DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS–SERIES B
Volume **4**, Number **3**, August **2004**

Website: http://AIMsciences.org

pp. **1143–1172**

CONVERGENCE ANALYSIS OF THE NUMERICAL METHOD FOR THE PRIMITIVE EQUATIONS FORMULATED IN MEAN VORTICITY ON A CARTESIAN GRID

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(Communicated by Jie Shen)

Abstract. A second order numerical method for the primitive equations (PEs) of large-scale oceanic flow formulated in mean vorticity is proposed and analyzed, and the full convergence in L^2 is established. In the reformulation of the PEs, the prognostic equation for the horizontal velocity is replaced by evolutionary equations for the mean vorticity field and the vertical derivative of the horizontal velocity. The total velocity field (both horizontal and vertical) is statically determined by differential equations at each fixed horizontal point. The standard centered difference approximation is applied to the prognostic equations and the determination of numerical values for the total velocity field is implemented by FFT-based solvers. Stability of such solvers are established and the convergence analysis for the whole scheme is provided in detail.

1. **Introduction.**

The primary purpose of this article is to give a detailed convergence analysis of a second order numerical method for the three-dimensional primitive equations (PEs) of large scale oceanic flow formulated in mean vorticity, using regular numerical grid.

The primitive equations (PEs) stand for one of the most fundamental governing equations for atmospheric and oceanic flow. A detailed derivation can be found at J. Pedlosky [19], J. L. Lions, R. Temam and S. Wang [12, 13, 16], etc. This system is derived from the 3-D incompressible NSEs under Boussinesq assumption that density variation is neglected except in the buoyancy term, combined with the asymptotic scaling such that the aspect ratio of the vertical to the horizontal length scale is small. The most distinguished feature of the PEs is that the hydrostatic balance replaces the momentum equation for the vertical velocity. As a result, the fast wave with respect to gravity effect is filtered out.

It is observed that the pressure gradient, the hydrostatic balance, are coupled together with the incompressibility of the 3-D velocity field $\boldsymbol{u}$. In addition, the vertical velocity has to be determined by an integration formula in terms of the divergence of the horizontal velocity field, since there is no momentum equation for the vertical velocity. As a result, the degree of nonlinearity of the primitive equations is even higher than that of the usual 3-D Navier-Stokes equations, due to lack of regularity for the vertical velocity field. Such nonlinearity is one of the main difficulties of the 3-D PEs, in both the PDE level and numerical analysis.

1991 *Mathematics Subject Classification.* 35Q35, 65M06, 65M12, 86A10.
Key words and phrases. Primitive equations, mean vorticity, mean stream function, convergence analysis.

The PDE analysis for the PEs can be found in earlier literatures [9, 10, 12, 13], etc. In those works the system is proven to be well-posed. Regarding the numerical issues, some computational methods based on velocity-pressure formulation have been proposed and analyzed in recent articles; see [22, 23], etc.

On the other hand, the development of a corresponding vorticity formulation for 3-D geophysical flow has not been as well studied. In the context of the 3-D PEs, the starting point of the vorticity formulation is the following: the averaged horizontal velocity field with respect to the vertical direction is divergence-free, namely (2.4) below. This allows the concept of a mean vorticity and mean stream function to be introduced so that the kinematic relationship between the two takes the form of a 2-D Poisson equation. In addition, by taking the vertical derivative to the original momentum equation and applying the hydrostatic balance, one can convert the pressure gradient into a density gradient, resulting in an evolution equation for $\boldsymbol{v}_z$. Thus, the entire PE system can be reformulated in terms of an evolution equation for the mean vorticity together with regular evolution equations for the density and $\boldsymbol{v}_z$. The total velocity in horizontal direction is then determined via a combination of its vertical derivative and its average from the top and bottom, which are updated using the dynamic evolution equations. The vertical velocity is then recovered from a second order ODE with homogeneous boundary conditions at the top and bottom at each fixed horizontal point. The above equations form an equivalent formulation of the PEs, namely the mean vorticity formulation. This formulation was reported in [27]. The derivation of the reformulation is reviewed in Section 2.

In Section 3, a second order centered-difference method based on the PEs formulated in mean vorticity is proposed and discussed in detail. The prognostic variables, including the mean vorticity field, the profile $\boldsymbol{v}_z$ and the density field, are updated by finite difference schemes applied to the dynamic equations. In turn, the total velocity field, both horizontal and vertical, are recovered by the mean velocity field and the dynamic variables $\boldsymbol{v}_z = (\xi, \zeta)$, using FFT-based solvers. The mean vorticity field on the lateral boundary are determined by the mean stream function field through a local formula. It can be seen that the regular numerical grid can be used for all physical variables, thus avoiding a more complicated staggered grid, such as 3-D MAC grid utilized in [22]. This is one of the main advantages of the mean vorticity formulation.

The basic idea of the convergence analysis for the proposed numerical scheme for the PEs in this alternate formulation is similar to that in [28]. There the 2-D NSEs formulated in terms of the vorticity are investigated. However, additional techniques are required due to special features of the reformulated PEs. The most distinguished feature of the numerical method is that the velocity field is not directly updated by an evolution equation. In Section 4, the accuracy analysis of the solvers for the velocity field is given. It is proven that the total velocity field $\boldsymbol{u} = (\boldsymbol{v}, w)$ generated by the scheme has the second order accuracy in $W^{1,\infty}$ norm.

In addition, lack of regularity for the vertical velocity in the PEs requires a more subtle consistency analysis. That is the main reason why the regularity requirement for the exact solution in Theorem 3.1 is higher than that in [28], where the usual 2-D NSEs are discussed. The details of the consistency analysis is given in Section 5. In the leading order expansion, the exact profile of the mean stream function and (ξ, ζ) are given, with the mean velocity and mean vorticity field determined by the finite differences of the mean stream function, and the approximate velocity profiles are constructed via the numerical procedure given in Section 3. Moreover, an $O(h^2)$ correction is added to the exact density profile to satisfy the discrete boundary condition to higher order. Due to the $W^{1,\infty}$ accuracy of the constructed velocity field, it is shown that the approximate solutions satisfy the numerical scheme of the PEs in the alternate formulation up to $O(h^2)$ error, including the boundary. Furthermore, higher order expansion is used so that the approximated solutions satisfy the numerical scheme up to $O(h^4)$ order. That makes possible the recovery of the L^∞ a-priori assumption, for both the horizontal and vertical velocity fields in the full nonlinear PE system, through the use of inverse inequalities in a 3-D setting.

Stability and error estimates are provided in Section 6, which show full second order convergence of the numerical scheme in L^2 norm. The analysis is based on the energy estimates for the error functions of the mean velocity, the profile (ξ, ζ) and the density. Standard summation by parts is applied on the regular numerical grid, with a careful treatment of the boundary conditions for the different physical variables. The inverse inequality is used to recover the L^∞ a-priori assumption for the numerical velocities.

2. Review of the mean vorticity formulation for the primitive equations.

The non-dimensional primitive equations for the atmosphere and ocean can be written in terms of the following system under proper scaling:

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + w\dfrac{\partial \boldsymbol{v}}{\partial z} + \dfrac{1}{Ro}\Big(fk\times\boldsymbol{v} + \nabla p\Big) = \Big(\dfrac{1}{Re_1}\triangle + \dfrac{1}{Re_2}\partial_z^2\Big)\boldsymbol{v}\,, \\ \dfrac{\partial p}{\partial z} = -\rho\,, \\ \nabla\cdot\boldsymbol{v} + \partial_z w = 0\,, \\ \rho_t + (\boldsymbol{v}\cdot\nabla)\rho + w\dfrac{\partial \rho}{\partial z} = \Big(\dfrac{1}{Rt_1}\triangle + \dfrac{1}{Rt_2}\partial_z^2\Big)\rho\,, \end{cases} \tag{2.1}$$

supplemented with the initial data

$$\boldsymbol{v}(x,y,0) = \boldsymbol{v}_0(x,y)\,, \quad \rho(x,y,0) = \rho_0(x,y)\,. \tag{2.2}$$

In the system (2.1), $\boldsymbol{u} = (\boldsymbol{v}, w) = (u, v, w)$ is the 3-D velocity vector field, $\boldsymbol{v} = (u, v)$ the horizontal velocity, ρ the density field, p the pressure, Ro the Rossby number. The term $fk \times \boldsymbol{v}$ corresponds to the Coriolis force in its $\beta-$plane approximation with $f = f_0 + \beta y$. The parameters Re_1, Re_2 represent the Reynolds numbers in horizontal and vertical directions respectively, which reflect different length scales and may also reflect the effects of eddy diffusion. Similarly, $\frac{1}{Rt_1}$ and $\frac{1}{Rt_2}$ stand for the horizontal and vertical heat conductivity coefficients. The operators ∇, $\nabla^\perp$, $\nabla\cdot$, $\triangle$ stand for the gradient, perpendicular gradient, divergence and Laplacian in horizontal plane, respectively. For simplicity of presentation below we denote $\nu_1 = \frac{1}{Re_1}$, $\nu_2 = \frac{1}{Re_1}$, $\kappa_1 = \frac{1}{Rt_1}$, $\kappa_2 = \frac{1}{Rt_1}$.

