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THE PRIMITIVE EQUATIONS FORMULATED IN MEAN VORTICITY

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Abstract. The primitive equations (PEs) of large-scale oceanic flow formulated in mean vorticity is proposed. In the reformulation of the PEs, the prognostic equation for the horizontal velocity is replaced by evolutionary equations for the mean vorticity field and the vertical derivative of the horizontal velocity. The total velocity field (both horizontal and vertical) is statically determined by differential equations at each fixed horizontal point. Its equivalence to the original formulation is also presented.

1. **Introduction.** The primary purpose of this article is to provide an alternate formulation of the three-dimensional primitive equations (PEs) of large scale oceanic flow in terms of mean vorticity and standard transport-type variables.

The primitive equations (PEs) stand for one of the most fundamental governing equations for atmospheric and oceanic flow. This system is derived from the 3-D incompressible NSE under Boussinesq assumption that density variation is neglected except in the buoyancy term, combined with the asymptotic scaling such that the aspect ratio of the vertical to the horizontal length scale is small. The most distinguished feature of the PEs is that the hydrostatic balance replaces the momentum equation for the vertical velocity. As a result, the fast wave with respect to gravity effect is filtered out.

The averaged horizontal velocity field with respect to the vertical direction is divergence-free, due to the incompressibility of the flow and the vanishing vertical velocity at the top and bottom. Therefore, the concept of mean vorticity and mean stream function can be introduced so that the kinematic relationship between them is a 2-D Poisson equation. In addition, taking the vertical derivative to the original momentum equation converts the pressure gradient into the density gradient by using hydrostatic balance and gives an evolution equation for $\boldsymbol{v}_z$. Thus, the whole system of the PEs can be reformulated in terms of mean vorticity evolution equation, together with regular evolution equations for density and $\boldsymbol{v}_z$. The total velocity in horizontal direction is determined by the combined data of its vertical derivative and vertical average, which are updated from the dynamic evolution equations. The vertical velocity is solved by a second order O.D.E. with homogeneous boundary condition at the top and bottom, at each fixed horizontal point.

The detailed derivation of the reformulation is given in Section 2. Its equivalence to the original formulation for smooth solutions is provided in Section 3.

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2. **Review of the Mean Vorticity Formulation for the Primitive Equations.** The non-dimensional primitive equations for the atmosphere and ocean can be written in the following system under proper scaling:

$$(2.1)\qquad \begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + w\dfrac{\partial \boldsymbol{v}}{\partial z} + \dfrac{1}{Ro}\Big(fk\times\boldsymbol{v} + \nabla p\Big) = \Big(\dfrac{1}{Re_1}\triangle + \dfrac{1}{Re_2}\partial_z^2\Big)\boldsymbol{v}\,, \\ \dfrac{\partial p}{\partial z} = -\rho\,, \\ \nabla\cdot\boldsymbol{v} + \partial_z w = 0\,, \\ \rho_t + (\boldsymbol{v}\cdot\nabla)\rho + w\dfrac{\partial\rho}{\partial z} = \Big(\dfrac{1}{Rt_1}\triangle + \dfrac{1}{Rt_2}\partial_z^2\Big)\rho\,, \end{cases}$$

supplemented with initial data

$$(2.2)\qquad \boldsymbol{v}(x,y,0) = \boldsymbol{v}_0(x,y)\,,\quad \rho(x,y,0) = \rho_0(x,y)\,.$$

