tag{3.25}$$

Likewise, we can define the discrete forms of $\widetilde{\mathbf{W}}_h$ and $\widetilde{\mathcal{P}}_h$.

Define $\widetilde{\mathbf{W}}_h := \{\mathbf{u}_{\Gamma,h} \in \mathbf{L}^2(\Gamma)|\ \langle \mathbf{u}_{\Gamma,h}, \nabla_\Gamma q_{\Gamma,h}\rangle = 0, \forall q_{\Gamma,h} \in S_h\}$. The projection operator $\widetilde{\mathcal{P}}_h : \mathbf{w}_\Gamma \in \mathbf{L}^2(\Gamma) \to \widetilde{\mathbf{W}}_h$ is given by

$$\widetilde{\mathcal{P}}_h(\mathbf{w}_\Gamma) = \nabla_\Gamma p_{\Gamma,h} + \mathbf{w}_\Gamma, \tag{3.26}$$

in which $p_{\Gamma,h} \in \mathring{S}_h$ is the unique solution of

$$(\nabla_\Gamma p_{\Gamma,h} + \mathbf{w}_\Gamma, \nabla_\Gamma q_{\Gamma,h}) = 0, \quad \forall q_{\Gamma,h} \in \mathring{S}_h.$$

Certainly, $\widetilde{\mathcal{P}}_h$ has similar properties as $\mathcal{P}_h$.

Lemma 3.7. *The projection $\widetilde{\mathcal{P}}_h$ is linear, and for any $\mathbf{w}_\Gamma \in \mathbf{L}^2(\Gamma)$, it follows that*

$$\big(\widetilde{\mathcal{P}}_h(\mathbf{w}_\Gamma) - \mathbf{w}_\Gamma, \mathbf{v}_{\Gamma,h}\big) = 0, \quad \forall \mathbf{v}_{\Gamma,h} \in \widetilde{\mathbf{W}}_h. \tag{3.27}$$

Due to $\widetilde{\mathcal{P}}_h(\mathbf{w}_\Gamma) \in \widetilde{\mathbf{W}}_h$, clearly the following result holds:

$$\big\|\widetilde{\mathcal{P}}_h(\mathbf{w}_\Gamma)\big\|_{\mathbf{L}^2(\Gamma)} \le \|\mathbf{w}_\Gamma\|_{\mathbf{L}^2(\Gamma)}. \tag{3.28}$$

Lemma 3.8. *Assume that* $\mathbf{w}_\Gamma \in \mathbf{H}^q(\Gamma)$ *satisfies the compatible boundary condition* $\mathbf{w}_\Gamma \cdot \mathbf{n}_\Gamma = 0$ *on* $\partial\Gamma$ *and* $p_\Gamma \in H^{q+1}(\Gamma)$, *then we have*

$$\big\|\widetilde{\mathcal{P}}_h(\mathbf{w}_\Gamma) - \widetilde{\mathcal{P}}(\mathbf{w}_\Gamma)\big\|_{\mathbf{L}^2(\Gamma)} = \big\|\nabla_\Gamma(p_\Gamma - p_{\Gamma,h})\big\|_{\mathbf{L}^2(\Gamma)} \le Ch^q|p_\Gamma|_{H^{q+1}(\Gamma)}. \tag{3.29}$$

Next, we provide the energy stability analysis of the fully discrete scheme. To facilitate the analysis, the following notations are introduced:

$$\hat{\mathbf{u}}_h^{n+1} := -\nabla p_h^n - \gamma\phi_h^n\nabla\mu_h^{n+1}, \qquad \hat{\mathbf{u}}_{\Gamma,h}^{n+1} := -\nabla_\Gamma p_{\Gamma,h}^n - \gamma\phi_{\Gamma,h}^n\nabla_\Gamma\mu_{\Gamma,h}^{n+1}, \tag{3.30}$$

$$\mathbf{u}_h^{n+1} := -\nabla p_h^{n+1} - \gamma\phi_h^n\nabla\mu_h^{n+1}, \quad \mathbf{u}_{\Gamma,h}^{n+1} := -\nabla_\Gamma p_{\Gamma,h}^{n+1} - \gamma\phi_{\Gamma,h}^n\nabla_\Gamma\mu_{\gamma,h}^{n+1}. \tag{3.31}$$

By (3.3), (3.6), we see that

$$\nabla\cdot\mathbf{u}_h^{n+1} = 0, \quad \nabla_\Gamma\cdot\mathbf{u}_{\Gamma,h}^{n+1} = 0. \tag{3.32}$$

Theorem 3.1. *Let* $(\phi_h^{n+1}, \mu_h^{n+1}, p_h^{n+1}, \phi_{\Gamma,h}^{n+1}, \mu_{\Gamma,h}^{n+1}, p_{\Gamma,h}^{n+1}, r_h^{n+1}, r_{\Gamma,h}^{n+1})$ *be the solution of the proposed numerical scheme* (3.1)-(3.8). *Then for any* $\Delta t > 0$ *and* $h > 0$, *the numerical solution satisfies the discrete energy dissipation law*

$$\begin{aligned} E_{SAV}^{n+1} - E_{SAV}^n \le& -\Delta t\epsilon\big\|\nabla\mu_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2 - \frac{\epsilon}{2}\big\|\nabla\phi_h^{n+1} - \nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2 - \big(r_h^{n+1} - r_h^n\big)^2 \\ &- \Delta t\epsilon\big\|\nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2 - \frac{\kappa\epsilon}{2}\big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1} - \nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2 - \big(r_{\Gamma,h}^{n+1} - r_{\Gamma,h}^n\big)^2 \\ &- \frac{\Delta t}{4\gamma}\big\|\nabla p_h^{n+1} - \nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)} - \frac{\Delta t}{4\gamma}\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1} - \nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)} \le 0, \end{aligned} \tag{3.33}$$

in which the modified discrete energy functional E_{SAV}^n *is defined as*

$$\begin{aligned} E_{SAV}^n =& \frac{\epsilon}{2}\big\|\nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2 + (r_h^n)^2 + \big\|\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)} \\ &+ \frac{\kappa\epsilon}{2}\big\|\nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2 + (r_{\Gamma,h}^n)^2 + \big\|\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}. \end{aligned} \tag{3.34}$$

Proof. From (3.30) and (3.31), the following identities are observed:

$$\mathbf{u}_h^{n+1} - \hat{\mathbf{u}}_h^{n+1} = -\nabla p_h^{n+1} + \nabla p_h^n, \quad \mathbf{u}_{\Gamma,h}^{n+1} - \hat{\mathbf{u}}_{\Gamma,h}^{n+1} = -\nabla_\Gamma p_{\Gamma,h}^{n+1} + \nabla_\Gamma p_{\Gamma,h}^n. \tag{3.35}$$

Taking $\chi_h = \Delta t\mu_h^{n+1}$ in (3.1), $\zeta_h = \phi_h^n - \phi_h^{n+1}$ in (3.2), $\psi_h = \Delta t\mu_{\Gamma,h}^{n+1}$ in (3.4), $\nu_h = \phi_{\Gamma,h}^n - \phi_{\Gamma,h}^{n+1}$ in (3.5), multiplying (3.7), (3.8) by $2r_h^{n+1}$ and $2r_{\Gamma,h}^{n+1}$, respectively, and adding them together, we get

$$\begin{aligned} &\Delta t\epsilon\big\|\nabla\mu_h^{n+1}\big\|_{L^2(\Omega)}^2 + \frac{\epsilon}{2}\Big(\big\|\nabla\phi_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2 - \big\|\nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2 + \big\|\nabla\phi_h^{n+1} - \nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2\Big) \\ &+ \Big((r_h^{n+1})^2 - (r_h^n)^2 + (r_h^{n+1} - r_h^n)^2\Big) + \Delta t\epsilon\big\|\nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2 \\ &+ \frac{\kappa\epsilon}{2}\Big(\big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2 - \big\|\nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2 + \big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1} - \nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2\Big) \\ &+ \Big((r_{\Gamma,h}^{n+1})^2 - (r_{\Gamma,h}^n)^2 + (r_{\Gamma,h}^{n+1} - r_{\Gamma,h}^n)^2\Big) - \Delta t\left(\phi_h^n\hat{\mathbf{u}}_h^{n+1}, \nabla\mu_h^{n+1}\right) \\ &- \Delta t\big\langle\phi_{\Gamma,h}^n\hat{\mathbf{u}}_{\Gamma,h}^{n+1}, \nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\rangle = 0. \end{aligned} \tag{3.36}$$

Taking the inner products with the two equations in (3.30) by $(\Delta t/\gamma)\hat{\mathbf{u}}_h^{n+1}$ and $(\Delta t/\gamma)\hat{\mathbf{u}}_{\Gamma,h}^{n+1}$, respectively, we have

$$\frac{\Delta t}{\gamma}\big\|\hat{\mathbf{u}}_h^{n+1}\big\|_{L^2(\Omega)} = -\frac{\Delta t}{\gamma}\big(\nabla p_h^n, \hat{\mathbf{u}}_h^{n+1}\big) - \gamma\big(\phi_h^n\nabla\mu_h^{n+1}, \hat{\mathbf{u}}_h^{n+1}\big), \tag{3.37}$$

$$\frac{\Delta t}{\gamma}\big\|\hat{\mathbf{u}}_{\Gamma,h}^{n+1}\big\|_{L^2(\Gamma)} = -\frac{\Delta t}{\gamma}\big\langle\nabla_\Gamma p_{\Gamma,h}^n, \hat{\mathbf{u}}_{\Gamma,h}^{n+1}\big\rangle - \gamma\big\langle\phi_{\Gamma,h}^n\nabla_\Gamma\mu_{\Gamma,h}^{n+1}, \hat{\mathbf{u}}_{\Gamma,h}^{n+1}\big\rangle. \tag{3.38}$$

A substitution of (3.35) in (3.3) and (3.6) gives

$$-\big(\hat{\mathbf{u}}_h^{n+1},\nabla q_h\big)+\big(\nabla p_h^{n+1}-\nabla p_h^n,\nabla q_h\big)=0, \tag{3.39}$$

$$-\big\langle\hat{\mathbf{u}}_{\Gamma,h}^{n+1},\nabla_\Gamma q_{\Gamma,h}\big\rangle+\big\langle\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n,\nabla_\Gamma q_{\Gamma,h}\big\rangle=0. \tag{3.40}$$

In turn, by choosing $q_h=(\Delta t/\gamma)p_h^{n+1}$ in (3.39) and $q_{\Gamma,h}=(\Delta t/\gamma)p_{\Gamma,h}^{n+1}$ in (3.40), we obtain

$$\begin{aligned}&-\frac{\Delta t}{\gamma}\big(\hat{\mathbf{u}}_h^{n+1},\nabla p_h^{n+1}\big)\\&+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla p_h^{n+1}-\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}^2+\big\|\nabla p_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2-\big\|\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}^2\Big)=0,\end{aligned} \tag{3.41}$$

$$\begin{aligned}&-\frac{\Delta t}{\gamma}\big\langle\hat{\mathbf{u}}_{\Gamma,h}^{n+1},\nabla_\Gamma p_{\Gamma,h}^{n+1}\big\rangle\\&+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2+\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2-\big\|\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2\Big)=0.\end{aligned} \tag{3.42}$$