The computational domain is taken as $\mathcal{M} = \mathcal{M}_0 \times [-H_0, 0]$, where $\mathcal{M}_0$ is the surface part of the ocean. The boundary condition for (2.1) is given by

$$\begin{aligned} &w = 0\,, \quad \nu_2\frac{\partial \boldsymbol{v}}{\partial z} = \tau_0\,, \quad \kappa_2\frac{\partial \rho}{\partial z} = \rho_f\,, \quad \text{at} \quad z = 0\,, \\ &w = 0\,, \quad \nu_2\frac{\partial \boldsymbol{v}}{\partial z} = 0\,, \quad \kappa_2\frac{\partial \rho}{\partial z} = 0\,, \quad \text{at} \quad z = -H_0\,, \end{aligned} \tag{2.3a}$$

$$\boldsymbol{v} = 0\,, \quad \text{and} \quad \frac{\partial \rho}{\partial \boldsymbol{n}} = 0\,, \quad \text{on } \partial\mathcal{M}_0 \times [-H_0, 0]\,, \tag{2.3b}$$

in which the term τ_0 represents the wind stress force, and ρ_f the heat flux at the surface of the ocean. The detailed description, derivation and analysis of the PEs in the above formulation were established by J. L. Lions, R. Temam and S. Wang in [12, 13, 14, 16, 15, 17], etc. In this paper, the numerical method is based on the above boundary conditions.

2.1. **Introduction of mean vorticity, mean stream function and mean velocity.** Motivated by the fact that

$$\int_{-H_0}^{0} (\nabla\cdot\boldsymbol{v})\,(x,y,\cdot)\,dz = 0\,, \quad \forall (x,y)\in\mathcal{M}_0\,, \tag{2.4}$$

which comes from the integration of the continuity equation and the boundary condition for w at $z=0,-H_0$, we arrive at the conclusion that the mean velocity field $\overline{\boldsymbol{v}}=(\overline{u},\overline{v})$ is divergence-free in (x,y) plane

$$(\nabla\cdot\overline{\boldsymbol{v}})\,(x,y) = 0\,, \quad \forall (x,y)\in\mathcal{M}_0\,. \tag{2.5}$$

The incompressibility of $\overline{\boldsymbol{v}}$ in the horizontal plane results in an introduction of the mean stream function $\overline{\psi}$, which is a 2-D field, such that

$$\overline{\boldsymbol{v}} = \nabla^{\perp}\overline{\psi} = (-\partial_y\overline{\psi},\partial_x\overline{\psi})\,. \tag{2.6}$$

The Dirichlet boundary condition $\overline{\boldsymbol{v}}\mid_{\partial\mathcal{M}_0}=0$ (because of the boundary condition for $\boldsymbol{v}$ on the lateral boundary section in (2.3b)) amounts to saying

$$\overline{\psi}=0\,, \quad \frac{\partial\overline{\psi}}{\partial\boldsymbol{n}}=0\,, \quad \text{on} \ \ \partial\mathcal{M}_0\,. \tag{2.7}$$

Accordingly, the mean vorticity is defined as

$$\overline{\omega} = \nabla\times\overline{\boldsymbol{v}} = -\partial_y\overline{u} + \partial_x\overline{v}\,. \tag{2.8}$$

Therefore, the kinematic relationship between the mean stream function and the mean vorticity can be expressed as the following 2-D Poisson equation

$$\triangle\overline{\psi} = \overline{\omega}\,. \tag{2.9}$$

Note that there are two boundary conditions for $\overline{\psi}$, including both Dirichlet and Neumann, as in (2.7). This issue will be discussed below.

2.2. **The reformulation of the PEs.** We have the following system of the PEs formulated in mean vorticity.
Mean vorticity equation

$$\begin{cases} \overline{\omega}_t + (\nabla^{\perp}\cdot\nabla\cdot)\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) + \dfrac{\beta}{Ro}\overline{v} = \nu_1\triangle\overline{\omega} + \dfrac{1}{H_0}\nabla^{\perp}\cdot\tau_0\,, \\ \triangle\overline{\psi} = \overline{\omega}\,, \\ \overline{\psi}=0\,, \quad \dfrac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0\,, \quad \text{on} \ \ \partial\mathcal{M}_0\,, \\ \overline{\boldsymbol{v}} = \nabla^{\perp}\overline{\psi} = (-\partial_y\overline{\psi},\partial_x\overline{\psi})\,, \end{cases} \tag{2.10a}$$

Evolutionary equation for $\boldsymbol{v}_z=(\xi,\zeta)$

$$\begin{cases} \boldsymbol{v}_{zt} + \begin{pmatrix} u\xi_x + v\xi_y + w\xi_z - v_y\xi + u_y\zeta \\ u\zeta_x + v\zeta_y + w\zeta_z - u_x\zeta + v_x\xi \end{pmatrix} + \dfrac{f}{Ro} k \times \boldsymbol{v}_z - \dfrac{1}{Ro}\nabla\rho \\ \qquad = \Big(\nu_1\triangle + \nu_2\partial_z^2\Big)\boldsymbol{v}_z \,, \\ \boldsymbol{v}_z \mid_{z=0} = \dfrac{1}{\nu_2}\tau_0 \,, \quad \boldsymbol{v}_z \mid_{z=-H_0} = 0 \,, \\ \boldsymbol{v}_z = 0 \,, \quad \text{on} \quad \partial\mathcal{M}_0 \times [-H_0, 0] \,, \end{cases} \tag{2.10b}$$

Recovery of the horizontal velocity

$$\partial_z u = \xi \,, \qquad \partial_z v = \zeta \,, \quad \frac{1}{H_0}\int_{-H_0}^{0} \boldsymbol{v}\, dz = \overline{\boldsymbol{v}} \,. \tag{2.10c}$$

Recovery of the vertical velocity

$$\begin{cases} \partial_z^2 w = -\nabla \cdot \boldsymbol{v}_z = -\xi_x - \zeta_y \,, \\ w = 0 \,, \qquad \text{at} \qquad z = 0 \,, -H_0 \,. \end{cases} \tag{2.10d}$$

Density transport equation

$$\begin{cases} \rho_t + (\boldsymbol{v}\cdot\nabla)\rho + w\dfrac{\partial\rho}{\partial z} = \Big(\kappa_1\triangle + \kappa_2\partial_z^2\Big)\rho \,, \\ \dfrac{\partial\rho}{\partial z} \mid_{z=0} = \dfrac{\rho_f}{\kappa_2} \,, \quad \dfrac{\partial\rho}{\partial z} \mid_{z=-H_0} = 0 \,, \quad \dfrac{\partial\rho}{\partial \boldsymbol{n}} \mid_{\partial\mathcal{M}_0\times[-H_0,0]} = 0 \,. \end{cases} \tag{2.10e}$$

The detailed derivation of the above reformulation can be found in [27]. The mean vorticity equation (2.10a) is obtained by taking the curl operator $\nabla^{\perp}\cdot$ to the average of the momentum equation in (2.1). The average of the velocity tensor product $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}$ is defined as

$$\overline{\boldsymbol{v}\otimes\boldsymbol{v}} = \begin{pmatrix} \overline{uu} & \overline{uv} \\ \overline{uv} & \overline{vv} \end{pmatrix} \,, \tag{2.11}$$

and the nonlinear convection term in (2.10a) can be rewritten as

$$\begin{aligned} (\nabla^{\perp}\cdot\nabla\cdot)\Big(\overline{\boldsymbol{v}\otimes\boldsymbol{v}}\Big) &= \begin{pmatrix} -\partial_{xy} & -\partial_y^2 \\ \partial_x^2 & \partial_{xy} \end{pmatrix} : \begin{pmatrix} \overline{uu} & \overline{uv} \\ \overline{uv} & \overline{vv} \end{pmatrix} \\ &= \partial_{xy}\Big(-\overline{uu} + \overline{vv}\Big) + \Big(\partial_x^2 - \partial_y^2\Big)\overline{uv} \,. \end{aligned} \tag{2.12}$$

It should be noted that (2.10a) is not a closed system for the mean profiles $\overline{\omega}$, $\overline{\psi}$, $\overline{\boldsymbol{v}}$, since the nonlinear convection term $\overline{\boldsymbol{v}\otimes\boldsymbol{v}}$ is not equal to $\overline{\boldsymbol{v}}\otimes\overline{\boldsymbol{v}}$.