In system (2.1), $\boldsymbol{u} = (\boldsymbol{v}, w) = (u,v,w)$ is the 3-D velocity vector field, $\boldsymbol{v} = (u,v)$ the horizontal velocity, ρ the density field, p the pressure, Ro the Rossby number. The term $fk\times\boldsymbol{v}$ corresponds to the Coriolis force in its $\beta-$plane approximation with $f = f_0 + \beta y$. The parameters Re_1, Re_2 represent the Reynolds numbers in horizontal and vertical directions respectively, which reflect different length scales and may also reflect the effects of eddy diffusion. Similarly, $\frac{1}{Rt_1}$ and $\frac{1}{Rt_2}$ stand for the horizontal and vertical heat conductivity coefficients. The operators ∇, $\nabla^\perp$, $\nabla\cdot$, $\triangle$ stand for the gradient, perpendicular gradient, divergence and Laplacian in horizontal plane, respectively. For simplicity of presentation below we denote $\nu_1 = \frac{1}{Re_1}$, $\nu_2 = \frac{1}{Re_1}$, $\kappa_1 = \frac{1}{Rt_1}$, $\kappa_2 = \frac{1}{Rt_1}$.

The system of the PEs (2.1) is derived from the Boussinesq approximation (i.e., the assumption that density variation is neglected except in the buoyancy term) of geophysical flow with asymptotic scaling. The most obvious difference between the PEs and the usual 3-D Boussinesq equations is the replacement of the momentum equation for the vertical velocity w by the hydrostatic balance $\frac{\partial p}{\partial z} = -\rho$.

The computational domain is taken as $\mathcal{M} = \mathcal{M}_0\times[-H_0, 0]$, where $\mathcal{M}_0$ is the surface part of the ocean. The boundary condition for (2.1) is given by

$$(2.3a)\qquad \begin{aligned} &w = 0\,,\quad \nu_2\frac{\partial\boldsymbol{v}}{\partial z} = \tau_0\,,\quad \kappa_2\frac{\partial\rho}{\partial z} = \rho_f\,,\quad \text{at}\quad z = 0\,,\\ &w = 0\,,\quad \nu_2\frac{\partial\boldsymbol{v}}{\partial z} = 0\,,\quad \kappa_2\frac{\partial\rho}{\partial z} = 0\,,\quad \text{at}\quad z = -H_0\,, \end{aligned}$$

$$(2.3b)\qquad \boldsymbol{v} = 0\,,\quad \text{and}\quad \frac{\partial\rho}{\partial\boldsymbol{n}} = 0\,,\quad \text{on}\ \partial\mathcal{M}_0\times[-H_0, 0]\,,$$

in which the term τ_0 represents the wind stress force, and ρ_f the heat flux at the surface of the ocean. The detailed description, derivation and analysis of the PEs in the above formulation were established by J. L. Lions, R. Temam and S. Wang in [6, 7, 8, 9, 10], etc.

2.1. **Evolution equation for mean vorticity.** Motivated by the fact that

$$(2.4)\qquad \int_{-H_0}^{0} (\nabla\cdot\boldsymbol{v})\,(x,y,\cdot)\,dz = 0\,,\quad \forall (x,y)\in\mathcal{M}_0\,,$$

which comes from the integration of the continuity equation and the boundary condition for w at $z = 0, -H_0$, we arrive at the conclusion that the mean velocity field $\overline{\boldsymbol{v}} = (\overline{u}, \overline{v})$ is divergence-free in (x, y) plane

$$(\nabla \cdot \overline{\boldsymbol{v}})\,(x, y) = 0\,, \quad \forall (x, y) \in \mathcal{M}_0\,. \tag{2.5}$$

The incompressibility of $\overline{\boldsymbol{v}}$ in the horizontal plane leads to the introduction of the mean stream function $\overline{\psi}$, which is a 2-D field, such that

$$\overline{\boldsymbol{v}} = \nabla^{\perp}\overline{\psi} = (-\partial_y\overline{\psi}, \partial_x\overline{\psi})\,. \tag{2.6}$$

The Dirichlet boundary condition $\overline{\boldsymbol{v}}\,|_{\partial\mathcal{M}_0} = 0$ (because of the boundary condition for $\boldsymbol{v}$ on the lateral boundary section in (2.3b)) amounts to saying

$$\overline{\psi} = 0\,, \quad \frac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0\,. \tag{2.7}$$