Combined with the following Cauchy inequalities

$$\frac{\Delta t}{\gamma}\big(\hat{\mathbf{u}}_h^{n+1},\nabla p_h^{n+1}-\nabla p_h^n\big)\le\frac{\Delta t}{\gamma}\big\|\hat{\mathbf{u}}_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2+\frac{\Delta t}{4\gamma}\big\|\nabla p_h^{n+1}-\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}^2, \tag{3.43}$$

$$\frac{\Delta t}{\gamma}\big\langle\hat{\mathbf{u}}_{\Gamma,h}^{n+1},\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n\big\rangle\le\frac{\Delta t}{\gamma}\big\|\hat{\mathbf{u}}_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2+\frac{\Delta t}{4\gamma}\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2, \tag{3.44}$$

we take the summation of (3.37), (3.38), (3.41) and (3.42) to further conclude that

$$\begin{aligned}&\frac{\Delta t}{4\gamma}\big\|\nabla p_h^{n+1}-\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla p_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}-\big\|\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}\Big)\\&\quad+\frac{\Delta t}{4\gamma}\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}-\big\|\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}\Big)\\&\le-\Delta t\big(\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\mu_h^{n+1}\big)-\Delta t\big\langle\phi_{\Gamma,h}^n\hat{\mathbf{u}}_{\Gamma,h}^{n+1},\nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\rangle.\end{aligned} \tag{3.45}$$

As a consequence, a combination of (3.36) and (3.45) leads to

$$\begin{aligned}&\Delta t\epsilon\big\|\nabla\mu_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2+\frac{\epsilon}{2}\Big(\big\|\nabla\phi_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2-\big\|\nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2+\big\|\nabla\phi_h^{n+1}-\nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2\Big)\\&\quad+\Big((r_h^{n+1})^2-(r_h^n)^2+(r_h^{n+1}-r_h^n)^2\Big)+\Delta t\epsilon\big\|\nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2\\&\quad+\frac{\kappa\epsilon}{2}\Big(\big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2-\big\|\nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2+\big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1}-\nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2\Big)\\&\quad+\Big((r_{\Gamma,h}^{n+1})^2-(r_{\Gamma,h}^n)^2+(r_{\Gamma,h}^{n+1}-r_{\Gamma,h}^n)^2\Big)\\&\quad+\frac{\Delta t}{4\gamma}\big\|\nabla p_h^{n+1}-\nabla p_h^n\big\|_{L^2(\Omega)}+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla p_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}-\big\|\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}\Big)\\&\quad+\frac{\Delta t}{4\gamma}\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}-\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}+\frac{\Delta t}{2\gamma}\Big(\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}-\big\|\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}\Big)\le0,\end{aligned} \tag{3.46}$$

which is equivalent to

$$\begin{aligned}&E_{SAV}^{n+1}-E_{SAV}^n+\Delta t\epsilon\big\|\nabla\mu_h^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2+\frac{\epsilon}{2}\big\|\nabla\phi_h^{n+1}-\nabla\phi_h^n\big\|_{\mathbf{L}^2(\Omega)}^2+\big(r_h^{n+1}-r_h^n\big)^2\\&\quad+\Delta t\epsilon\big\|\nabla_\Gamma\mu_{\Gamma,h}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}^2+\frac{\kappa\epsilon}{2}\big\|\nabla_\Gamma\phi_{\Gamma,h}^{n+1}-\nabla_\Gamma\phi_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}^2+\big(r_{\Gamma,h}^{n+1}-r_{\Gamma,h}^n\big)^2\\&\quad+\frac{\Delta t}{4\gamma}\big\|\nabla p_h^{n+1}-\nabla p_h^n\big\|_{\mathbf{L}^2(\Omega)}+\frac{\Delta t}{4\gamma}\big\|\nabla_\Gamma p_{\Gamma,h}^{n+1}+\nabla_\Gamma p_{\Gamma,h}^n\big\|_{\mathbf{L}^2(\Gamma)}\le0.\end{aligned} \tag{3.47}$$

This completes the proof of Theorem 3.1. □

Remark 3.1. Since the proposed SAV numerical scheme (3.1)-(3.8) is linear, the energy dissipation estimate (3.33)-(3.34) reveals that, the linear system corresponding to homogeneous part of the numerical scheme (3.1)-(3.8) has a trivial solution. As a result, the unique solvability of the linear numerical scheme (3.1)-(3.8) is theoretically justified.

4. The Error Estimate

For the convergence analysis in this section, we assume that the weak solution $(\phi,\mu,p,\phi_\Gamma,\mu_\Gamma,p_\Gamma,r,r_\Gamma)$ of CHHS system satisfies the following regularity conditions:

$$\begin{aligned}
&\phi\in H^2\big(0,T;L^2(\Omega)\big)\cap L^\infty\big(0,T;W^{1,6}(\Omega)\big)\cap H^1\big(0,T;H^{q+1}(\Omega)\big),\\
&\mu\in L^\infty\big(0,T;H^1(\Omega)\big)\cap L^2\big(0,T;H^{q+1}(\Omega)\big),\quad \nabla\mu\in L^\infty\big(0,T;H^q(\Omega)\big),\\
&\mathbf{u}\in L^\infty\big(0,T;H^q(\Omega)\big),\quad \mathbf{u}_\Gamma\in L^\infty\big(0,T;H^q(\Gamma)\big),\\
&\phi_\Gamma\in H^2\big(0,T;L^2(\Gamma)\big)\cap L^\infty\big(0,T;W^{1,6}(\Gamma)\big)\cap H^1\big(0,T;H^{q+1}(\Gamma)\big),\\
&\mu_\Gamma\in L^\infty\big(0,T;H^1(\Gamma)\big)\cap L^2\big(0,T;H^{q+1}(\Gamma)\big),\quad \nabla_\Gamma\mu_\Gamma\in L^\infty\big(0,T;H^q(\Gamma)\big).
\end{aligned}\tag{4.1}$$

Define the backward difference operator

$$D_t\phi^{n+1}:=\frac{\phi^{n+1}-\phi^n}{\Delta t},$$

and the approximation errors as follows:

$$\begin{aligned}
&\eta_\phi^{n+1}=\phi_h^{n+1}-R_h\phi^{n+1}, && \eta_\mu^{n+1}=\mu_h^{n+1}-R_h\mu^{n+1}, && \eta_p^{n+1}=p_h^{n+1}-R_hp^{n+1},\\
&\theta_\phi^{n+1}=\phi^{n+1}-R_h\phi^{n+1}, && \theta_\mu^{n+1}=\mu^{n+1}-R_h\mu^{n+1}, && \theta_p^{n+1}=p^{n+1}-R_hp^{n+1},\\
&\eta_{\phi_\Gamma}^{n+1}=\phi_{\Gamma,h}^{n+1}-R_h\phi_\Gamma^{n+1}, && \eta_{\mu_\Gamma}^{n+1}=\mu_{\Gamma,h}^{n+1}-R_h\mu_\Gamma^{n+1}, && \eta_{p_\Gamma}^{n+1}=p_{\Gamma,h}^{n+1}-R_hp_\Gamma^{n+1},\\
&\theta_{\phi_\Gamma}^{n+1}=\phi_\Gamma^{n+1}-R_h\phi_\Gamma^{n+1}, && \theta_{\mu_\Gamma}^{n+1}=\mu_\Gamma^{n+1}-R_h\mu_\Gamma^{n+1}, && \theta_{p_\Gamma}^{n+1}=p_\Gamma^{n+1}-R_hp_\Gamma^{n+1},\\
&\sigma_\phi^{n+1}=\partial_t\phi^{n+1}-D_tR_h\phi^{n+1}, && \sigma_{\phi_\Gamma}^{n+1}=\partial_t\phi_\Gamma^{n+1}-D_tR_h\phi_\Gamma^{n+1},\\
&\rho_r^{n+1}=r_h^{n+1}-r^{n+1}, && \rho_{r_\Gamma}^{n+1}=r_{\Gamma,h}^{n+1}-r_\Gamma^{n+1}.
\end{aligned}$$

By the definition of Ritz projection, subtracting (3.1)-(3.8) from (2.21)-(2.28) (at $t=t_{n+1}$), we obtain

$$\big(D_t\eta_\phi^{n+1},\chi\big)+\epsilon\big(\nabla\eta_\mu^{n+1},\nabla\chi\big)=\big(\sigma_\phi^{n+1},\chi\big)+\big(\phi^{n+1}\mathbf{u}^{n+1}-\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\chi\big),\tag{4.2}$$

$$\big\langle D_t\eta_{\phi_\Gamma}^{n+1},\psi\big\rangle+\epsilon\big\langle\nabla_\Gamma\eta_{\mu_\Gamma}^{n+1},\nabla_\Gamma\psi\big\rangle=\big\langle\sigma_{\phi_\Gamma}^{n+1},\psi\big\rangle+\big\langle\phi_\Gamma^{n+1}\mathbf{u}_\Gamma^{n+1}-\phi_{\Gamma,h}^n\hat{\mathbf{u}}_{\Gamma,h}^{n+1},\nabla_\Gamma\psi\big\rangle,\tag{4.3}$$

$$\begin{aligned}
&\big(\eta_\mu^{n+1},\nu\big)-\epsilon\big(\nabla\eta_\phi^{n+1},\nabla\nu\big)+\big\langle\eta_{\mu_\Gamma}^{n+1},\nu\big\rangle-\kappa\epsilon\big\langle\nabla_\Gamma\eta_{\phi_\Gamma}^{n+1},\nabla_\Gamma\nu\big\rangle\\
&=\big(\theta_\mu^{n+1},\nu\big)-r^{n+1}\left(\frac{f(\phi^{n+1})}{\sqrt{E_1[\phi^{n+1}]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},\nu\right)+\rho_r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}},\nu\right)\\
&\quad-r^{n+1}\left(\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}-\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}},\nu\right)+\langle\theta_{\mu_\Gamma}^{n+1},\nu\rangle\\
&\quad-r_\Gamma^{n+1}\left\langle\frac{g(\phi_\Gamma^{n+1})}{\sqrt{E_1[\phi_\Gamma^{n+1}]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}},\nu\right\rangle+\rho_{r_\Gamma}^{n+1}\left\langle\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}},\nu\right\rangle\\
&\quad-r_\Gamma^{n+1}\left\langle\frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}}-\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}},\nu\right\rangle,
\end{aligned}\tag{4.4}$$

$$
\begin{aligned}
D_t\rho_r^{n+1}+D_tr^{n+1}-\partial_tr^n=\frac{1}{2}\Bigg\{&\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}},D_t\eta_\phi^{n+1}\right)-\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}},D_t\theta_\phi^{n+1}\right)\\
&+\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}},\frac{\phi^{n+1}-\phi^n}{\Delta t}-\partial_t\phi^n\right)+\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},D_t\phi^n\right)\Bigg\},
\end{aligned}\tag{4.5}
$$

$$
\begin{aligned}
D_t\rho_{r_\Gamma}^{n+1}+D_tr_\Gamma^{n+1}-\partial_tr_\Gamma^n=\frac{1}{2}\Bigg\{&\left\langle\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}},D_t\eta_{\phi_\Gamma}^{n+1}\right\rangle-\left\langle\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}},D_t\theta_{\phi_\Gamma}^{n+1}\right\rangle\\
&+\left\langle\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}},\frac{\phi_\Gamma^{n+1}-\phi_\Gamma^n}{\Delta t}-\partial\phi_\Gamma^n\right\rangle+\left\langle\frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}},D_t\phi_\Gamma^n\right\rangle\Bigg\}.
\end{aligned}\tag{4.6}
$$