Taking the vertical derivative of the momentum equation leads to system (2.10b) for $\boldsymbol{v}_z = (\xi, \zeta)$, with Dirichlet boundary condition on all boundary sections.

With the combined data of $\overline{\boldsymbol{v}}$ and $\boldsymbol{v}_z$ at hand, which can be obtained by solving (2.10a), (2.10b), respectively, the horizontal velocity field can be determined by (2.10c), a system of ordinary differential equations.

In addition, by taking the vertical derivative of the continuity equation $\nabla \cdot \boldsymbol{v} + \partial_z w = 0$, we arrive at (2.10d), a system of second order ODE for the vertical velocity with the vanishing Dirichlet boundary condition. Both (2.10c) and (2.10d) can be solved at any fixed horizontal point (x, y).

The density transport equation (2.10e) is the same as that in (2.1)-(2.3). This finishes the derivation of the reformulation (2.10).

3. **The numerical scheme on the regular grid.**

For simplicity of presentation we consider the case of $\mathcal{M}_0 = [0,1]^2$, $H_0 = 1$. The regular uniform grid $\{x_i = i/N, y_j = j/N, z_k = k/N - 1,\ i,j,k = 0,1,\cdots,N\}$, with mesh size $\triangle x = \triangle y = \triangle z = h = \frac{1}{N}$, is used. Let $\tilde{D}_x$, $\tilde{D}_y$ $\tilde{D}_z$ represent the standard second-order centered-difference approximation to ∂_x, ∂_y and ∂_z, D_x^2, D_y^2, D_z^2 be second-order centered-difference approximations to ∂_x^2, ∂_y^2, ∂_z^2, $\triangle_h = D_x^2 + D_y^2$ the standard five-point Laplacian. In addition, the vertical average of any variable f being evaluated on regular grid points (x_i, y_j, z_k) is defined by trapezoid rule

$$\overline{\overline{f}}_{i,j} = \frac{1}{H_0} \sum_{k=0}^{N_z-1} \left(\frac{1}{2} \triangle z \, (f_{i,j,k} + f_{i,j,k+1}) \right). \tag{3.1}$$

3.1. **The numerical scheme.**

The application of second order centered-difference approximations to the reformulated PEs (2.10) leads to the following system. For simplicity of presentation below we set $\tau_0 = 0$.

$$\begin{cases} \partial_t \overline{\omega} + \tilde{D}_x \tilde{D}_y (\overline{\overline{vv}} - \overline{\overline{uu}}) + (D_x^2 - D_y^2)(\overline{\overline{uv}}) + \dfrac{\beta}{Ro} \overline{v} = \nu_1 \triangle_h \overline{\omega} \,, \\ \triangle_h \overline{\psi} = \omega \,, \qquad \overline{\psi} \mid_{\Gamma} = 0 \,, \\ \overline{u} = -\tilde{D}_y \overline{\psi} \,, \qquad \overline{v} = \tilde{D}_x \overline{\psi} \,, \\ \overline{\omega}_{i,0} = \dfrac{1}{h^2} (4 \overline{\psi}_{i,1} - \dfrac{1}{2} \overline{\psi}_{i,2}) \,, \end{cases} \tag{3.2a}$$

$$\begin{cases} \partial_t \xi + u \tilde{D}_x \xi + v \tilde{D}_y \xi + w \tilde{D}_z \xi - (\tilde{D}_y v) \xi + (\tilde{D}_y u) \zeta - \dfrac{f}{Ro} \zeta - \dfrac{1}{Ro} \tilde{D}_x \rho \\ \qquad = (\nu_1 \triangle_h + \nu_2 D_z^2) \xi \,, \\ \partial_t \zeta + u \tilde{D}_x \zeta + v \tilde{D}_y \zeta + w \tilde{D}_z \zeta - (\tilde{D}_x u) \zeta + (\tilde{D}_x v) \xi + \dfrac{f}{Ro} \xi - \dfrac{1}{Ro} \tilde{D}_y \rho \\ \qquad = (\nu_1 \triangle_h + \nu_2 D_z^2) \zeta \,, \\ \xi \mid_{z=-H_0} = 0 \,, \quad \zeta \mid_{z=-H_0} = 0 \,, \quad \xi \mid_{z=0} = 0 \,, \quad \zeta \mid_{z=0} = 0 \,, \\ \xi = 0 \,, \quad \zeta = 0 \,, \quad \text{on} \ \ \partial \mathcal{M}_0 \times [-H_0, 0] \,, \end{cases} \tag{3.2b}$$

$$\begin{cases} \tilde{D}_z u = \xi \,, \qquad \tilde{D}_z v = \zeta \,, \qquad \text{at} \qquad (i,j,k) \,, \quad 1 \le k \le N-1 \,, \\ \overline{\overline{u}}_{i,j} = \dfrac{1}{H_0} \displaystyle\sum_{k=0}^{N_z-1} \left(\dfrac{1}{2} \triangle z \, (u_{i,j,k} + u_{i,j,k+1}) \right) = \overline{u}_{i,j} \,, \\ \overline{\overline{v}}_{i,j} = \dfrac{1}{H_0} \displaystyle\sum_{k=0}^{N_z-1} \left(\dfrac{1}{2} \triangle z \, (v_{i,j,k} + v_{i,j,k+1}) \right) = \overline{v}_{i,j} \,, \\ (\tilde{D}_z \boldsymbol{v})_{i,j,0} = (\tilde{D}_z \boldsymbol{v})_{i,j,N} = 0 \,, \quad \text{i.e.,} \quad \boldsymbol{v}_{i,j,-1} = \boldsymbol{v}_{i,j,1} \,, \ \boldsymbol{v}_{i,j,N+1} = \boldsymbol{v}_{i,j,N-1} \,, \end{cases} \tag{3.2c}$$

$$\begin{cases} D_z^2 w = -\tilde{D}_x \xi - \tilde{D}_y \zeta \,, \qquad \text{at} \qquad (i,j,k) \,, \quad 1 \le k \le N_z - 1 \,, \\ w_{i,j,0} = w_{i,j,N} = 0 \,, \end{cases} \tag{3.2d}$$

$$\begin{cases} \partial_t\rho + u\tilde{D}_x\rho + v\tilde{D}_y\rho + w\tilde{D}_z\rho = \Big(\kappa_1\triangle_h + \kappa_2 D_z^2\Big)\rho\,, \\ \tilde{D}_z\rho\mid_{z=0} = \dfrac{\rho_f}{\kappa_2}\,, \quad \tilde{D}_z\rho\mid_{z=-H_0} = 0\,, \\ \dfrac{\partial\rho}{\partial\boldsymbol{n}} = 0\,, \quad \text{on } \partial\mathcal{M}_0\times[-H_0,0]\,. \end{cases} \tag{3.2e}$$

The dynamic equation for the mean vorticity is updated by (3.2a) at interior grid points (x_i, y_j), $i,j = 1,...N-1$. The profile $\boldsymbol{v}_z = (\xi,\zeta)$ is updated by (3.2b) at 3-D interior grid points (x_i, y_j, z_k), $i,j,k = 1,...N-1$. Note that Dirichlet boundary conditions for $\boldsymbol{v}_z$ are imposed on the six boundary sections. The discretization for the density equation can be implemented as in (3.2e) at 3-D numerical grid points (x_i, y_j, z_k), $i,j,k = 0,...N$, due to the Neumann boundary condition imposed for density field.