Accordingly, the mean vorticity is defined by

$$\overline{\omega} = \nabla \times \overline{\boldsymbol{v}} = -\partial_y\overline{u} + \partial_x\overline{v}\,. \tag{2.8}$$

Therefore, the kinematic relationship between the mean stream function and the mean vorticity turns out to be the following 2-D Poisson equation

$$\triangle\overline{\psi} = \overline{\omega}\,. \tag{2.9}$$

Note that there are two boundary conditions for $\overline{\psi}$, both Dirichlet and Neumann, as in (2.7). Some relevant issues in the usual 2-D NSE have been discussed in earlier references [1, 2, 3, 11, 15, 16, 17], etc.

To obtain the dynamic equation for the mean vorticity, we observe that the momentum equations in (2.1) can be rewritten as

$$\begin{cases} u_t + (uu)_x + (uv)_y + (uw)_z - \dfrac{f}{Ro}v + \dfrac{1}{Ro}\partial_x p = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)u\,, \\ v_t + (uv)_x + (vv)_y + (vw)_z + \dfrac{f}{Ro}u + \dfrac{1}{Ro}\partial_y p = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)v\,, \end{cases} \tag{2.10}$$

because of the incompressibility of $\boldsymbol{u} = (u, v, w)$. Taking the average of the above equation over $[-H_0, 0]$ gives the dynamic equations for mean velocity field $\overline{\boldsymbol{v}}$

$$\overline{\boldsymbol{v}}_t + \nabla\cdot\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \frac{f}{Ro}k \times \overline{\boldsymbol{v}} + \frac{1}{Ro}\nabla\overline{p} = \nu_1\triangle\overline{\boldsymbol{v}} + \frac{1}{H_0}\tau_0\,, \tag{2.11}$$

where the average of the velocity tensor product $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}$ is defined by

$$\overline{\boldsymbol{v}\otimes\boldsymbol{v}} = \begin{pmatrix} \overline{uu} & \overline{uv} \\ \overline{uv} & \overline{vv} \end{pmatrix}. \tag{2.12}$$

Note that the terms $\overline{(uw)_z}$, $\overline{(vw)_z}$ disappear because of the boundary condition for w at $z = 0, -H_0$. Integration by parts, combined with the Neumann boundary condition for $\boldsymbol{v}$ at the top and bottom sections, was used to evaluate $\overline{\partial_z^2\boldsymbol{v}}$, which turns out to be the force term in (2.11).

Taking the curl operator $\nabla^{\perp}\cdot$ to (2.11), we cancel the mean pressure gradient term and arrive at the evolution equation for the mean vorticity

$$\overline{\omega}_t + (\nabla^{\perp}\cdot\nabla\cdot)\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \nabla^{\perp}\cdot\Big(\frac{f}{Ro}k \times \overline{\boldsymbol{v}}\Big) = \nu_1\triangle\overline{\omega} + \frac{1}{H_0}\,\nabla^{\perp}\cdot\tau_0\,, \tag{2.13}$$

which is a scalar equation. The nonlinear convection term can be written in the form of matrix product

$$(2.14)\qquad \begin{aligned} (\nabla^\perp\cdot\nabla\cdot)\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) &= \begin{pmatrix} -\partial_{xy} & -\partial_y^2 \\ \partial_x^2 & \partial_{xy} \end{pmatrix} : \begin{pmatrix} \overline{uu} & \overline{uv} \\ \overline{uv} & \overline{vv} \end{pmatrix} \\ &= \partial_{xy}\Big(-\overline{uu}+\overline{vv}\Big) + \Big(\partial_x^2-\partial_y^2\Big)\overline{uv}\,. \end{aligned}$$

The term corresponding to the Coriolis force can be evaluated by

$$(2.15)\qquad \nabla^\perp\cdot\Big(\frac{f}{Ro}k\times\overline{\boldsymbol{v}}\Big) = \frac{1}{Ro}\nabla^\perp\cdot\Big(\beta y k\times\overline{\boldsymbol{v}}\Big) = \frac{1}{Ro}\Big((\beta y\overline{v})_y + (\beta y\overline{u})_x\Big) = \frac{\beta}{Ro}\overline{v}\,,$$

where the expression of the parameter $f = f_0 + \beta y$ and the incompressibility of $\overline{\boldsymbol{v}}$ were used.