Substituting $\chi=\eta_\mu^{n+1},\eta_\phi^{n+1}$ into (4.2), $\psi=\eta_{\mu_\Gamma}^{n+1},\eta_{\phi_\Gamma}^{n+1}$ into (4.3), $\nu=-D_t\eta_\phi^{n+1},\eta_\mu^{n+1}$ into (4.4), multiplying (4.5), (4.6) by $2\rho_r^{n+1}$ and $2\rho_{r_\Gamma}^{n+1}$, respectively, and summarizing all equalities, we immediately get the following equation:

$$
\begin{aligned}
&\Delta t\epsilon\big\|\eta_\mu^{n+1}\big\|_{H^1(\Omega)}^2+\Delta t\epsilon\big\|\eta_{\mu_\Gamma}^{n+1}\big\|_{H^1(\Gamma)}^2\\
&+\frac{\epsilon}{2}\left(\big\|\eta_\phi^{n+1}-\eta_\phi^n\big\|_{H^1(\Omega)}^2+\big\|\eta_\phi^{n+1}\big\|_{H^1(\Omega)}^2-\big\|\eta_\phi^n\big\|_{H^1(\Omega)}^2\right)\\
&+\frac{\kappa\epsilon}{2}\left(\big\|\eta_{\phi_\Gamma}^{n+1}-\eta_{\phi_\Gamma}^n\big\|_{H^1(\Gamma)}^2+\big\|\eta_{\phi_\Gamma}^{n+1}\big\|_{H^1(\Gamma)}^2-\big\|\eta_{\phi_\Gamma}^n\big\|_{H^1(\Gamma)}^2\right)\\
&+\left(\big(\rho_r^{n+1}-\rho_r^n\big)^2+\big(\rho_r^{n+1}\big)^2-\big(\rho_r^n\big)^2\right)\\
&+\left(\big(\rho_{r_\Gamma}^{n+1}-\rho_{r_\Gamma}^n\big)^2+\big(\rho_{r_\Gamma}^{n+1}\big)^2-\big(\rho_{r_\Gamma}^n\big)^2\right)=\Delta t\sum_{i=1}^{6}(A_i+I_i)
\end{aligned}\tag{4.7}
$$

with

$$A_1=\big(\sigma_\phi^{n+1},\eta_\mu^{n+1}\big)+\epsilon\big(\sigma_\phi^{n+1},\eta_\phi^{n+1}\big),\tag{4.8}$$

$$I_1=\big\langle\sigma_{\phi_\Gamma}^{n+1},\eta_{\mu_\Gamma}^{n+1}\big\rangle+\kappa\epsilon\big\langle\sigma_{\phi_\Gamma}^{n+1},\eta_{\phi_\Gamma}^{n+1}\big\rangle,\tag{4.9}$$

$$A_2=-\big(\theta_\mu^{n+1},D_t\eta_\phi^{n+1}\big)+\epsilon\big(\theta_\mu^{n+1},\eta_\mu^{n+1}\big),\tag{4.10}$$

$$I_2=-\big\langle\theta_{\mu_\Gamma}^{n+1},D_t\eta_{\phi_\Gamma}^{n+1}\big\rangle+\epsilon\big\langle\theta_{\mu_\Gamma}^{n+1},\eta_{\mu_\Gamma}^{n+1}\big\rangle,\tag{4.11}$$

$$
\begin{aligned}
A_3=&\,r^{n+1}\left(\frac{f(\phi^{n+1})}{\sqrt{E_1[\phi^{n+1}]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},D_t\eta_\phi^{n+1}\right)\\
&+\epsilon r^{n+1}\left(\frac{f(\phi^{n+1})}{\sqrt{E_1[\phi^{n+1}]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},\eta_\mu^{n+1}\right),
\end{aligned}\tag{4.12}
$$

$$
\begin{aligned}
I_3=&\,r_\Gamma^{n+1}\left\langle\frac{g(\phi_\Gamma^{n+1})}{\sqrt{E_1[\phi_\Gamma^{n+1}]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}},D_t\eta_{\phi_\Gamma}^{n+1}\right\rangle\\
&+\epsilon r_\Gamma^{n+1}\left\langle\frac{g(\phi_\Gamma^{n+1})}{\sqrt{E_1[\phi_\Gamma^{n+1}]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}},\eta_{\mu_\Gamma}^{n+1}\right\rangle,
\end{aligned}\tag{4.13}
$$

and

$$A_4=-r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},D_t\eta_\phi^{n+1}\right)+\rho_r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},D_t\phi^{n+1}\right)$$

$$
- \epsilon r^{n+1} \left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}} - \frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}, \eta_\mu^{n+1} \right), \tag{4.14}
$$

$$
\begin{aligned}
I_4 = {} & r_\Gamma^{n+1} \left\langle \frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}} - \frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}}, D_t \eta_{\phi_\Gamma}^{n+1} \right\rangle + \rho_{r_\Gamma}^{n+1} \left\langle \frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}} - \frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}}, D_t \phi_\Gamma^{n+1} \right\rangle \\
& - \epsilon r_\Gamma^{n+1} \left\langle \frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}} - \frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}}, \eta_{\mu_\Gamma}^{n+1} \right\rangle,
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
A_5 = {} & \rho_r^{n+1} \left(\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}, D_t \phi^{n+1} - \partial_t \phi^n \right) - \rho_r^{n+1} \left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}, D_t \theta_\phi^{n+1} \right) \\
& - 2\gamma \cdot \rho_r^{n+1} \left(D_t r^{n+1} - \partial_t r^n \right) + \epsilon \rho_r^{n+1} \left(\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}, \eta_\mu^{n+1} \right),
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
I_5 = {} & \rho_{r_\Gamma}^{n+1} \left\langle \frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}}, D_t \phi_\Gamma^{n+1} - \partial_t \phi_\Gamma^n \right\rangle - \rho_{r_\Gamma}^{n+1} \left\langle \frac{g(\phi_{\Gamma,h}^n)}{\sqrt{E_1[\phi_{\Gamma,h}^n]}}, D_t \theta_{\phi_\Gamma}^{n+1} \right\rangle \\
& - 2\gamma \cdot \rho_{r_\Gamma}^{n+1} \left(D_t r_\Gamma^{n+1} - \partial_t r_\Gamma^n \right) + \epsilon \rho_{r_\Gamma}^{n+1} \left\langle \frac{g(\phi_\Gamma^n)}{\sqrt{E_1[\phi_\Gamma^n]}}, \eta_{\mu_\Gamma}^{n+1} \right\rangle,
\end{aligned} \tag{4.17}
$$

$$
A_6 = \left(\phi^{n+1} \mathbf{u}^{n+1} - \phi_h^n \hat{\mathbf{u}}_h^{n+1}, \nabla \eta_\mu^{n+1} \right) + \epsilon \left(\phi^{n+1} \mathbf{u}^{n+1} - \phi_h^n \hat{\mathbf{u}}_h^{n+1}, \nabla \eta_\phi^{n+1} \right), \tag{4.18}
$$

$$
I_6 = \left\langle \phi_\Gamma^{n+1} \mathbf{u}_\Gamma^{n+1} - \phi_{\Gamma,h}^n \hat{\mathbf{u}}_{\Gamma,h}^{n+1}, \nabla_\Gamma \eta_{\mu_\Gamma}^{n+1} \right\rangle + \kappa \epsilon \left\langle \phi_\Gamma^{n+1} \mathbf{u}_\Gamma^{n+1} - \phi_{\Gamma,h}^n \mathbf{u}_{\Gamma,h}^{n+1}, \nabla_\Gamma \eta_{\phi_\Gamma}^{n+1} \right\rangle. \tag{4.19}
$$

For the estimates of (4.8)-(4.19), we start with the following lemma.

Lemma 4.1 ([19]). *If the weak solution* $(\phi, \mu, p, \phi_\Gamma, \mu_\Gamma, p_\Gamma, r, r_\Gamma)$ *of* (2.21)-(2.28) *satisfies the regularity assumption* (4.1), *we have*

$$
\begin{aligned}
\left\| \sigma_\phi^{n+1} \right\|_{L^2(\Omega)}^2 \leq {} & C \frac{h^{2q+2}}{\Delta t} \int_{t^n}^{t^{n+1}} \left\| \partial_s \phi(s) \right\|_{H^{q+1}(\Omega)}^2 ds \\
& + \frac{\Delta t}{3} \int_{t^n}^{t^{n+1}} \left\| \partial_{ss} \phi(s) \right\|_{L^2(\Omega)}^2 ds, \quad \forall t \in (0, T],
\end{aligned} \tag{4.20}
$$

where $C > 0$ *is a constant independent of* h *and* Δt.

Similarly, we are able to derive the following estimate:

$$
\begin{aligned}
\left\| \sigma_{\phi_\Gamma}^{n+1} \right\|_{L^2(\Gamma)}^2 \leq {} & C \frac{h^{2q+2}}{\Delta t} \int_{t^n}^{t^{n+1}} \left\| \partial_s \phi_\Gamma(s) \right\|_{H^{q+1}(\Gamma)}^2 ds \\
& + \frac{\Delta t}{3} \int_{t^n}^{t^{n+1}} \left\| \partial_{ss} \phi_\Gamma(s) \right\|_{L^2(\Gamma)}^2 ds, \quad \forall t \in (0, T].
\end{aligned} \tag{4.21}
$$

Moreover, we assume that the potentials F and G are bounded from below:

$$
F''(s) = f'(s) \leq -\widetilde{C}_1, \quad s f(s) \geq b |s|^{p_1} - \tilde{c}_1, \tag{4.22}
$$

$$
\left| f'(s) \right| < C \left(|x|^p + 1 \right), \quad \begin{cases} \forall p > 0, & \text{if} \quad d = 1, 2, \\ 0 < p < 4, & \text{if} \quad d = 3, \end{cases} \tag{4.23}
$$

$$\left|f''(s)\right| < C\big(|x|^q + 1\big), \quad \begin{cases} \forall q > 0, & \text{if } \ d = 1, 2, \\ 0 < q < 3, & \text{if } \ d = 3, \end{cases} \tag{4.24}$$

$$G''(s) = g'(s) \leq -\widetilde{C}_2, \quad sg(s) \geq b|s|^{p_2} - \tilde{c}_2, \tag{4.25}$$

$$\left|g'(s)\right| < C\big(|x|^p + 1\big), \quad \begin{cases} \forall p > 0, & \text{if } \ d = 1, 2, \\ 0 < p < 4, & \text{if } \ d = 3, \end{cases} \tag{4.26}$$

$$\left|g''(s)\right| < C\big(|x|^q + 1\big), \quad \begin{cases} \forall q > 0, & \text{if } \ d = 1, 2, \\ 0 < q < 3, & \text{if } \ d = 3, \end{cases} \tag{4.27}$$

where $C > 0, \widetilde{C}_1 > 0, \widetilde{C}_2 > 0, p_1 > 0$ and $p_2 > 0$ are all constants [60].