3.2. **Local boundary condition for the mean vorticity.** The difficulty of the treatment of the mean vorticity equation is similar to that of the standard 2-D NSEs formulated in vorticity-stream function. There are two boundary conditions for $\overline{\psi}$, including both Dirichlet and Neumann, as shown in (2.10a); while there is no explicit boundary condition for the vorticity, which is needed in the scheme (3.2a). A similar procedure is adopted to overcome this difficulty. First, the interior value of mean vorticity is updated by using (3.2a) with all the terms, including the nonlinear term, viscosity term and Coriolis term, treated explicitly. Subsequently, the mean stream function is solved by the implementation of the Dirichlet boundary condition $\overline{\psi} = 0$ on $\partial\mathcal{M}_0$

$$\begin{cases} \triangle_h\overline{\psi} = \overline{\omega}\,, \\ \overline{\psi}\mid_{\partial\mathcal{M}_0} = 0\,, \end{cases} \tag{3.3}$$

where only Sine transformation is needed. It can be seen that the Neumann boundary condition, $\frac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0$, cannot be enforced directly. Yet, it could be converted into the boundary condition for mean vorticity $\overline{\omega}$. Motivated by the fact that $\overline{\psi}\mid_{\partial\mathcal{M}_0} = 0$, we have the approximation for the mean vorticity on the boundary (say on Γ_x, $j = 0$)

$$\overline{\omega}_{i,0} = D_y^2\overline{\psi}_{i,0} = \frac{1}{h^2}(\overline{\psi}_{i,1} + \overline{\psi}_{i,-1}) = \frac{2\overline{\psi}_{i,1}}{h^2} - \frac{2}{h}\frac{\overline{\psi}_{i,1} - \overline{\psi}_{i,-1}}{2h}\,, \tag{3.4}$$

where $(i,-1)$ refers to a "ghost" grid point out of the computational domain. Taking the approximation identity $\dfrac{\overline{\psi}_{i,1} - \overline{\psi}_{i,-1}}{2h} = 0$, which implies that $\overline{\psi}_{i-1} = \overline{\psi}_{i,1}$, as a second order normal boundary condition for $(\partial_y\overline{\psi})_{i,0} = 0$, we arrive at **Thom's formula**

$$\overline{\omega}_{i,0} = \frac{2\overline{\psi}_{i,1}}{h^2}\,. \tag{3.5}$$

A similar derivation can be found in [26].

The vorticity boundary condition can also be determined by other approximation of $\overline{\psi}_{i,-1}$. For example, using a third order one-sided approximation for the normal boundary condition $\frac{\partial\overline{\psi}}{\partial\boldsymbol{n}} = 0$ gives

$$(\partial_y\overline{\psi})_{i,0} = \frac{-\overline{\psi}_{i,-1} + 3\overline{\psi}_{i,1} - \frac{1}{2}\overline{\psi}_{i,2}}{3h} = 0\,, \qquad \text{i.e. ,} \quad \overline{\psi}_{i,-1} = 3\overline{\psi}_{i,1} - \frac{1}{2}\overline{\psi}_{i,2}\,. \tag{3.6}$$

The substitution of (3.6) into the difference formula $\omega_{i,0} = \dfrac{1}{h^2}(\psi_{i,1} + \psi_{i,-1})$ as in (3.4) results in **Wilkes-Pearson's formula**

$$\overline{\omega}_{i,0} = \frac{1}{h^2}\Big(4\overline{\psi}_{i,1} - \frac{1}{2}\overline{\psi}_{i,2}\Big) . \tag{3.7}$$

A crucial point worthy of mention is that Thom's formula is only first order accurate for $\overline{\omega}$ on the boundary, while Wilkes' formula gives second order accuracy, by formal Taylor expansion. More sophisticated consistency analysis assures that both formulas combined with second order centered difference scheme is indeed second order accurate. See the relevant analysis of the 2-D NSEs in [8, 28]. This article will give a detailed analysis of such boundary condition in the context of 3-D PEs.

3.3. **Recovery for the horizontal velocity field.** The remaining work focuses on the discrete recovery of the horizontal and vertical velocity field, based on the differential equations (2.10c, d), which play a role of bridge between the total velocity field and the dynamic variables $\overline{\omega}$, ξ, ζ. Such a discretization is given by the numerical scheme (3.2c), which forms a linear system to solve for $u_{i,j,k}$, $v_{i,j,k}$, $0 \le k \le N$.

Note that for either velocity component, there are $N+1$ unknowns: $u_{i,j,k}$ or $v_{i,j,k}$, $k = 0, 1, 2, ...N$ at each fixed horizontal point (i,j), yet there are only N equations: $N-1$ in the first equation and 1 in the second one. Meanwhile, the last equation in (3.2c) only provides a choice for the numerical value for $\boldsymbol{v}$ at "ghost" computational points. We will give a methodology to overcome this difficulty below, and the second order accuracy (up to its finite difference) will also be shown.

System (3.2c) can be solved by using FFT. The Neumann type boundary condition (in a discrete level) for $\boldsymbol{v}$ at $k = 0, N$ suggests making Cosine transformation in z direction for each fixed grid (i,j), i.e.,

$$\begin{aligned} u_{i,j,k} &= \frac{1}{\sqrt{2N_z}}\left[\hat{u}_{i,j,0} + \sum_{l=1}^{N_z-1}(2\hat{u}_{i,j,l})\cos\Big(\frac{lk\pi}{N_z}\Big) + (-1)^k\hat{u}_{i,j,N}\right] , \\ v_{i,j,k} &= \frac{1}{\sqrt{2N_z}}\left[\hat{v}_{i,j,0} + \sum_{l=1}^{N_z-1}(2\hat{v}_{i,j,l})\cos\Big(\frac{lk\pi}{N_z}\Big) + (-1)^k\hat{v}_{i,j,N}\right] . \end{aligned} \tag{3.8}$$

Clearly the the horizontal velocity field satisfies the boundary condition in (3.2c). Then the remaining work focuses on the determination of the Fourier modes $\hat{\boldsymbol{v}}_{i,j,l}$, $l = 0, 1, ...N_z$, at each fixed grid point (i,j) by utilizing the difference equation and the constraint in (3.2c) for the mean velocity field. The application of the centered difference operator $\tilde{D}_z$ to each basis function in the Fourier expansion gives

$$\tilde{D}_z\hat{\boldsymbol{v}}_{i,j,0} = 0 , \qquad \tilde{D}_z\cos\Big(lk\pi h\Big) = f_l \cdot \sin\Big(lk\pi h\Big) , \qquad \tilde{D}_z\Big((-1)^k\Big) = 0 , \tag{3.9a}$$

in which the coefficient f_l turns out to be

$$f_l = -\frac{\sin\Big(l\pi h\Big)}{\triangle z} . \tag{3.9b}$$

The insertion of (3.9) into (3.8) results in

$$(\tilde{D}_z u)_{i,j,k} = \frac{2}{\sqrt{2N_z}} \sum_{l=1}^{N_z-1} f_l \hat{u}_{i,j,l} \sin\Big(\frac{lk\pi}{N_z}\Big) , \quad (\tilde{D}_z v)_{i,j,k} = \frac{2}{\sqrt{2N_z}} \sum_{l=1}^{N_z-1} f_l \hat{v}_{i,j,l} \sin\Big(\frac{lk\pi}{N_z}\Big) . \tag{3.10}$$

Meanwhile, the homogeneous Dirichlet boundary condition for ξ, ζ at $k = 0, N$ implies their Sine transformation in Fourier space

$$\xi_{i,j,k} = \sum_{l=1}^{N_z-1} \frac{2\hat{\xi}_{i,j,l}}{\sqrt{2N_z}} \sin\Big(\frac{lk\pi}{N_z}\Big) , \quad \zeta_{i,j,k} = \sum_{l=1}^{N_z-1} \frac{2\hat{\zeta}_{i,j,l}}{\sqrt{2N_z}} \sin\Big(\frac{lk\pi}{N_z}\Big) . \tag{3.11}$$

The comparison of (3.10) with (3.11) shows that the difference equation (3.8) is exactly satisfied if we set

$$\hat{u}_{i,j,l} = \frac{\hat{\xi}_{i,j,l}}{f_l} , \quad \hat{v}_{i,j,l} = \frac{\hat{\zeta}_{i,j,l}}{f_l} , \qquad \text{for} \qquad 1 \le l \le N_z - 1 , \tag{3.12}$$

with f_l given by (3.9b).