Then we have the following system in terms of mean vorticity and mean stream function

$$(2.16)\qquad \begin{cases} \overline{\omega}_t + (\nabla^\perp\cdot\nabla\cdot)\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \dfrac{\beta}{Ro}\overline{v} = \nu_1\triangle\overline{\omega} + \dfrac{1}{H_0}\nabla^\perp\cdot\tau_0\,, \\ \triangle\overline{\psi} = \overline{\omega}\,, \\ \overline{\psi} = 0\,, \quad \dfrac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0\,, \quad \text{on} \ \ \partial\mathcal{M}_0\,, \\ \overline{\boldsymbol{v}} = \nabla^\perp\overline{\psi} = (-\partial_y\overline{\psi}, \partial_x\overline{\psi})\,. \end{cases}$$

It should be noted that (2.16) is not a closed system for the mean profiles $\overline{\omega}$, $\overline{\psi}$, $\overline{\boldsymbol{v}}$, since in the nonlinear convection term, $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}$ is not equal to $\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}$. To update the total velocity field $\boldsymbol{v}$, we need additional information of $\boldsymbol{v}_z$.

2.2. **Evolutionary equation for $\boldsymbol{v}_z = (\xi,\zeta)$.** Taking the vertical derivative of the momentum equations (2.10) leads to the following system for $\boldsymbol{v}_z = (\xi,\zeta)$, with Dirichlet boundary condition on all boundary sections

$$(2.17a)\qquad \begin{cases} \partial_t\boldsymbol{v}_z + \mathcal{N}LF + \dfrac{f}{Ro}k\times\boldsymbol{v}_z - \dfrac{1}{Ro}\nabla\rho = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)\boldsymbol{v}_z\,, \\ \boldsymbol{v}_z\mid_{z=0} = \dfrac{1}{\nu_1}\tau_0\,, \quad \boldsymbol{v}_z\mid_{z=-H_0} = 0\,, \\ \boldsymbol{v}_z = 0\,, \quad \text{on} \quad \partial\mathcal{M}_0\times[-H_0,0]\,. \end{cases}$$

where the nonlinear term $\mathcal{N}LF = (f_1, f_2)$ is evaluated as the following by using the incompressibility condition $u_x + v_y + w_z = 0$:

$$(2.17b)\qquad \begin{aligned} f_1 &= \partial_z\Big(uu_x + vu_y + wu_z\Big) = u\xi_x + v\xi_y + w\xi_z - v_y\xi + u_y\zeta\,, \\ f_2 &= \partial_z\Big(uv_x + vv_y + wv_z\Big) = u\zeta_x + v\zeta_y + w\zeta_z - u_x\zeta + v_x\xi\,. \end{aligned}$$

2.3. **Recovery of total velocity field $\boldsymbol{u} = (\boldsymbol{v}, w)$.** Given the combined data of $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}_z$, which can be obtained by solving (2.16), (2.17), respectively, the horizontal velocity field can be determined by the following system of ordinary differential equations:

$$
\left\{
\begin{aligned}
&\partial_z \boldsymbol{v} = \partial_z \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \\
&\frac{1}{H_0}\int_{-H_0}^{0} \boldsymbol{v}\, dz = \overline{\boldsymbol{v}}\,.
\end{aligned}
\right. \tag{2.18}
$$

In addition, by taking the z direction derivative of the continuity equation $\nabla\cdot\boldsymbol{v}+\partial_z w = 0$, we arrive at the following system of second order O.D.E. for the vertical velocity with the vanishing Dirichlet boundary condition

$$
\left\{
\begin{aligned}
&\partial_z^2 w = -\nabla\cdot\boldsymbol{v}_z = -\xi_x - \zeta_y\,, \\
&w = 0\,, \qquad \text{at} \qquad z = 0\,, -H_0\,.
\end{aligned}
\right. \tag{2.19}
$$

Both (2.18) and (2.19) can be solved at any fixed horizontal point (x, y).