Last but not least, by the a-priori estimates [48], the following bounds are available:

$$\left\|\phi_h^n\right\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty\left(L^\infty(\Omega)\right)} + 1, \quad \left\|\phi_{\Gamma,h}^n\right\|_{L^\infty(\Gamma)} \leq \|\phi_\Gamma\|_{L^\infty\left(L^\infty(\Gamma)\right)} + 1. \tag{4.28}$$

The proof is left to interested readers.

Theorem 4.1. *Suppose that* $(\phi_h^{n+1}, \mu_h^{n+1}, p_h^{n+1}, \phi_{\Gamma,h}^{n+1}, \mu_{\Gamma,h}^{n+1}, p_{\Gamma,h}^{n+1}, r_h^{n+1}, r_{\Gamma,h}^{n+1})$ *and* $(\phi^{n+1}, \mu^{n+1}, p^{n+1}, \phi_\Gamma^{n+1}, \mu_\Gamma^{n+1}, p_\Gamma^{n+1}, r^{n+1}, r_\Gamma^{n+1})$ *be the solution of* (3.1)-(3.8) *and* (2.3)-(2.12), *respectively. Under the regularity assumption* (4.1), *we have the error estimate*

$$\begin{aligned} \max_{0\leq l\leq N-1} \Bigg\{ & \left\|\phi^{l+1} - \phi_h^{l+1}\right\|_{H^1(\Omega)}^2 + \left\|\phi_\Gamma^{l+1} - \phi_{\Gamma,h}^{l+1}\right\|_{H^1(\Gamma)}^2 + \big(r^{l+1} - r_h^{l+1}\big)^2 + \big(r_\Gamma^{l+1} - r_{\Gamma,h}^{l+1}\big)^2 \\ & + \Delta t \sum_{n=1}^{l} \Big(\left\|\mu^{n+1} - \mu_h^{n+1}\right\|_{H^1(\Omega)}^2 + \left\|\mu_\Gamma^{n+1} - \mu_{\Gamma,h}^{n+1}\right\|_{H^1(\Gamma)}^2 \Big) \Bigg\} \\ \leq C\big((\Delta t)^2 + h^{2q}\big), & \end{aligned} \tag{4.29}$$

where $C > 0$ *is a constant independent of* $h, \Delta t$.

Proof. Using the error estimate (4.20) and Young's inequality, we are able to derive

$$A_1 \leq C\big(h^{2q+2} + (\Delta t)^2\big) + \frac{\epsilon}{16}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2 + \frac{\epsilon}{16}\left\|\eta_\phi^{n+1}\right\|_{H^1(\Omega)}^2. \tag{4.30}$$

Taking a similar approach, by virtue of (4.21), it is easy to get

$$I_1 \leq C\big(h^{2q+2} + (\Delta t)^2\big) + \frac{\epsilon}{16}\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2 + \frac{\kappa\epsilon}{16}\left\|\eta_{\phi_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2. \tag{4.31}$$

By employing (3.11), (4.22) and Young's inequality, we obtain

$$\begin{aligned} A_2 &\leq \left\|\theta_\mu^{n+1}\right\|_{H^1(\Omega)}\left\|D_t\eta_\phi^{n+1}\right\|_{H^{-1}(\Omega)} + \left\|\theta_\mu^{n+1}\right\|_{H^{-1}(\Omega)}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)} \\ &\leq Ch^{2q} + \frac{1}{3}\left\|D_t\eta_\phi^{n+1}\right\|_{H^{-1}(\Omega)}^2 + \frac{\epsilon}{16}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2. \end{aligned} \tag{4.32}$$

In a similar way, by applying (3.16), (4.23), we see that

$$\begin{aligned} I_2 &\leq \left\|\theta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}\left\|D_t\eta_{\phi_\Gamma}^{n+1}\right\|_{H^{-1}(\Gamma)} + \epsilon\left\|\theta_{\mu_\Gamma}^{n+1}\right\|_{H^{-1}(\Gamma)}\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)} \\ &\leq Ch^{2q} + \frac{1}{3}\left\|D_t\eta_{\phi_\Gamma}^{n+1}\right\|_{H^{-1}(\Gamma)}^2 + \frac{\epsilon}{16}\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2. \end{aligned} \tag{4.33}$$

To proceed further, from (4.23) and (4.26), we get the following estimates:

$$\begin{aligned}&\left\|r^{n+1}\left(\frac{f(\phi^{n+1})}{\sqrt{E_1[\phi^{n+1}]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right)\right\|_{H^s(\Omega)}\\&\le|r^{n+1}|\left\|\frac{f(\phi^{n+1})}{\sqrt{E_1[\phi^{n+1}]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right\|_{H^s(\Omega)}\\&\le\sup_{t\in[0,T]}|r(t)|\Bigg(\|f(\phi_h^n)\|_{H^s(\Omega)}\frac{|E_1[\phi^n]-E_1[\phi^{n+1}]|}{\sqrt{E_1[\phi^n]E_1[\phi^{n+1}](E_1[\phi^n]+E_1[\phi^{n+1}])}}\\&\qquad+\frac{\|f(\phi^{n+1})-f(\phi^n)\|_{H^s(\Omega)}}{\sqrt{E_1[\phi^n]}}\Bigg)\le C\Delta t,\end{aligned}\tag{4.34}$$

and

$$\begin{aligned}&\left\|r_\Gamma^{n+1}\left(\frac{g(\phi_\Gamma^{n+1})}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^{n+1}]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^n]}}\right)\right\|_{H^s(\Gamma)}\\&\le|r_\Gamma^{n+1}|\left\|\frac{g(\phi_\Gamma^{n+1})}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^{n+1}]}}-\frac{g(\phi_\Gamma^n)}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^n]}}\right\|_{H^s(\Gamma)}\\&\le\sup_{t\in[0,T]}|r_\Gamma(t)|\Bigg(\|g(\phi_{\Gamma,h}^n)\|_{H^s(\Gamma)}\frac{|E_{\Gamma,1}[\phi_\Gamma^n]-E_{\Gamma,1}[\phi_\Gamma^{n+1}]|}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^n]E_{\Gamma,1}[\phi_\Gamma^{n+1}](E_{\Gamma,1}[\phi_\Gamma^n]+E_{\Gamma,1}[\phi_\Gamma^{n+1}])}}\\&\qquad+\frac{\|g(\phi_\Gamma^{n+1})-g(\phi_\Gamma^n)\|_{H^s(\Gamma)}}{\sqrt{E_{\Gamma,1}[\phi_\Gamma^n]}}\Bigg)\le C\Delta t.\end{aligned}\tag{4.35}$$

Then, from (4.34) and Young's inequality, we see that

$$A_3\le C(\Delta t)^2+\frac{1}{3}\|D_t\eta_\phi^{n+1}\|_{H^{-1}(\Omega)}^2+\frac{\epsilon}{16}\|\eta_\mu^{n+1}\|_{H^1(\Omega)}^2.\tag{4.36}$$

Likewise, from (4.35) and Young's inequality, we discover that

$$I_3\le C(\Delta t)^2+\frac{1}{3}\|D_t\eta_{\phi_\Gamma}^{n+1}\|_{H^{-1}(\Gamma)}^2+\frac{\epsilon}{16}\|\eta_{\mu_\Gamma}^{n+1}\|_{H^1(\Gamma)}^2.\tag{4.37}$$

Meanwhile, it is observed that

$$\|D_t\phi^{n+1}\|_{H^{-1}(\Omega)}\le(\Delta t)^{-1}\int_{t^n}^{t^{n+1}}\|\partial_t\phi^{n+1}\|_{H^{\max(1,q-1)}(\Omega)}ds\le C.\tag{4.38}$$

By employing (4.38) and Young's inequality, we get

$$\begin{aligned}A_4\le&\left\|r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right)\right\|_{H^1(\Omega)}\|D_t\eta_\phi^{n+1}\|_{H^{-1}(\Omega)}\\&+\left\|\rho_r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right)\right\|_{H^1(\Omega)}\|D_t\phi^{n+1}\|_{H^{-1}(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&+\epsilon\left\|\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right\|_{L^2(\Omega)}\left\|\eta_\mu^{n+1}\right\|_{L^2(\Omega)}\\
&\leq C(\rho_r^{n+1})^2+\frac{1}{3}\left\|D_t\eta_\phi^{n+1}\right\|_{H^{-1}(\Omega)}^2+\frac{\epsilon}{16}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2\\
&+C\left\|\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right\|_{H^1(\Omega)}^2. \qquad (4.39)
\end{aligned}
$$

In order to estimate A_4, we rewrite the gradient of last term on the right-hand side as follows:

$$
\begin{aligned}
&\nabla\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right)\\
&=\nabla f(\phi^n)\frac{E_1[\phi^n]-E_1[\phi_h^n]}{\sqrt{E_1[\phi^n]E_1[\phi_h^n](E_1[\phi^n]+E_1[\phi_h^n])}}+\frac{\nabla f(\phi_h^n)-\nabla f(\phi^n)}{\sqrt{E_1[\phi_h^n]}}\\
&=: Q_1+Q_2. \qquad (4.40)
\end{aligned}
$$

Using the fact that $E_1[\phi_h^n]\geq C_1$ with (4.23), combined with (4.24) and (4.28), we get

$$
\|Q_1\|_{\mathbf{L}^2(\Omega)}\leq C\|\nabla f(\phi_h^n)\|_{\mathbf{L}^2(\Omega)}\|\phi^n-\phi_h^n\|_{L^2(\Omega)}\leq C\left(\left\|\eta_\phi^n\right\|_{L^2(\Omega)}+\left\|\theta_\phi^n\right\|_{L^2(\Omega)}\right), \qquad (4.41)
$$

$$
\begin{aligned}
\left\|Q_2\right\|_{\mathbf{L}^2(\Omega)}&\leq C\left\|\nabla f(\phi_h^n)-\nabla f(\phi^n)\right\|_{\mathbf{L}^2(\Omega)}\\
&\leq C\left(\left\|\left(f'(\phi_h^n)-f'(\phi^n)\right)\nabla\phi^n\right\|_{\mathbf{L}^2(\Omega)}+\left\|f'(\phi_h^n)\nabla\eta_\phi^n\right\|_{\mathbf{L}^2(\Omega)}+\left\|f'(\phi_h^n)\nabla\theta_\phi^n\right\|_{\mathbf{L}^2(\Omega)}\right)\\
&\leq C\left(\left\|\nabla\eta_\phi^n\right\|_{\mathbf{L}^2(\Omega)}+\left\|\nabla\theta_\phi^n\right\|_{\mathbf{L}^2(\Omega)}+\left\|\eta_\phi^n\right\|_{L^2(\Omega)}+\left\|\theta_\phi^n\right\|_{L^2(\Omega)}\right). \qquad (4.42)
\end{aligned}
$$

Hence, from (4.40)-(4.42), we have

$$
\begin{aligned}
&\left\|\nabla\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right)\right\|_{\mathbf{L}^2(\Omega)}\\
&\leq Ch^q\left\|\phi^n\right\|_{H^{q+1}(\Omega)}+C\left(\left\|\nabla\eta_\phi^n\right\|_{\mathbf{L}^2(\Omega)}+\left\|\eta_\phi^n\right\|_{L^2(\Omega)}\right). \qquad (4.43)
\end{aligned}
$$