To obtain the 0-th Fourier mode coefficient for $\boldsymbol{v}$ at each fixed grid point (i, j), we note that the substitution of (3.8) into the constraint (3.2c) leads to

$$\frac{1}{H_0} N_z \triangle z \cdot \frac{1}{\sqrt{2N_z}} \hat{u}_{i,j,0} = \overline{u}_{i,j} , \qquad \frac{1}{H_0} N_z \triangle z \cdot \frac{1}{\sqrt{2N_z}} \hat{v}_{i,j,0} = \overline{v}_{i,j} , \tag{3.13}$$

due to the property that the basis functions $\cos\Big(kl\pi h\Big)$ have vanishing discrete average in the vertical direction, for $l = 1, 2, ...N_z$, provided that N_z is even. A direct indication of (3.13) gives

$$\hat{u}_{i,j,0} = \sqrt{2N_z}\overline{u}_{i,j} , \qquad \hat{v}_{i,j,0} = \sqrt{2N_z}\overline{v}_{i,j} . \tag{3.14}$$

In addition, it is well-known that the Fourier mode coefficient $\hat{\boldsymbol{v}}_{i,j,l}$ decays exponentially as l increases to N_z under suitable regularity assumption for the velocity field. As a result, the coefficient $\hat{\boldsymbol{v}}_{i,j,N}$ can be set to be 0, i.e.,

$$\hat{u}_{i,j,N} = \hat{v}_{i,j,N} = 0 . \tag{3.15}$$

Therefore, the combination of (3.8), (3.11), (3.12), (3.14) and (3.15) gives the procedure to solve for (3.2c).

3.4. **Recovery for the vertical velocity field.** The vertical velocity w can be solved by the centered-difference approximation to the second order O.D.E. (2.10d), namely (3.2d). At each fixed horizontal grid point (i, j), there are $N - 1$ equations and $N - 1$ unknowns: $w_{i,j,k}$ at interior grid points $1 \le k \le N_z - 1$. Moreover, the complete set of the eigenvalues corresponding to the operator D_z^2 (under the homogeneous Dirichlet boundary condition for w) is given by

$$\lambda_l = -\frac{4}{\triangle z^2} \sin^2\Big(\frac{l\pi}{2N_z}\Big) , \qquad \text{for} \quad 1 \le l \le N_z - 1 , \tag{3.16}$$

which are non-zero. Hence (3.2d) is a non-singular linear system at each fixed horizontal grid point. Such system can be solved by using FFT in vertical direction.

3.5. **Main theorem and some notation.** The notation of L^2 norms in a discrete level needs to be introduced before the statement of the main convergence theorem in this paper.

Notation 3.1 *For any pair of variables f, g which are evaluated at the 3-D mesh points, the following discrete L^2-inner product are given*

$$\langle f\,,\,g\rangle_0 = h^3 \sum_{k=1}^{N-1}\sum_{j=1}^{N-1}\sum_{i=1}^{N-1} f_{i,j,k}\, g_{i,j,k}\,, \tag{3.17a}$$

$$\begin{aligned}\langle f\,,\,g\rangle_3 = \quad & h^2\Big(\sum_{j=1}^{N-1}\sum_{i=1}^{N-1}\langle f\,,\,g\rangle_{i,j} + \frac{1}{2}\sum_{i=1}^{N-1}\langle f\,,\,g\rangle_{i,0} + \frac{1}{2}\sum_{i=1}^{N-1}\langle f\,,\,g\rangle_{i,N} \\ & + \frac{1}{2}\sum_{j=1}^{N-1}\langle f\,,\,g\rangle_{0,j} + \frac{1}{2}\sum_{j=1}^{N-1}\langle f\,,\,g\rangle_{N,j} \\ & + \frac{1}{4}(\langle f\,,\,g\rangle_{0,0} + \langle f\,,\,g\rangle_{0,N} + \langle f\,,\,g\rangle_{N,0} + \langle f\,,\,g\rangle_{N,N})\Big)\,,\end{aligned} \tag{3.17b}$$

in which

$$\langle f\,,\,g\rangle_{i,j} = \triangle z\Big(\frac{1}{2}(f_{i,j,0}g_{i,j,0} + f_{i,j,N_z}g_{i,j,N_z}) + \sum_{k=1}^{N_z-1} f_{i,j,k}g_{i,j,k}\Big)\,. \tag{3.17c}$$

Their L^2 norms in 3-D can be defined accordingly.

Notation 3.2 *For any pair of variables f, g which are evaluated at the 2-D mesh points (i,j), the following discrete L^2-inner product are given*

$$\langle f\,,\,g\rangle_2 = h^2 \sum_{j=1}^{N-1}\sum_{i=1}^{N-1} f_{i,j}\, g_{i,j}\,, \tag{3.18}$$

and the corresponding L^2 norm in 2-D can be similarly defined.

The following is the main theorem in this paper.

Theorem 3.1. *Let $\boldsymbol{u}_e = (\boldsymbol{v}_e, w_e)$, ρ_e be the exact solution of the PEs (2.1), (2.2) satisfying*

$$\|\boldsymbol{u}_e\|_{L^\infty(0,T;C^{7,\alpha}(\overline{\mathcal{M}}))} < +\infty\,, \quad \|\rho_e\|_{L^\infty(0,T;C^{7,\alpha}(\overline{\mathcal{M}}))} < +\infty\,, \tag{3.19}$$

for any $0 < \alpha < 1$, and let $(\boldsymbol{v}_h, w_h, \rho_h)$ be the numerical solution formulated in (3.2). Then we have

$$\|\boldsymbol{v}_e - \boldsymbol{v}_h\|_{L^\infty(0,T;L^2)} + \|\rho_e - \rho_h\|_{L^\infty(0,T;L^2)} \le Ch^2\,, \tag{3.20a}$$

where the constant C depends only on the regularity of the exact solution

$$C = C\Big(\|\boldsymbol{u}_e\|_{L^\infty(0,T;C^{7,\alpha}(\overline{\mathcal{M}}))}\,, \|\rho_e\|_{L^\infty(0,T;C^{7,\alpha}(\overline{\mathcal{M}}))}\Big)\,, \tag{3.20b}$$

for any $0 < \alpha < 1$.

The PDE analysis of the PEs can be found in some recent articles. In [12, 13] by J. L. Lions, R. Temam and S. Wang, global existence of weak solutions (in $L^\infty(0,t_1;L^2(\mathcal{M})) \cap L^2(0,t_1;H^1(\mathcal{M}))$) and local existence of strong solutions (in $L^\infty(0,t_1;H^1(\mathcal{M}))\cap L^2(0,t_1;H^2(\mathcal{M}))$) were established. In addition, a unique global strong solution was proven to exist under a small depth assumption (i.e., $H_0 \le \epsilon_0$, where ϵ_0 is a critical value depending on the physical parameters, such as ν, κ, f_0, β, etc.) in [10] by C. Hu, R. Temam and M. Ziane. The regularity assumption (3.19) for the exact solution is valid by local estimates if the initial data is smooth and the domain $\mathcal{M}$ is regular enough. See some relevant discussions in [1, 2, 7, 9, 13], etc. Yet, it should be noted that the exact solution does not generally satisfy (3.19) in a cubic domain, which is a shortcoming of all convergence proofs for finite difference methods. Nevertheless, in many cases, such as periodic oceanic flow (in horizontal direction) or geophysical flow in which the lateral boundary layer is not dominate, the solution does possess the required regularity.