2.4. **The reformulation of the PEs.** Then we have the following system of the PEs formulated in mean vorticity.

Mean vorticity equation

$$
\left\{
\begin{aligned}
&\overline{\omega}_t + (\nabla^\perp\cdot\nabla\cdot)\left(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\right) + \frac{\beta}{Ro}\overline{v} = \nu_1\triangle\overline{\omega} + \frac{1}{H_0}\,\nabla^\perp\cdot\tau_0\,, \\
&\triangle\overline{\psi} = \overline{\omega}\,, \\
&\overline{\psi} = 0\,, \quad \frac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0\,, \quad \text{on} \ \ \partial\mathcal{M}_0\,, \\
&\overline{\boldsymbol{v}} = \nabla^\perp\overline{\psi} = (-\partial_y\overline{\psi}, \partial_x\overline{\psi})\,,
\end{aligned}
\right. \tag{2.20a}
$$

Evolutionary equation for $\boldsymbol{v}_z = (\xi, \zeta)$

$$
\left\{
\begin{aligned}
&\boldsymbol{v}_{zt} + \mathcal{N}LF + \frac{f}{Ro}k\times\boldsymbol{v}_z - \frac{1}{Ro}\nabla\rho = \left(\nu_1\triangle + \nu_2\partial_z^2\right)\boldsymbol{v}_z\,, \\
&\boldsymbol{v}_z\mid_{z=0} = \frac{1}{\nu_2}\tau_0\,, \quad \boldsymbol{v}_z\mid_{z=-H_0} = 0\,, \\
&\boldsymbol{v}_z = 0\,, \quad \text{on} \quad \partial\mathcal{M}_0\times[-H_0, 0]\,,
\end{aligned}
\right. \tag{2.20b}
$$

Recovery of the horizontal velocity

$$
\left\{
\begin{aligned}
&\partial_z u = \xi\,, \qquad \partial_z v = \zeta\,, \\
&\frac{1}{H_0}\int_{-H_0}^{0}\boldsymbol{v}\,dz = \overline{\boldsymbol{v}}\,.
\end{aligned}
\right. \tag{2.20c}
$$

Recovery of the vertical velocity

$$
\left\{
\begin{aligned}
&\partial_z^2 w = -\nabla\cdot\boldsymbol{v}_z = -\xi_x - \zeta_y\,, \\
&w = 0\,, \qquad \text{at} \qquad z = 0\,, -H_0\,.
\end{aligned}
\right. \tag{2.20d}
$$

Density transport equation

$$\text{(2.20e)} \qquad \begin{cases} \rho_t + (\boldsymbol{v}\cdot\nabla)\rho + w\dfrac{\partial \rho}{\partial z} = \Big(\kappa_1\triangle + \kappa_2\partial_z^2\Big)\rho\,, \\ \dfrac{\partial \rho}{\partial z}\,|_{z=0} = \dfrac{\rho_f}{\kappa_2}\,, \quad \dfrac{\partial \rho}{\partial z}\,|_{z=-H_0} = 0\,, \quad \dfrac{\partial \rho}{\partial \boldsymbol{n}}\,|_{\partial\mathcal{M}_0\times[-H_0,0]} = 0\,. \end{cases}$$