Similarly, the following estimate is valid:

$$
\left\|\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}-\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right\|_{L^2(\Omega)}\leq Ch^q\left\|\phi^n\right\|_{H^{q+1}(\Omega)}+C\left\|\eta_\phi^n\right\|_{L^2(\Omega)}. \qquad (4.44)
$$

It follows from (4.40)-(4.44) that

$$
A_4\leq Ch^{2q}+C\left(\rho_r^{n+1}\right)^2+\frac{1}{3}\left\|D_t\eta_\phi^{n+1}\right\|_{H^{-1}(\Omega)}^2+\frac{\epsilon}{16}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2+\frac{\epsilon}{16}\left\|\eta_\phi^n\right\|_{H^1(\Omega)}^2. \qquad (4.45)
$$

Similarly, we are able to derive the following estimate

$$
I_4\leq Ch^{2q}+C\left(\rho_{r_\Gamma}^{n+1}\right)^2+\frac{1}{3}\left\|D_t\eta_{\phi_\Gamma}^{n+1}\right\|_{H^{-1}(\Gamma)}^2+\frac{\epsilon}{16}\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2+\frac{\epsilon}{16}\left\|\eta_{\phi_\Gamma}^n\right\|_{H^1(\Gamma)}^2. \qquad (4.46)
$$

To obtain an estimate for A_5, we see that

$$
\begin{aligned}
\left|E_r^n\right|&=\left|\left(\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}},D_t\phi^{n+1}-\partial_t\phi^n\right)-2\left(D_tr^{n+1}-\partial_tr^n\right)\right|\\
&\leq C\left(\int_{t^n}^{t^{n+1}}\int_\Omega\left|\partial_{tt}\phi(s,x)\right|dxds+\int_{t^n}^{t^{n+1}}\left|\partial_{tt}r(s)\right|ds\right)\leq C\Delta t. \qquad (4.47)
\end{aligned}
$$

A combination of (4.28) and (4.47) results in

$$\begin{aligned} A_5 &\le \left|\rho_r^{n+1}\left(\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}, D_t\theta_\phi^{n+1}\right)\right| + \left|\ \rho_r^{n+1}E_r^n\right| + \epsilon\left|\rho_r^{n+1}\left(\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}, \eta_\mu^{n+1}\right)\right| \\ &\le C\left\|\rho_r^{n+1}\frac{f(\phi_h^n)}{\sqrt{E_1[\phi_h^n]}}\right\|_{H^1(\Omega)}\left\|D_t\theta_\phi^{n+1}\right\|_{H^{-1}(\Omega)} + C\left|\rho_r^{n+1}\right|\left|E_r^n\right| \\ &\quad +C\left\|\rho_r^{n+1}\frac{f(\phi^n)}{\sqrt{E_1[\phi^n]}}\right\|_{L^2(\Omega)}\left\|\eta_\phi^{n+1}\right\|_{L^2(\Omega)} \\ &\le C\big(h^{2q}+(\Delta t)^2\big) + C\big(\rho_r^{n+1}\big)^2 + \frac{\epsilon}{16}\left\|\eta_\phi^{n+1}\right\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.48}$$

Applying the above two inequalities (4.47) and (4.48), we find that

$$I_5 \le C\big(h^{2q}+(\Delta t)^2\big) + C\big(\rho_{r_\Gamma}^{n+1}\big)^2 + \frac{\epsilon}{16}\left\|\eta_{\phi_\Gamma}^{n+1}\right\|_{L^2(\Gamma)}^2. \tag{4.49}$$

In the estimate of A_6, the first part is rewritten as

$$\begin{aligned} A_6^1 &= \big(\phi^{n+1}\mathbf{u}^{n+1} - \phi_h^n\hat{\mathbf{u}}_h^{n+1}, \nabla\eta_\mu^{n+1}\big) \\ &= \Big(\phi^{n+1}\mathbf{u}^{n+1} - R_h\phi^{n+1}\mathbf{u}^{n+1} + R_h\phi^{n+1}\mathbf{u}^{n+1} - R_h\phi^n\mathbf{u}^{n+1} \\ &\qquad + R_h\phi^n\mathbf{u}^{n+1} - \phi_h^n\mathbf{u}^{n+1} + \phi_h^n\mathbf{u}^{n+1} - \phi_h^n\hat{\mathbf{u}}_h^{n+1}, \nabla\eta_\mu^{n+1}\Big) \\ &= \Big(\theta_\phi^{n+1}\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big) + \Big(\Delta t D_t R_h\phi^{n+1}\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big) \\ &\qquad -\Big(\eta_\phi^n\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big) + \Big(\phi_h^n\big(\mathbf{u}^{n+1} - \hat{\mathbf{u}}_h^{n+1}\big), \nabla\eta_\mu^{n+1}\Big). \end{aligned} \tag{4.50}$$

By using (3.10), it is observed that

$$\begin{aligned} &\left|\Big(\theta_\phi^{n+1}\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big)\right| \\ &\le C\left\|\theta_\phi^{n+1}\right\|_{L^3(\Omega)}\left\|\mathbf{u}^{n+1}\right\|_{L^6(\Omega)}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)} \\ &\le Ch^{2q} + \frac{\epsilon}{16}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \tag{4.51}$$

In virtue of (3.11), $H^1(\Omega) \hookrightarrow L^3(\Omega)$ and the Taylor formula, we are able to acquire

$$\begin{aligned} &\left|\Big(\Delta t D_t R_h\phi^{n+1}\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big)\right| \\ &\le C\left\|\Delta t D_t R_h\phi^{n+1}\right\|_{L^3(\Omega)}\left\|\mathbf{u}^{n+1}\right\|_{\mathbf{L}^6(\Omega)}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)} \\ &\le C\Delta t\left\|D_t\phi^{n+1}\right\|_{H^1(\Omega)}^2 + \frac{\epsilon}{16}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2 \\ &\le C\left(\int_{t^n}^{t^{n+1}}\left\|\partial_t\phi\right\|_{H^1(\Omega)}ds\right)^2 + \frac{\epsilon}{16}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2 \\ &\le C(\Delta t)^2 + \frac{\epsilon}{16}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \tag{4.52}$$

Similar to (4.52), the following inequality could be derived:

$$\begin{aligned} &\left|\Big(\eta_\phi^n\mathbf{u}^{n+1}, \nabla\eta_\mu^{n+1}\Big)\right| \\ &\le C\left\|\eta_\phi^n\right\|_{L^3(\Omega)}\left\|\mathbf{u}^{n+1}\right\|_{\mathbf{L}^6(\Omega)}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)} \\ &\le C\left\|\eta_\phi^n\right\|_{H^1(\Omega)}^2 + \frac{\epsilon}{16}\left\|\nabla\eta_\mu^{n+1}\right\|_{\mathbf{L}^2(\Omega)}^2. \end{aligned} \tag{4.53}$$

Just for the sake of the next process, we introduce

$$\tilde{\mathbf{u}}_h^{n+1} = -\nabla p_h^{n+1} - \gamma\phi_h^n\nabla\mu_h^{n+1}. \tag{4.54}$$

By the the definition of projections $\mathcal{P}$ and $\mathcal{P}_h$, we get

$$\begin{aligned}
&-\big(\mathbf{u}^{n+1} - \hat{\mathbf{u}}_h^{n+1}\big)\\
&= -\Big(\nabla p^{n+1} + \gamma\phi^{n+1}\nabla\mu^{n+1} - \big(\nabla p_h^n + \gamma\phi_h^n\nabla\mu_h^{n+1}\big)\Big)\\
&= -\Big(\mathbf{u}^{n+1} - \tilde{\mathbf{u}}_h^{n+1} + \nabla(p_h^{n+1} - p_h^n)\Big)\\
&= \mathcal{P}\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi_h^n\nabla\mu_h^{n+1}\big) - \nabla\big(p_h^{n+1} - p_h^n\big)\\
&= \mathcal{P}\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) + \mathcal{P}_h\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big)\\
&\quad -\mathcal{P}_h\big(\gamma\phi^n\nabla\mu^{n+1}\big) + \mathcal{P}_h\big(\gamma\phi^n\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi_h^n\nabla\mu^{n+1}\big)\\
&\quad +\mathcal{P}_h\big(\gamma\phi_h^n\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi_h^n\nabla\mu_h^{n+1}\big) - \nabla\big(p_h^{n+1} - p_h^n\big)\\
&= \mathcal{P}\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) + \mathcal{P}\big(\gamma\Delta t D_t\phi^{n+1}\nabla\mu^{n+1}\big)\\
&\quad +\mathcal{P}\big(\gamma(\theta_\phi^n + \eta_\phi^n)\nabla\mu^{n+1}\big) + \mathcal{P}\big(\gamma\phi_h^n\nabla(\theta_\mu^n + \eta_\mu^n)\big) - \nabla\big(p_h^{n+1} - p_h^n\big)\\
&= Q_5 + \mathcal{P}\big(\gamma\phi_h^n\nabla(\theta_\mu^n + \eta_\mu^n)\big).
\end{aligned} \tag{4.55}$$

A combination of (3.18), (3.24) and (3.25) yields

$$\begin{aligned}
\|Q_5\|_{L^2(\Omega)}^2 &\le 4\,\big\|\mathcal{P}\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big) - \mathcal{P}_h\big(\gamma\phi^{n+1}\nabla\mu^{n+1}\big)\big\|_{\mathbf{L}^2(\Omega)}^2 + 4\,\big\|\mathcal{P}\big(\gamma\Delta t D_t\phi^{n+1}\nabla\mu^{n+1}\big)\big\|_{\mathbf{L}^2(\Omega)}^2\\
&\quad +4\,\big\|\mathcal{P}\big(\gamma(\theta_\phi^n + \eta_\phi^n)\nabla\mu^{n+1}\big)\big\|_{\mathbf{L}^2(\Omega)}^2 + 4\,\big\|-\nabla(p_h^{n+1} - p_h^n)\big\|_{\mathbf{L}^2(\Omega)}^2\\
&\le Ch^{2q}|p|_{H^{q+1}(\Omega)}^2 + C(\Delta t)^2\big\|D_t\phi^{n+1}\big\|_{L^3(\Omega)}^2\big\|\nabla\mu^{n+1}\big\|_{\mathbf{L}^6(\Omega)} + 8\big\|\theta_\phi^n\nabla\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2\\
&\quad +8\big\|\eta_\phi^n\nabla\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2 + C\left\|\int_{t^n}^{t^{n+1}}\partial_t\big(\nabla p_h^{n+1}\big)\right\|_{\mathbf{L}^2(\Omega)}^2\\
&\le C\left(h^{2q} + (\Delta t)^2\right) + \frac{\epsilon}{16}\big\|\eta_\phi^n\big\|_{\mathbf{L}^2(\Omega)}^2.
\end{aligned} \tag{4.56}$$