The assumption (3.19) is required for the convergence analysis, in particular for the estimates involving high order expansions which arise in the consistency analysis. Although such an assumption is not optimal, the convergence theorem provides theoretical evidence for the performance of the proposed numerical scheme.

The basic methodology of the convergence analysis is similar to that in [28], in which the 2-D NSEs formulated in term of the vorticity were investigated. The Strang convergence theorem [25] states that if a solution of a nonlinear hyperbolic system is sufficiently smooth and the linearization of the corresponding numerical scheme is L^2 stable then the scheme for the nonlinear problem is strongly convergent. Such an idea can be generalized to analyze the nonlinear system of fluid equations, such as the NSEs or the PEs. More precisely, we can achieve the convergence result by the following procedure: first, construct approximate profiles based on the exact solution so that these profiles satisfy the numerical scheme up to a local truncation error; then provide an estimate for the error functions, which comes from the L^2 stability of the linearized scheme.

Additional techniques are needed to analyze (3.2), due to special features of the reformulated PEs. Since the velocity field is not directly updated by an evolution equation, an estimate of the solvers for the velocity field is necessary to the convergence analysis. It is proven in Section 4 that the total velocity field $\boldsymbol{u} = (\boldsymbol{v}, w)$ generated by (3.2c), (3.2d) has the second order accuracy in $W^{1,\infty}$ norm. Also, lack of regularity for the vertical velocity in the PEs results in a more subtle consistency analysis in Section 5 which is the main reason why the regularity requirement (3.19) for the exact solution is higher than that in [28], where $\boldsymbol{u}_e \in L^\infty(0, T; C^{5,\alpha})$ was imposed. Subsequently, stability and error estimates are given in Section 6, which show the full second order convergence of the numerical scheme.

4. Analysis of the solvers to determine the velocity field.

One prominent feature of the PEs formulated in mean vorticity is that the velocity field is not directly updated by evolution equations. Instead, the mean vorticity field and the profile $\partial_z \boldsymbol{v} = (\xi, \zeta)$ are dynamic variables, as can be seen in (2.10a), (2.10b). In turn, the horizontal and vertical velocity field are determined by the system of ODEs (2.10c), (2.10d). The solvers to recover horizontal and vertical velocities stand for the discrete realization of such recovery procedure. In this section we give an accuracy analysis of the procedures. In more details, we are going to show that the total velocity field $\boldsymbol{u} = (\boldsymbol{v}, w)$ generated by the two procedures does have the second order accuracy if the exact values of (ξ, ζ) and $\overline{\boldsymbol{v}}$ are given.

4.1. Analysis of the solver for the vertical velocity.

At each grid point (i, j, k), $1 \le k \le N_z - 1$, the exact solution $(\boldsymbol{v}_e, w_e)$ satisfies

$$\partial_z^2 w_e = -\partial_x(\partial_z u_e) - \partial_y(\partial_z v_e) = -\partial_x \xi_e - \partial_y \zeta_e \,. \tag{4.1}$$

Meanwhile, local Taylor expansion for the exact solution reads

$$
\begin{aligned}
&D_z^2 w_e = \partial_z^2 w_e + O(h^2)\|w_e\|_{C^4} \,, \\
&-\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e = -\partial_x \xi_e - \partial_y \zeta_e + O(h^2)\|\partial_z \boldsymbol{v}_e\|_{C^3} \,,
\end{aligned} \tag{4.2}
$$

at (x_i, y_j, z_k). Then we arrive at

$$
\begin{aligned}
D_z^2 w_e = & \quad -\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e + O(h^2)\Big(\|w_e\|_{C^4} + \|\partial_z \boldsymbol{v}_e\|_{C^2}\Big) \\
= & \quad -\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e + O(h^2)\|\boldsymbol{v}_e\|_{C^5} \,,
\end{aligned} \tag{4.3}
$$

in which the last equality is due to the fact that

$$
\|w_e\|_{C^4} \le C\|\nabla \cdot \boldsymbol{v}_e\|_{C^4} \le C\|\boldsymbol{v}_e\|_{C^5} \,. \tag{4.4}
$$

Analogous to (3.2d), the following system of difference equations satisfied by the exact solution can be derived

$$
\begin{cases}
D_z^2 w_e = -\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e + h^2 \boldsymbol{f}_w \,, & \text{at} \quad (i,j,k) \,, \quad 1 \le k \le N_z - 1 \,, \\
(w_e)_{i,j,0} = (w_e)_{i,j,N} = 0 \,,
\end{cases} \tag{4.5}
$$

in which $\boldsymbol{f}_w \le C\|\boldsymbol{v}_e\|_{C^5}$ and the vanishing boundary condition for the exact vertical velocity field is used.

Subtracting (3.2d) from (4.5) and denoting the error function as

$$
\tilde{w} = w_e - w \,, \quad \tilde{\xi} = \xi_e - \xi \,, \quad \tilde{\zeta} = \zeta_e - \zeta \,, \qquad \text{at} \quad (i,j,k) \,, \tag{4.6}
$$

we obtain the error system

$$
\begin{cases}
D_z^2 \tilde{w} = -\tilde{D}_x \tilde{\xi} - \tilde{D}_y \tilde{\zeta} + h^2 \boldsymbol{f}_w \,, & \text{at} \quad (i,j,k) \,, \quad 1 \le k \le N_z - 1 \,, \\
\tilde{w}_{i,j,0} = \tilde{w}_{i,j,N} = 0 \,.
\end{cases} \tag{4.7}
$$

Due to the homogeneous Dirichlet boundary condition for $\tilde{w}$ at $k = 0, N$, we apply the maximum principle to (4.7) and get

$$
\begin{aligned}
\|\tilde{w}\|_{L^\infty(i,j)} \le & \quad C\|D_z^2 \tilde{w}\|_{L^\infty(i,j)} \\
\le & \quad C\Big(\|\tilde{D}_x \tilde{\xi}\|_{L^\infty(i,j)} + \|\tilde{D}_y \tilde{\zeta}\|_{L^\infty(i,j)}\Big) + Ch^2\|\boldsymbol{f}_w\|_{L^\infty(i,j)} \,,
\end{aligned} \tag{4.8}
$$

in which $\|\cdot\|_{L^\infty(i,j)}$ represents the discrete L^∞ norm at the given grid point (i,j). As a result, if the numerical values of ξ and ζ are given as the same as the exact solution, i.e.,

$$
\xi_{i,j,k} = (\xi_e)_{i,j,k} \,, \qquad \zeta_{i,j,k} = (\zeta_e)_{i,j,k} \,, \tag{4.9}
$$

the second order accuracy in L^∞ norm for the solver (3.2d) is preserved

$$
\|w_h - w_e\|_{L^\infty(i,j)} \le Ch^2\|\boldsymbol{v}_e\|_{C^5} \,. \tag{4.10}
$$

In addition, we need to consider the discrete gradient (in horizontal direction) of $\tilde{w}$. Similar results to (4.10) can be derived (the detail is omitted)

$$
\|\nabla_h (w_h - w_e)\|_{L^\infty(i,j)} \le C\|\nabla_h D_z^2 \tilde{w}\|_{L^\infty(i,j)} \le Ch^2\|\boldsymbol{v}_e\|_{C^6} \,. \tag{4.11}
$$

Note that the estimates (4.10), (4.11) are valid for each fixed grid point (i,j). Then we arrive at

$$
\|w_h - w_e\|_{W^{1,\infty}(\mathcal{M}_0)} \le Ch^2\|\boldsymbol{v}_e\|_{C^6} \,. \tag{4.12}
$$

in which $\|\cdot\|_{W^{m,\infty}(\mathcal{M}_0)}$ represents the maximum value at of the given function up to m-th order finite-difference, over the 2-D domain $\mathcal{M}_0$ in the horizontal direction.