3. **Equivalence of the PEs in Original and Mean Vorticity Formulation.** From the derivation in Section 1, it is obvious that the any smooth solution of the PEs formulated in (2.1)-(2.3) satisfies the system of the alternate formulation (2.20). To prove the converse, we need to apply the recovery equation (2.20c) for $\boldsymbol{v}$, along with the evolution equations for $\overline{\boldsymbol{v}}$, $\boldsymbol{v}_z$, respectively, and derive the corresponding momentum equation for $\boldsymbol{v}$. We see that the mean vorticity equation is the same as

$$\text{(3.1)} \qquad \nabla^{\perp}\cdot\left\{\overline{\boldsymbol{v}}_t + \nabla\cdot\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \frac{1}{Ro}(fk\times\overline{\boldsymbol{v}}) - \nu_1\triangle\overline{\boldsymbol{v}} - \nu_2\partial_z^2\overline{\boldsymbol{v}}\right\} = 0\,,$$

where the kinematic relationship that $\overline{\omega} = \nabla\times\overline{\boldsymbol{v}}$, the integration of $\partial_z^2\overline{\boldsymbol{v}}$ in vertical direction and the boundary condition of $\boldsymbol{v}_z$ in (2.20b) are used. Note that the term inside the bracket of (3.1) is a 2-D vector field. Then we conclude from (3.1) that the term must be a gradient, i.e.,

$$\text{(3.2)} \qquad \overline{\boldsymbol{v}}_t + \nabla\cdot\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \frac{1}{Ro}(fk\times\overline{\boldsymbol{v}}) - \nu_1\triangle\overline{\boldsymbol{v}} - \nu_2\partial_z^2\overline{\boldsymbol{v}} + \nabla\hat{p} = 0\,.$$

Later we will show that $\hat{p}$, which is a 2-D scalar field, is exactly $\dfrac{\overline{p}}{Ro}$ up to a constant.

The recovery equations (2.20c), (2.20d) indicate

$$\text{(3.3)} \qquad \partial_z(\nabla\cdot\boldsymbol{v} + \partial_z w) = \nabla\cdot(\partial_z\boldsymbol{v}) + \partial_z^2 w = \partial_x\xi + \partial_y\zeta + \partial_z^2 w = 0\,.$$

Meanwhile, the average of the term inside the parenthesis of (3.3) also vanishes at each horizontal point

$$\text{(3.4)} \qquad \int_{-H_0}^{0}(\nabla\cdot\boldsymbol{v} + \partial_z w)\,dz = H_0\,\nabla\cdot\overline{\boldsymbol{v}} + w(\cdot,0) - w(\cdot,-H_0) = H_0\,\nabla\cdot\overline{\boldsymbol{v}} = 0\,,$$

in which the second step comes from the vanishing boundary condition for the vertical velocity given by (2.20d), the last step is based on the kinematic relationship $\overline{\boldsymbol{v}} = (-\partial_y\overline{\psi}, \partial_x\overline{\psi})$. The combination of (3.3) and (3.4) results in

$$\text{(3.5)} \qquad \nabla\cdot\boldsymbol{v} + \partial_z w = 0\,,$$

i.e., the 3-D total velocity vector field $\boldsymbol{u} = (\boldsymbol{v}, w)$ is divergence-free and it is compatible with the homogeneous boundary condition for the vertical velocity field given in (2.20d). As a result of (3.5), we have the following identities

$$\text{(3.6)} \qquad \begin{aligned} \overline{(uu)_x + (uv)_y} &= \overline{(uu)_x + (uv)_y + (uw)_z} = \overline{uu_x + vu_y + wu_z}\,, \\ \overline{(uv)_x + (vv)_y} &= \overline{(uv)_x + (vv)_y + (vw)_z} = \overline{vu_x + vv_y + wv_z}\,. \end{aligned}$$