In turn, the last term of A_6^1 could be bounded as follows:

$$\begin{aligned}
&\Big|\Big(\phi_h^n\big(\mathbf{u}^{n+1} - \mathbf{u}_h^{n+1}\big), \nabla\eta_\mu^{n+1}\Big)\Big|\\
&\le \Big|\Big(\phi_h^n Q_5, \nabla\eta_\mu^{n+1}\Big)\Big| + \Big|\Big(\phi_h^n\mathcal{P}\big(\gamma\phi_h^n\nabla\big(\theta_\mu^n + \eta_\mu^n\big)\big), \nabla\eta_\mu^{n+1}\Big)\Big|\\
&\le C\big\|\phi_h^n\big\|_{L^\infty(\Omega)}\big\|Q_5\big\|_{L^2(\Omega)}\big\|\nabla\eta_\mu^{n+1}\big\|_{L^2(\Omega)} + C\big\|\phi_h^n\big\|_{L^\infty(\Omega)}\big\|\phi_h^n\nabla(\theta_\mu^n + \eta_\mu^n)\big\|_{L^2(\Omega)}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}\\
&\le C\big\|Q_5\big\|_{L^2(\Omega)}^2 + C\Big(\big\|\theta_\mu^n\big\|_{L^2(\Omega)}^2 + \big\|\eta_\mu^n\big\|_{L^2(\Omega)}^2\Big) + \frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2\\
&\le C\left(h^{2q} + (\Delta t)^2\right) + \frac{\epsilon}{16}\big\|\eta_\phi^n\big\|_{L^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\eta_\mu^n\big\|_{L^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2.
\end{aligned} \tag{4.57}$$

Therefore, we get the estimate of A_6^1

$$A_6^1 \le C\big(h^{2q} + (\Delta t)^2\big) + \frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\eta_\phi^n\big\|_{L^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\eta_\mu^n\big\|_{L^2(\Omega)}^2. \tag{4.58}$$

The other part of A_6 could be analyzed as follows:

$$\begin{aligned}
A_6^2 &= \epsilon\left(\phi^{n+1}\mathbf{u}^{n+1} - \phi_h^n\hat{\mathbf{u}}_h^{n+1}, \nabla\eta_\phi^{n+1}\right)\\
&\le C\big(h^{2q} + (\Delta t)^2\big) + \frac{\epsilon}{16}\big\|\nabla\eta_\phi^{n+1}\big\|_{\mathbf{L}^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\eta_\phi^n\big\|_{L^2(\Omega)}^2 + \frac{\epsilon}{16}\big\|\eta_\mu^n\big\|_{L^2(\Omega)}^2.
\end{aligned} \tag{4.59}$$

Repeating a similar process as (4.50)-(4.59), we get

$$\begin{aligned} I_6 \ &\le C\big(h^{2q}+(\Delta t)^2\big)+\frac{\kappa\epsilon}{16}\big\|\nabla_\Gamma\eta_{\mu_\Gamma}^{n+1}\big\|_{L^2(\Gamma)}^2+\frac{\kappa\epsilon}{16}\big\|\eta_{\phi_\Gamma}^{n}\big\|_{L^2(\Gamma)}^2 \\ &\quad+\frac{\kappa\epsilon}{16}\big\|\eta_{\mu_\Gamma}^{n}\big\|_{L^2(\Gamma)}^2+\frac{\kappa\epsilon}{16}\big\|\nabla_\Gamma\eta_{\phi_\Gamma}^{n+1}\big\|_{L^2(\Gamma)}^2. \end{aligned} \tag{4.60}$$

Besides, we have to estimate $\|D_t\eta_\phi^{n+1}\|_{H^{-1}(\Omega)}$ in (4.32), (4.36) and (4.45).

The method outlined in [24] is adopted. Define $\mathcal{Q}_h$ as the standard L^2 projection operator into M_h. For any $\chi\in H^1(\Omega)$, setting $\chi_h=\mathcal{Q}_h\chi$ in (4.2), by using Young's inequality and (4.20), we have

$$\begin{aligned} &\big(D_t\eta_\phi^{n+1},\chi\big)=\big(D_t\eta_\phi^{n+1},\chi_h\big) \\ &=-\epsilon\big(\nabla\eta_\mu^{n+1},\nabla\chi_h\big)+\big(\sigma_\phi^{n+1},\chi_h\big)+\big(\phi^{n+1}\mathbf{u}^{n+1}-\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\chi_h\big) \\ &\le\frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)}+\big\|\sigma_\phi^{n+1}\big\|_{L^2(\Omega)}\big\|\chi_h\big\|_{L^2(\Omega)}+\big(\phi^{n+1}\mathbf{u}^{n+1}-\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\chi_h\big) \\ &\le C\left(\frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}+h^{q+1}+\Delta t\right)\ \big\|\chi_h\big\|_{H^1(\Omega)}+\big(\phi^{n+1}\mathbf{u}^{n+1}-\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\chi_h\big). \end{aligned} \tag{4.61}$$

Recalling the techniques applied in the process of proving A_6^1, we can give an estimate of the last term in (4.61) as follows:

$$\begin{aligned} &\big(\phi^{n+1}\mathbf{u}^{n+1}-\phi_h^n\hat{\mathbf{u}}_h^{n+1},\nabla\chi_h\big) \\ &=\big(\theta_\phi^{n+1}\mathbf{u}^{n+1},\nabla\chi_h\big)+\big(\Delta tD_tR_h\phi^{n+1}\mathbf{u}^{n+1},\nabla\chi_h\big) \\ &\quad-\big(\eta_\phi^n\mathbf{u}^{n+1},\nabla\chi_h\big)-\big(\phi_h^nQ_5,\nabla\chi_h\big) \\ &\quad-\Big(\phi_h^n\mathcal{P}\big(\gamma\phi_h^n\nabla(\theta_\mu^n+\eta_\mu^n)\big),\nabla\chi_h\Big) \\ &\le C\big\|\theta_\phi^{n+1}\big\|_{L^3(\Omega)}\big\|\mathbf{u}^{n+1}\big\|_{\mathbf{L}^6(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)} \\ &\quad+C\big\|\Delta tD_tR_h\phi^{n+1}\big\|_{L^3(\Omega)}\big\|\mathbf{u}^{n+1}\big\|_{\mathbf{L}^6(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)} \\ &\quad+C\big\|\eta_\phi^n\big\|_{L^3(\Omega)}\big\|\mathbf{u}^{n+1}\big\|_{\mathbf{L}^6(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)} \\ &\quad+C\big\|\phi_h^n\big\|_{L^\infty(\Omega)}\big\|Q_5\big\|_{L^2(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)} \\ &\quad+C\big\|\phi_h^n\big\|_{L^\infty(\Omega)}\big\|\gamma\phi_h^n\nabla(\theta_\mu^n+\eta_\mu^n)\big\|_{\mathbf{L}^2(\Omega)}\big\|\nabla\chi_h\big\|_{\mathbf{L}^2(\Omega)} \\ &\le\Big(C\big(h^q+\Delta t+\big\|\eta_\phi^n\big\|_{H^1(\Omega)}\big)+\frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}\Big)\ \big\|\chi_h\big\|_{H^1(\Omega)}. \end{aligned} \tag{4.62}$$

A combination of (4.61) and (4.62) yields

$$\big\|D_t\eta_\phi^{n+1}\big\|_{H^{-1}(\Omega)}\le C\left(h^q+\Delta t+\big\|\eta_\phi^n\big\|_{H^1(\Omega)}\right)+\frac{\epsilon}{16}\big\|\nabla\eta_\mu^{n+1}\big\|_{\mathbf{L}^2(\Omega)}. \tag{4.63}$$

As for the estimate of $\|D_t\eta_{\phi_\Gamma}^{n+1}\|_{H^{-1}(\Gamma)}$ in (4.33), (4.37) and (4.46), repeating the process (4.61)-(4.63), we are able to derive

$$\big\|D_t\eta_{\phi_\Gamma}^{n+1}\big\|_{H^{-1}(\Gamma)}\le C\left(h^q+\Delta t+\big\|\eta_{\phi_\Gamma}^n\big\|_{H^1(\Gamma)}\right)+\frac{\epsilon}{16}\big\|\nabla_\Gamma\eta_{\mu_\Gamma}^{n+1}\big\|_{\mathbf{L}^2(\Gamma)}. \tag{4.64}$$

With the help of all estimates of A_i,I_i $(1\le i\le 6)$ and (4.63), (4.64), we obtain

$$\begin{aligned} &\Delta t\epsilon\big\|\eta_\mu^{n+1}\big\|_{H^1(\Omega)}^2+\Delta t\epsilon\big\|\eta_{\mu_\Gamma}^{n+1}\big\|_{H^1(\Gamma)}^2+\frac{\epsilon}{2}\left(\big\|\eta_\phi^{n+1}\big\|_{H^1(\Omega)}^2-\big\|\eta_\phi^n\big\|_{H^1(\Omega)}\right) \\ &\quad+\frac{\kappa\epsilon}{2}\left(\big\|\eta_{\phi_\Gamma}^{n+1}\big\|_{H^1(\Gamma)}-\big\|\eta_{\phi_\Gamma}^n\big\|_{H^1(\Gamma)}\right)+\big(\rho_r^{n+1}\big)^2-\big(\rho_r^n\big)^2+\big(\rho_{r_\Gamma}^{n+1}\big)^2-\big(\rho_{r_\Gamma}^n\big)^2 \end{aligned}$$

$$
\begin{aligned}
&\le C\left(h^{2q}+(\Delta t)^2\right)+\Delta t\left\{\frac{3\epsilon}{5}\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2+\frac{\epsilon}{5}\left\|\eta_\mu^{n}\right\|_{H^1(\Omega)}^2+\frac{\epsilon}{5}\left\|\eta_\phi^{n+1}\right\|_{H^1(\Omega)}^2\right.\\
&\qquad+\frac{\epsilon}{2}\left\|\eta_\phi^{n}\right\|_{H^1(\Omega)}^2+\frac{3\epsilon}{5}\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2+\frac{\epsilon}{16}\left\|\eta_{\mu_\Gamma}^{n}\right\|_{H^1(\Gamma)}^2\\
&\qquad\left.+\frac{\kappa\epsilon}{5}\left\|\eta_{\phi_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2+\frac{3\kappa\epsilon}{10}\left\|\eta_{\phi_\Gamma}^{n}\right\|_{H^1(\Gamma)}^2\right\}. \qquad (4.65)
\end{aligned}
$$

For any positive integer l $(0\le l\le N-1)$, summing (4.65) up from $n=0$ to l and applying the discrete Gronwall's inequality, we arrive at

$$
\begin{aligned}
&\frac{\epsilon}{2}\left\|\eta_\phi^{l+1}\right\|_{H^1(\Omega)}^2+\frac{\kappa\epsilon}{2}\left\|\eta_{\phi_\Gamma}^{l+1}\right\|_{H^1(\Gamma)}^2+\left(\rho_r^{l+1}\right)^2+\left(\rho_{r_\Gamma}^{l+1}\right)^2\\
&\quad+\frac{\Delta t\epsilon}{5}\sum_{n=1}^{l}\left(\left\|\eta_\mu^{n+1}\right\|_{H^1(\Omega)}^2+\left\|\eta_{\mu_\Gamma}^{n+1}\right\|_{H^1(\Gamma)}^2\right)\\
&\le C\big(h^{2q}+(\Delta t)^2\big). \qquad (4.66)
\end{aligned}
$$

Finally, with the help of the error estimates of Ritz projection, we complete the proof of Theorem 4.1. □

5. Numerical Results

In this section we perform a numerical accuracy check for the proposed numerical scheme (3.1)-(3.8). The computational domain is chosen as $\Omega=(0,1)^2$, and the exact profiles for the phase variable, the velocity vector, the pressure field, are set to be

$$
\begin{aligned}
\phi_e(x,y,t)&=\frac{1}{2\pi}\big(\sin(2\pi x)\cos(2\pi y),\cos(2\pi x)\sin(2\pi y)\big)^T\cos(t),\\
\mathbf{u}_e(x,y,t)&=\frac{1}{2\pi}\big(-\sin(2\pi x)\cos(2\pi y),\cos(2\pi x)\sin(2\pi y)\big)^T\cos(t), \qquad (5.1)\\
p_e(x,y,t)&=\frac{1}{2\pi}\cos(2\pi x)\cos(2\pi y)\cos(t).
\end{aligned}
$$

Of course, the chemical potential is given by $\mu_e=-\epsilon\Delta\phi_e+f(\phi_e)$, and the boundary profiles are determined as $(\phi|_\Gamma)_e=(\phi_e)_\Gamma$ and $(\mu|_\Gamma)_e=(\mu_e)_\Gamma$. The physical parameters are taken as $\varepsilon=0.5$, $\kappa=1$ and $\gamma=1$. To make $(\phi_e,\mathbf{u}_e,p_e)$ satisfy the original PDE system (1.9)-(1.16), we have to add an artificial, time-dependent forcing term. Due to the square domain, the linear element with a uniform triangular mesh is used. In this set-up, the proposed scheme (3.1)-(3.8) could be efficiently implemented with the help of FFT.