4.2. **Analysis of the solver for the horizontal velocity.** To compare the horizontal velocity field determined by (3.2c) and the exact velocity, we denote $U^e_{i,j,k} = u_e(x_i, y_j, z_k)$, $V^e_{i,j,k} = v_e(x_i, y_j, z_k)$, for $0 \le k \le N_z$, and define the "ghost" computational point values for $\boldsymbol{V}^e = (U^e, V^e)$ by applying the no-flux boundary condition in a discrete level in the same way as in the last equation of (3.2c):

$$\begin{aligned} &(\tilde{D}_z \boldsymbol{V}^e)_{i,j,0} = (\tilde{D}_z \boldsymbol{V}^e)_{i,j,N} = 0\,, \\ &\text{i.e.,} \qquad \boldsymbol{V}^e_{i,j,-1} = \boldsymbol{V}^e_{i,j,1}\,,\ \boldsymbol{V}^e_{i,j,N+1} = \boldsymbol{V}^e_{i,j,N-1}\,. \end{aligned} \tag{4.13}$$

Note that $\boldsymbol{V}^e$ share the same values with $\boldsymbol{v}_e$ at interior numerical grids (i,j,k), $0 \le k \le N_z$. Taylor expansion of $\boldsymbol{v}_e$ in vertical direction gives

$$\begin{aligned} \tilde{D}_z U^e = \tilde{D}_z u_e = \partial_z u_e + O(h^2)\|\partial_z u_e\|_{C^3} = \xi_e + O(h^2)\|\boldsymbol{v}_e\|_{C^4}\,, \\ \tilde{D}_z V^e = \tilde{D}_z v_e = \partial_z v_e + O(h^2)\|\partial_z v_e\|_{C^3} = \zeta_e + O(h^2)\|\boldsymbol{v}_e\|_{C^4}\,, \end{aligned} \tag{4.14}$$

at (i,j,k) with $1 \le k \le N-1$. In addition, the trapezoid rule (3.1) is of second order accurate, thus we have

$$\frac{1}{H_0} \sum_{k=0}^{N_z-1} \Big(\frac{1}{2} \triangle z \, (\boldsymbol{V}^e_{i,j,k} + \boldsymbol{U}^e_{i,j,k+1}) \Big) = (\overline{\boldsymbol{v}}_e)_{i,j} + O(h^2)\|\boldsymbol{v}_e\|_{C^2}\,. \tag{4.15}$$

Subtracting (3.2c) from (4.13)-(4.15) and denoting the error functions

$$\tilde{\boldsymbol{v}} = (\tilde{u}, \tilde{v}) = (u_e - u, v_e - v)\,, \quad \tilde{\overline{\boldsymbol{v}}} = (\tilde{\overline{u}}, \tilde{\overline{v}}) = (\overline{u}_e - \overline{u}, \overline{v}_e - \overline{v})\,, \quad \text{at} \quad (i,j,k)\,, \tag{4.16}$$

we arrive at the following system

$$\begin{cases} \tilde{D}_z \tilde{u} = \tilde{\xi} + h^2 \boldsymbol{f}_{uz}\,, \quad \tilde{D}_z \tilde{v} = \tilde{\zeta} + h^2 \boldsymbol{f}_{vz}\,, \quad \text{at} \quad (i,j,k)\,,\ 1 \le k \le N-1\,, \\ \overline{\overline{\tilde{u}}}_{i,j} = \tilde{\overline{u}}_{i,j} + h^2 \boldsymbol{f}_{bu}\,, \quad \overline{\overline{\tilde{v}}}_{i,j} = \tilde{\overline{v}}_{i,j} + h^2 \boldsymbol{f}_{bv}\,, \\ \tilde{\boldsymbol{v}}_{i,j,-1} = \tilde{\boldsymbol{v}}_{i,j,1}\,,\ \tilde{\boldsymbol{v}}_{i,j,N+1} = \tilde{\boldsymbol{v}}_{i,j,N-1}\,, \end{cases} \tag{4.17}$$

with the local truncation error terms

$$|\boldsymbol{f}_{uz}|, |\boldsymbol{f}_{vz}| \le C\|\boldsymbol{v}_e\|_{C^4}\,, \qquad |\boldsymbol{f}_{bu}|, |\boldsymbol{f}_{bv}| \le C\|\boldsymbol{v}_e\|_{C^2}\,. \tag{4.18}$$

As a result of its boundary condition given by (4.17), the Cosine transformation in z direction can be made for $\tilde{\boldsymbol{v}}$ at each fixed horizontal (i,j)

$$\begin{aligned} \tilde{u}_{i,j,k} &= \frac{1}{\sqrt{2N_z}} \left[\hat{\tilde{u}}_{i,j,0} + \sum_{l=1}^{N_z-1} (2\hat{\tilde{u}}_{i,j,l}) \cos\Big(\frac{lk\pi}{N_z}\Big) + (-1)^k \hat{\tilde{u}}_{i,j,N} \right]\,, \\ \tilde{v}_{i,j,k} &= \frac{1}{\sqrt{2N_z}} \left[\hat{\tilde{v}}_{i,j,0} + \sum_{l=1}^{N_z-1} (2\hat{\tilde{v}}_{i,j,l}) \cos\Big(\frac{lk\pi}{N_z}\Big) + (-1)^k \hat{\tilde{v}}_{i,j,N} \right]\,. \end{aligned} \tag{4.19}$$

The Parseval equality shows that

$$\begin{aligned} \frac{1}{2}(\tilde{u}_{i,j,0})^2 + \sum_{k=1}^{N-1} (\tilde{u}_{i,j,k})^2 + \frac{1}{2}(\tilde{u}_{i,j,N})^2 = \frac{1}{2}\Big(\hat{\tilde{u}}_{i,j,0}\Big)^2 + \sum_{l=1}^{N-1} \Big(\hat{\tilde{u}}_{i,j,l}\Big)^2 + \frac{1}{2}\Big(\hat{\tilde{u}}_{i,j,N}\Big)^2\,, \\ \frac{1}{2}(\tilde{v}_{i,j,0})^2 + \sum_{k=1}^{N-1} (\tilde{v}_{i,j,k})^2 + \frac{1}{2}(\tilde{v}_{i,j,N})^2 = \frac{1}{2}\Big(\hat{\tilde{v}}_{i,j,0}\Big)^2 + \sum_{l=1}^{N-1} \Big(\hat{\tilde{v}}_{i,j,l}\Big)^2 + \frac{1}{2}\Big(\hat{\tilde{v}}_{i,j,N}\Big)^2\,. \end{aligned} \tag{4.20}$$

By the choice of the numerical values for $\hat{u}_{i,j,N}$, $\hat{v}_{i,j,N}$ in the procedure (3.15), we have

$$\hat{\hat{u}}_{i,j,N} = \hat{U^e}_{i,j,N} \,, \qquad \hat{\hat{v}}_{i,j,N} = \hat{V^e}_{i,j,N} \,, \tag{4.21}$$

in which $\hat{U^e}_{i,j,N}$, $\hat{V^e}_{i,j,N}$ stand for the N-th mode Fourier mode (in Cosine transformation) coefficients for U^e, V^e, respectively, at the fixed horizontal grid (i,j). Under some suitable regularity requirement for $\boldsymbol{V}^e$, which have the same numerical values with $\boldsymbol{v}_e$, we can assume

$$|\hat{U^e}_{i,j,N}| \,, \ |\hat{V^e}_{i,j,N}| \le Ch^4 \,. \tag{4.22}$$

To estimate the terms $\hat{u}_{i,j,0}$, $\hat{v}_{i,j,0}$, we note that the application of trapezoid rule to $\tilde{u}_{i,j,k}$, $\tilde{v}_{i,j,k}$ which is expanded in Fourier series (4.19) leads to

$$\begin{aligned} \frac{1}{H_0} \sum_{k=0}^{N_z-1} \Big(\frac{1}{2} \triangle z \left(\tilde{u}_{i,j,k} + \tilde{u}_{i,j,k+1} \right) \Big) &= \frac{1}{\sqrt{2N_z}} \hat{\tilde{u}}_{i,j,0} \,, \\ \frac{1}{H_0} \sum_{k=0}^{N_z-1} \Big(\frac{1}{2} \triangle z \left(\tilde{v}_{i,j,k} + \tilde{v}_{i,j,k+1} \right) \Big) &= \frac{1}{\sqrt{2N_z}} \hat{\tilde{v}}_{i,j,0} \,, \end{aligned} \tag{4.23}$$

due to the property that the basis functions $\cos\Big(kl\pi h\Big)$ have vertically vanishing discrete average for $l = 1, 2, ...N_z$, provided that N_z is even. The combination of (4.23) and (4.17) gives us

$$\begin{aligned} (\hat{\tilde{u}}_{i,j,0})^2 &= 2N_z (\overline{\overline{\tilde{u}}}_{i,j})^2 = 2N_z \Big(\overline{\tilde{u}}_{i,j} + h^2 \boldsymbol{f}_{bu} \Big)^2 \,, \\ (\hat{\tilde{v}}_{i,j,0})^2 &= 2N_z (\overline{\overline{\tilde{v}}}_{i,j})^2 = 2N_z \Big(\overline{\tilde{v}}_{i,j} + h^2 \boldsymbol{f}_{bv} \Big)^2 \,. \end{aligned} \tag{4.24}$$