The first step is true since $\overline{(uw)_z} = \overline{(uw)_z} = 0$ due to the boundary condition for w at $z = 0, -H_0$; the second step is true because of the incompressibility of $\boldsymbol{u} = (u, v, w)$. The combination of (3.2) and (3.6) gives

$$\text{(3.7)} \qquad \overline{\boldsymbol{v}}_t + \begin{pmatrix} \overline{uu_x + vu_y + wu_z} \\ \overline{vu_x + vv_y + wv_z} \end{pmatrix} + \frac{1}{Ro}(fk\times\overline{\boldsymbol{v}}) + \nabla\hat{p} = \nu_1\triangle\overline{\boldsymbol{v}} + \nu_2\partial_z^2\overline{\boldsymbol{v}}\,.$$

On the other hand, the incompressibility of $\boldsymbol{u} = (u, v, w)$ indicates that the nonlinear term $\mathcal{N}LF$ appearing in the dynamic equation for $\boldsymbol{v}_z = (\xi, \zeta)$ can be expressed in the form as in (2.17b), thus (2.20b) can be rewritten as

$$(3.8)\quad \partial_t \boldsymbol{v}_z + \begin{pmatrix} \partial_z\Big(uu_x + vu_y + wu_z\Big) \\ \partial_z\Big(uv_x + vv_y + wv_z\Big) \end{pmatrix} + \frac{f}{Ro} k \times \boldsymbol{v}_z - \frac{1}{Ro}\nabla\rho = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)\boldsymbol{v}_z\,.$$

In addition, the recovery equation (2.20c) amounts to saying

$$(3.9)\quad \boldsymbol{v}(x,y,z) = \int_{-H_0}^{z} \boldsymbol{v}_z(x,y,s)\,ds + \overline{\boldsymbol{v}}(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \boldsymbol{v}_z(x,y,s)\,ds\,dz\,,$$

since every variable can be uniquely determined by the combined data of its average in vertical direction and its vertical derivative via an integration formula. Taking the temporal derivative of (3.9) gives
(3.10)

$$\boldsymbol{v}_t(x,y,z) = \int_{-H_0}^{z} \boldsymbol{v}_{zt}(x,y,s)\,ds + \overline{\boldsymbol{v}}_t(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \boldsymbol{v}_{zt}(x,y,s)\,ds\,dz\,.$$

The combination of (3.2), (3.8), (3.10) leads to

$$(3.11a)\qquad \boldsymbol{v}_t + \begin{pmatrix} uu_x + vu_y + wu_z \\ uv_x + vv_y + wv_z \end{pmatrix} + \frac{f}{Ro} k \times \boldsymbol{v} - \frac{1}{Ro}\nabla p = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)\boldsymbol{v}\,,$$

where the total pressure p turns out to be
(3.11b)

$$p(x,y,z) = -\int_{-H_0}^{z} \rho(x,y,s)\,ds + Ro\,\hat{p}(x,y) + \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \rho(x,y,s)\,ds\,dz\,.$$

It can be seen that if $\hat{p}(x,y)$ appearing in (3.7) is denoted as $\frac{1}{Ro}\overline{p}(x,y)$, the expression of pressure variable p stands for exactly its representation formula. Then we proved the equivalence of (3.11) and the momentum equation in the original PEs (2.1).

The hydrostatic balance $\frac{\partial p}{\partial z} = -\rho$ is a direct consequence of the representation formula for the total pressure in (3.11b). The incompressibility $\nabla \cdot \boldsymbol{v} + \partial_z w = 0$ was proven in (3.5). The density equation in (2.20e) is the same as the one in (2.1). The boundary conditions for the density field on all the boundary sections as in (2.3a), (2.3b) are included in system (2.20). The boundary condition for the horizontal velocity field, including both the Neumann-type at the top and bottom as in (2.3a) and Dirichlet-type as in (2.3b), is indicated by the combination of the recovery equation for $\boldsymbol{v}$ in (2.20c) and the Dirichlet boundary condition for $\boldsymbol{v}_z$ in the evolutionary equation (2.20b). The vanishing boundary condition for the vertical velocity at the top and bottom boundary sections as shown in (2.3a) is also given in the recovery equation (2.20d). Thus the equivalence between the original and alternate formulations of the PEs is proven.

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