In the accuracy check for the temporal accuracy, we fix the spatial resolution as $N=256$ (with $h=1/256$), so that there are 2×256^2 uniform triangular meshes. Because of this fine mesh, the spatial numerical error is negligible. The final time is set as $T=1$. Naturally, a sequence of time step sizes are taken as $\Delta t=T/N_T$, with $N_T=100:100:1000$. The expected temporal numerical accuracy assumption $e=C\Delta t$ indicates that $\ln|e|=\ln(CT)-\ln N_T$, so that we plot $\ln|e|$ vs. $\ln N_T$ to demonstrate the temporal convergence order. The fitted line displayed in Fig. 5.1 shows an approximate slope of -0.9757, -0.9993, -0.9993, for the phase variable and the two velocity component variables, respectively. This which in turn verifies a very nice first order temporal convergence order, for all the physical variables, in both the discrete L^2 and L^∞ norms.

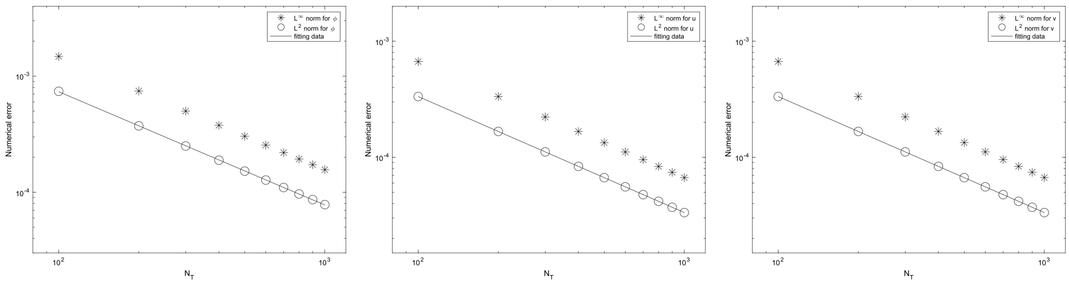

Fig. 5.1. The discrete L^2 and L^∞ numerical errors vs. temporal resolution N_T for $N_T = 100 : 100 : 1000$, with a spatial resolution $N = 256$, so that there are 2×256^2 uniform triangular meshes. The numerical results are obtained by the computation using the proposed scheme (3.1)-(3.8). The physical parameters are taken as $\varepsilon = 0.5$, $\kappa = 1$ and $\gamma = 1$. The numerical errors for the three physical variables ϕ, u and v, are displayed. The data lie roughly on curves CN_T^{-1}, for appropriate choices of C, confirming the full first-order temporal accuracy of the scheme.

Notice that the dynamical boundary condition has been applied in the numerical scheme (3.1)-(3.8), although the exact profile (5.1) also satisfies the homogeneous Neumann boundary condition. In fact, if the homogeneous Neumann boundary condition is imposed for the physical system, the corresponding numerical system becomes much simpler, and no coupling between the interior solution and boundary profile is needed. With either the homogeneous Neumann boundary condition and the dynamical boundary condition, the full first order temporal accuracy is valid, while the one with the dynamical boundary condition corresponds to a larger convergence constant, as expected.

Another interesting issue is the dependence of the numerical solution on the interfacial width parameter ϵ. Our numerical experiments have revealed that, the perfect convergence rate still preserves up to $\epsilon = 0.2$ at the final time $T = 1$, while such a perfect convergence rate is not available for smaller ϵ. Meanwhile, as validated by the theoretical analysis, the numerical stability is still preserved for smaller ϵ, although a perfect numerical convergence rate may not be valid at $O(1)$ time scale. Also see the related work [69], which reported that a smaller time step size is needed for the SAV numerical simulation of certain challenging physical model (such as functionalized Cahn-Hilliard gradient flow) to achieve a desired numerical accuracy.

6. Conclusions

In this paper, the Cahn-Hilliard-Hele-Shaw (CHHS) system is considered with dynamic boundary conditions, in which some influences of the boundary to the bulk dynamics are taken into account. To facilitate the nonlinear analysis of the physical model, we adopt the SAV formulation to construct an equivalent system (2.3)-(2.12), which is equivalent to the original PDE system (1.9)-(1.16). The mass conservation identity and and energy dissipation law could be similarly established at a PDE level. Furthermore, we propose the fully discrete SAV numerical scheme, with mixed finite element spatial approximation. The discrete mass conservation and energy dissipation law have been proved by a careful analysis. In addition, the convergence analysis and error estimate is performed for the proposed finite element scheme, with the help of Ritz projection estimate, a few appropriate regularization assumptions, as well as detailed stability analysis for the nonlinear error terms.

Acknowledgments. C. Yao is supported by NSFC (Grant No. 11871441), and C. Wang is supported by NSF (Grant No. DMS-2012669).

References

[1] X. Antoine, J. Shen and Q.L. Tang, Scalar Auxiliary Variable/Lagrange multiplier based pseudospectral schemes for the dynamics of nonlinear Schrödinger/Gross-Pitaevskii equations, *J. Comput. Phys.*, **437** (2021), Paper No. 110328, 19.

[2] X.L. Bao and H. Zhang, Numerical approximations and error analysis of the Cahn-Hilliard equation with dynamic boundary conditions, *Commun. Math. Sci.*, **19**:3 (2021), 663–685.

[3] X.L. Bao and H. Zhang, Numerical approximations and error analysis of the Cahn-Hilliard equation with reaction rate dependent dynamic boundary conditions, *J. Sci. Comput.*, **87**:3 (2021), Paper No. 72, 32. DOI:10.1007/s10915-021-01475-2.

[4] A. Baskaran, J.S. Lowengrub, C. Wang and S.M. Wise, Convergence analysis of a second order convex splitting scheme for the modified phase field crystal equation, *SIAM J. Numer. Anal.*, **51**:5 (2013), 2851–2873.

[5] P.W. Bates and P.C. Fife, The dynamics of nucleation for the Cahn-Hilliard equation, *SIAM J. Appl. Math.*, **53**:4 (1993), 990–1008.

[6] E. Celledoni, V. Grimm, R.I. McLachlan, D.I. McLaren, D. O'Neale, B. Owren and G.R.W. Quispel, Preserving energy resp. dissipation in numerical PDEs using the "average vector field" method, *J. Comput. Phys.*, **231**:20 (2012), 6770–6789.

[7] H.T. Chen and J.J. Mao and J. Shen, Optimal error estimates for the scalar auxiliary variable finite-element schemes for gradient flows, *Numer. Math.*, **145**:1 (2020), 167–196.

[8] W.B. Chen, W.Q. Feng, Y. Liu, C. Wang and S.M. Wise, A second order energy stable scheme for the Cahn-Hilliard-Hele-Shaw equations, *Discrete Contin. Dyn. Syst. Ser. B*, **24**:1 (2019), 149–182.

[9] W.B. Chen, D.Z. Han, C. Wang, S.F. Wang, X.M. Wang and Y.C. Zhang, Error estimate of a decoupled numerical scheme for the Cahn-Hilliard-Stokes-Darcy system, *IMA J. Numer. Anal.*, 2021, accepted and published online, 00, 1–35, `https://doi.org/10.1093/imanum/drab046`.

[10] W.B. Chen, W.J. Li, Z.W. Luo, C. Wang and X.M. Wang, A stabilized second order exponential time differencing multistep method for thin film growth model without slope selection, *Math. Model. Numer. Anal.*, **54**:3 (2020), 727–750.

[11] W.B. Chen, W.J. Li, C. Wang, S.F. Wang and X.M. Wang, Energy stable higher order linear ETD multi-step methods for gradient flows: application to thin film epitaxy, *Res. Math. Sci.*, **7**:3 (2020), Paper No. 13, 27.

[12] W.B. Chen, Y. Liu, C. Wang and S.M. Wise, Convergence analysis of a fully discrete finite difference scheme for Cahn-Hilliard-Hele-Shaw equation, *Math. Comp.*, **85**:301 (2016), 2231–2257.

[13] K.L. Cheng, Z.H. Qiao and C. Wang, A third order exponential time differencing numerical scheme for no-slope-selection epitaxial thin film model with energy stability, *J. Sci. Comput.*, **81**:1 (2019), 154–185.

[14] Q. Cheng and C. Wang, Error estimate of a second order accurate scalar auxiliary variable (SAV) numerical method for the epitaxial thin film equation, *Adv. Appl. Math. Mech.*, **13**:6 (2021), 1318–1354.

[15] Q. Cheng and J. Shen, Global constraints preserving scalar auxiliary variable schemes for gradient flows, *SIAM J. Sci. Comput.*, **42**:4 (2020), A2489–A2513.

[16] P.G. Ciarlet and J.-L. Lions, Handbook of Numerical Analysis. Vol. II, Finite Element Methods. Part 1, North-Holland, Amsterdam, (1991).

[17] C. Collins, J. Shen and S.M. Wise, An efficient, energy stable scheme for the Cahn-Hilliard-Brinkman system, *Commun. Comput. Phys.*, **13**:4 (2013), 929–957.

[18] P. Colli, T. Fukao and K.F. Lam, On a coupled bulk-surface Allen-Cahn system with an affine linear transmission condition and its approximation by a Robin boundary condition, *Nonlinear*

Anal., **184** (2019), 116–147.
[19] A.E. Diegel, X.B. Feng and S.M. Wise, Analysis of a mixed finite element method for a Cahn-Hilliard-Darcy-Stokes system, *SIAM J. Numer. Anal.*, **53**:1 (2015), 127–152.
[20] A.E. Diegel, C. Wang and S.M. Wise, Stability and convergence of a second order mixed finite element method for the Cahn-Hilliard equation, *IMA J. Numer. Anal.*, **36**:4 (2016), 1867–1897.
[21] A.E. Diegel, C. Wang, X.M. Wang and S.M. Wise, Convergence analysis and error estimates for a second order accurate finite element method for the Cahn-Hilliard-Navier-Stokes system, *Numer. Math.*, **137**:3 (2017), 495–534.
[22] Q. Du, L.L. Ju, X. Li and Z.H. Qiao, Maximum principle preserving exponential time differencing schemes for the nonlocal Allen-Cahn equation, *SIAM J. Numer. Anal.*, **57**:2 (2019), 875–898.
[23] X.L. Fan, J.S. Kou, Z.H. Qiao and S.Y. Sun, A componentwise convex splitting scheme for diffuse interface models with van der Waals and Peng-Robinson equations of state, *SIAM J. Sci. Comput.*, **39**:1 (2017), B1–B28.
[24] X.B. Feng and S. Wise, Analysis of a Darcy-Cahn-Hilliard diffuse interface model for the Hele-Shaw flow and its fully discrete finite element approximation, *SIAM J. Numer. Anal.*, **50**:3 (2012), 1320–1343.
[25] H.B. Frieboes, F. Jin, Y.L. Chuang, S.M. Wise, J.S. Lowengrub and V. Cristini, Three-dimensional multispecies nonlinear tumor growth—II:Tumor invasion and angiogenesis, *J. Theoret. Biol.*, **264**:4 (2010), 1254–1278.
[26] H. Garcke and P. Knopf, Weak solutions of the Cahn-Hilliard system with dynamic boundary conditions: a gradient flow approach, *SIAM J. Math. Anal.*, **52**:1 (2020), 340–369.
[27] Y.Z. Gong, J. Zhao and Q. Wang, Arbitrarily high-order unconditionally energy stable SAV schemes for gradient flow models, *Comput. Phys. Commun.*, **249** (2020), 107033, 11.
[28] Z. Guan, J.S. Lowengrub, C. Wang and S.M. Wise, Second order convex splitting schemes for periodic nonlocal Cahn-Hilliard and Allen-Cahn equations, *J. Comput. Phys.*, **277** (2014), 48–71.
[29] J. Guo, C. Wang, S.M. Wise and X.Y. Yue, An H^2 convergence of a second-order convex-splitting, finite difference scheme for the three-dimensional Cahn-Hilliard equation, *Commun. Math. Sci.*, **14**:2 (2016), 489–515.
[30] J. Guo, C. Wang, S.M. Wise and X.Y. Yue, An improved error analysis for a second-order numerical scheme for the Cahn-Hilliard equation, *J. Comput. Appl. Math.*, **388** (2021), Paper No. 113300, 16.
[31] R.H. Guo, Y.H. Xia and Y. Xu, An efficient fully-discrete local discontinuous Galerkin method for the Cahn-Hilliard-Hele-Shaw system, *J. Comput. Phys.*, **264** (2014), 23–40.
[32] J. Guzmán, D. Leykekhman, J. Rossmann and A.H. Schatz, Hölder estimates for Green's functions on convex polyhedral domains and their applications to finite element methods, *Numer. Math.*, **112**:2 (2009), 221–243.
[33] D.Z. Han, A decoupled unconditionally stable numerical scheme for the Cahn-Hilliard-Hele-Shaw system, *J. Sci. Comput.*, **66**:3 (2016), 1102–1121.
[34] D.Z. Han, A. Brylev, X.F. Yang and Z.J. Tan, Numerical analysis of second order, fully discrete energy stable schemes for phase field models of two-phase incompressible flows, *J. Sci. Comput.*, **70**:3 (2017), 965–989.
[35] F.K. Huang, J. Shen and Z.G. Yang, A highly efficient and accurate new scalar auxiliary variable approach for gradient flows, *SIAM J. Sci. Comput.*, **42**:4 (2020), A2514–A2536.
[36] L.L. Ju, X. Li, Z.H. Qiao and H. Zhang, Energy stability and error estimates of exponential time differencing schemes for the epitaxial growth model without slope selection, *Math. Comp.*, **87**:312 (2018), 1859–1885.
[37] L.L. Ju, J. Zhang and Q. Du, Fast and accurate algorithms for simulating coarsening dynamics of Cahn-CHilliard equations, *Comput. Mater. Sci.*, **108** (2015), 272–282.
[38] D. Kay, V. Styles and R Welford, Finite element approximation of a Cahn-Hilliard-Navier-Stokes system, *Interfaces Free Bound.*, **10**:1 (2008), 15–43.

[39] J. Kim, A diffuse-interface model for axisymmetric immiscible two-phase flow, *Appl. Math. Comput.*, **160**:2 (2005), 589–606.

[40] P. Knopf and A. Signori, On the nonlocal Cahn-Hilliard equation with nonlocal dynamic boundary condition and boundary penalization, *J. Differential Equations*, **280** (2021), 236–291.

[41] P. Knopf, K.F. Lam, C. Liu and S. Metzger, Phase-field dynamics with transfer of materials: the Cahn-Hillard equation with reaction rate dependent dynamic boundary conditions, *ESAIM Math. Model. Numer. Anal.*, **55**:1 (2021), 229–282.

[42] H.G. Lee, J.S. Lowengrub and J. Goodman, Modeling pinchoff and reconnection in a Hele-Shaw cell.I. The models and their calibration, *Phys. Fluids*, **14**:2 (2002), 492–513.

[43] H.G. Lee, J.S. Lowengrub and J. Goodman, Modeling pinchoff and reconnection in a Hele-Shaw cell.II. Analysis and simulation in the nonlinear regime, *Phys. Fluids*, **14**:2 (2002), 514–545.

[44] X. Li, L.L. Ju and X.C. Meng, Convergence analysis of exponential time differencing schemes for the Cahn-Hilliard equation, *Commun. Comput. Phys.*, **26**:5 (2019), 1510–1529.

[45] X. Li, Z.H. Qiao and H. Zhang, A second-order convex splitting scheme for a Cahn-Hilliard equation with variable interfacial parameters, *J. Comput. Math.*, **35**:6 (2017), 693–710.

[46] M. Liero, Passing from bulk to bulk-surface evolution in the Allen-Cahn equation, *NoDEA Nonlinear Differential Equations Appl.*, **20**:3 (2013), 919–942.

[47] C. Liu and H. Wu, An energetic variational approach for the Cahn-Hilliard equation with dynamic boundary condition: Model derivation and mathematical analysis, *Arch. Rational Mech. Anal.*, **233**:1 (2019), 167–247.

[48] Q. Liu and D.Y. Shi, Superconvergence analysis of a two-grid method for an energy-stable Ciarlet-Raviart type scheme of Cahn-Hilliard equation, *Numer. Algorithms*, **85**:2 (2020), 607–622.

[49] Y. Liu, W. B. Chen, C. Wang and S.M. Wise, Error analysis of a mixed finite element method for a Cahn-Hilliard-Hele-Shaw system, *Numer. Math.*, **135**:3 (2017), 679–709.

[50] J.S. Lowengrub, J. Goodman, H. Lee, E.K. Longmire, M.J. Shelley and L. Truskinovsky, Topological transitions in liquid/liquid interfaces, Free boundary problems: theory and applications (Crete,1997), **409** (1999), 221–236.

[51] X.C. Meng, T.T.P. Hoang, Z. Wang and L.L. Ju, Localized exponential time differencing method for shallow water equations: algorithms and numerical study, *Commun. Comput. Phys.*, **29**:1 (2021), 80–110.

[52] S. Metzger, An efficient and convergent finite element scheme for Cahn-Hilliard equations with dynamic boundary conditions, *SIAM J. Numer. Anal.*, **59**:1 (2021), 219–248.

[53] A. Miranville and S. Zelik, The Cahn-Hilliard equation with singular potentials and dynamic boundary conditions, *Discrete Contin. Dyn. Syst.*, **28**:1 (2010), 275–310.

[54] Z.H. Qiao, C. Wang, S.M. Wise and Z.R. Zhang, Error analysis of a finite difference scheme for the epitaxial thin film model with slope selection with an improved convergence constant, *Int. J. Numer. Anal. Model.*, **14**:2 (2017), 283–305.

[55] G.R.W. Quispel and D.I. McLaren, A new class of energy-preserving numerical integration methods, *J. Phys. A*, **41**:4 (2008), 045206, 7.

[56] J. Shen, C. Wang, X.M. Wang and S.M. Wise, Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: application to thin film epitaxy, *SIAM J. Numer. Anal.*, **50**:1 (2012), 105–125.

[57] J. Shen, J. Xu and J. Yang, A new class of efficient and robust energy stable schemes for gradient flows, *SIAM Rev.*, **61**:3 (2019), 474–506.

[58] J. Shen, J. Xu and J. Yang, The scalar auxiliary variable (SAV) approach for gradient flows, *J. Comput. Phys.*, **353** (2018), 407–416.

[59] J. Shen and X.F. Yang, A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities, *SIAM J. Sci. Comput.*, **32**:3 (2010), 1159–1179.

[60] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Applied Mathe-

matical Sciences, Springer-Verlag, New York, 1997.

[61] V. Thomée, Galerkin Finite Element Methods for Parabolic Problems, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, 2006.

[62] M. Wang, C. Wang, Cheng and Q.M. Huang, A second order accurate scalar auxiliary variable (SAV) numerical method for the square phase field crystal equation, *J. Sci. Comput.*, **88**:2 (2021), Paper No. 33, 36.

[63] S.M. Wise, Unconditionally stable finite difference, nonlinear multigrid simulation of the Cahn-Hilliard-Hele-Shaw system of equations, *J. Sci. Comput.*, **44**:1 (2010), 38–68.

[64] S.M. Wise, J.S. Lowengrub, H.B. Frieboes and V. Cristini, Three-dimensional multispecies nonlinear tumor growth—I:Model and numerical method, *J. Theoret. Biol.*, **253**:3 (2008), 524–543.

[65] H. Wu and S.M. Zheng, Convergence to equilibrium for the Cahn-Hilliard equation with dynamic boundary conditions, *J. Differential Equations*, **204**:2 (2004), 511–531.

[66] X.F. Yang, Linear, first and second-order, unconditionally energy stable numerical schemes for the phase field model of homopolymer blends, *J. Comput. Phys.*, **327** (2016), 294–316.

[67] X.F. Yang and L.L. Ju, Efficient linear schemes with unconditional energy stability for the phase field elastic bending energy model, *Comput. Methods Appl. Mech. Engrg.*, **315** (2017), 691–712.

[68] X.F. Yang and L.L. Ju, Linear and unconditionally energy stable schemes for the binary fluid-surfactant phase field model, *Comput. Methods Appl. Mech. Engrg.*, **318** (2017), 1005–1029.

[69] C.H. Zhang, J. Ouyang, C. Wang and S.M. Wise, Numerical comparison of modified-energy stable SAV-type schemes and classical BDF methods on benchmark problems for the functionalized Cahn-Hilliard equation, *J. Comput. Phys.*, **423** (2020), 109772, 35.

[70] J. Zhao, Q. Wang and X.F. Yang, Numerical approximations for a phase field dendritic crystal growth model based on the invariant energy quadratization approach, *Internat. J. Numer. Methods Engrg.*, **110**:3 (2017), 279–300.