To deal with the terms $\hat{u}_{i,j,l}$, $\hat{v}_{i,j,l}$ for $1 \le l \le N-1$, we take the centered-difference $\tilde{D}_z$ operator to $\tilde{\boldsymbol{v}}$ represented in Fourier expansion (4.19), and arrive at

$$\begin{aligned} (\tilde{D}_z \tilde{u})_{i,j,k} &= \frac{2}{\sqrt{2N_z}} \sum_{l=1}^{N_z-1} f_l \hat{\tilde{u}}_{i,j,l} \sin\Big(\frac{lk\pi}{N_z} \Big) \,, \\ (\tilde{D}_z \tilde{v})_{i,j,k} &= \frac{2}{\sqrt{2N_z}} \sum_{l=1}^{N_z-1} f_l \hat{\tilde{v}}_{i,j,l} \sin\Big(\frac{lk\pi}{N_z} \Big) \,, \end{aligned} \tag{4.25}$$

for $1 \le k \le N-1$, by using the same arguments in (3.10) and the coefficients f_l given in (3.9b). Again, the Parseval equality gives

$$\sum_{k=1}^{N-1} (\tilde{D}_z \tilde{u})^2_{i,j,k} = \sum_{l=1}^{N-1} f_l^2 \cdot \Big(\hat{\tilde{u}}_{i,j,l} \Big)^2 , \quad \sum_{k=1}^{N-1} (\tilde{D}_z \tilde{v})^2_{i,j,k} = \sum_{l=1}^{N-1} f_l^2 \cdot \Big(\hat{\tilde{v}}_{i,j,l} \Big)^2 . \tag{4.26}$$

Moreover, it is straightforward to verify that

$$\frac{1}{f_l^2} \le \frac{2\,h^2}{l^2\,\pi^2\,h^2} \le \frac{2}{\pi^2} \,, \qquad \text{for} \qquad 1 \le l \le N_z - 1 \,. \tag{4.27}$$

The combination of (4.27) and (4.26) shows that

$$\sum_{l=1}^{N-1} \Big(\hat{\tilde{u}}_{i,j,l} \Big)^2 \le \frac{2}{\pi^2} \sum_{k=1}^{N-1} (\tilde{D}_z \tilde{u})^2_{i,j,k} \,, \qquad \sum_{l=1}^{N-1} \Big(\hat{\tilde{v}}_{i,j,l} \Big)^2 \le \frac{2}{\pi^2} \sum_{k=1}^{N-1} (\tilde{D}_z \tilde{v})^2_{i,j,k} \,. \tag{4.28}$$

Moreover, the application of the first two equations in (4.17) indicates

$$\sum_{l=1}^{N-1}\Big(\hat{\tilde{u}}_{i,j,l}\Big)^2 \le \frac{2}{\pi^2}\sum_{k=1}^{N-1}\Big(\tilde{\xi}+h^2\boldsymbol{f}_{uz}\Big)^2_{i,j,k}, \quad \sum_{l=1}^{N-1}\Big(\hat{\tilde{v}}_{i,j,l}\Big)^2 \le \frac{2}{\pi^2}\sum_{k=1}^{N-1}\Big(\tilde{\zeta}+h^2\boldsymbol{f}_{vz}\Big)^2_{i,j,k}. \tag{4.29}$$

The combination of (4.20), (4.22), (4.24) and (4.29) results in the following L^2 estimate (at a fixed grid point (i,j)) of the error function for the horizontal velocity

$$\begin{aligned}\|\tilde{\boldsymbol{v}}\|^2_{L^2(i,j)} \le \quad & C\Big(\|\tilde{\xi}\|^2_{L^2(i,j)}+\|\tilde{\zeta}\|^2_{L^2(i,j)}+(\tilde{\overline{u}}_{i,j})^2+(\tilde{\overline{v}}_{i,j})^2\Big)\\ & +Ch^4\Big(\|\boldsymbol{f}_{uz}\|^2_{L^2(i,j)}+\|\boldsymbol{f}_{vz}\|^2_{L^2(i,j)}+\|\boldsymbol{f}_{bu}\|^2_{L^2(i,j)}+\|\boldsymbol{f}_{bv}\|^2_{L^2(i,j)}\Big),\end{aligned} \tag{4.30}$$

in which $\|\cdot\|_{L^2(i,j)}$ represents the 1-D discrete L^2 norm at the given grid point (i,j). Therefore, if the numerical values of ξ, ζ, $\overline{u}$, $\overline{v}$ are given as the same as the exact solution, the second order accuracy for system (3.2c) can be derived

$$\|\boldsymbol{v}_h-\boldsymbol{v}_e\|_{L^2(i,j)} \le Ch^2\|\boldsymbol{v}_e\|_{C^4}. \tag{4.31}$$

In addition, we notice that

$$\tilde{D}_z(\boldsymbol{v}-\boldsymbol{v}_e)=\tilde{D}_z\boldsymbol{v}-\tilde{D}_z\boldsymbol{v}_e=(\xi,\zeta)-(\xi_e,\zeta_e)+O(h^2)\|\boldsymbol{v}_e\|_{C^3}=O(h^2)\|\boldsymbol{v}_e\|_{C^3}. \tag{4.32}$$

As a result,

$$\|\tilde{D}_z(\boldsymbol{v}_h-\boldsymbol{v}_e)\|_{L^2(i,j)} \le Ch^2\|\boldsymbol{v}_e\|_{C^3}. \tag{4.33}$$

The combination of (4.31) and (4.33) shows that

$$\|\boldsymbol{v}_h-\boldsymbol{v}_e\|_{L^\infty(i,j)} \le C\Big(\|\tilde{\boldsymbol{v}}\|_{L^2(i,j)}+\|\tilde{D}_z\tilde{\boldsymbol{v}}\|_{L^2(i,j)}\Big) \le Ch^2\|\boldsymbol{v}_e\|_{C^4}, \tag{4.34}$$

where discrete Sobolev inequality is used.

Similar to (4.11), the discrete gradient (in horizontal direction) of $\tilde{\boldsymbol{v}}$ can be estimated as

$$\|\nabla_h(\boldsymbol{v}_h-\boldsymbol{v}_e)\|_{L^2(i,j)}+\|\tilde{D}_z\nabla_h(\boldsymbol{v}_h-\boldsymbol{v}_e)\|_{L^2(i,j)} \le Ch^2\|\boldsymbol{v}_e\|_{C^5}. \tag{4.35}$$

Consequently, we get

$$\|\nabla_h(\boldsymbol{v}_h-\boldsymbol{v}_e)\|_{L^\infty(i,j)} \le C\Big(\|\nabla_h\tilde{\boldsymbol{v}}\|_{L^2(i,j)}+\|\tilde{D}_z\nabla_h\tilde{\boldsymbol{v}}\|_{L^2(i,j)}\Big) \le Ch^2\|\boldsymbol{v}_e\|_{C^5}. \tag{4.36}$$

The combination of (4.36) and (4.34) results in

$$\|\boldsymbol{v}_h-\boldsymbol{v}_e\|_{W^{1,\infty}(\mathcal{M}_0)} \le Ch^2\|\boldsymbol{v}_e\|_{C^5}, \tag{4.37}$$

since both estimates are valid for each fixed grid point (i,j).

Similar estimate can be applied to the second order difference operator of the horizontal velocity. We finally arrive at

$$\|\boldsymbol{v}_h-\boldsymbol{v}_e\|_{W^{2,\infty}(\mathcal{M}_0)} \le Ch^2\|\boldsymbol{v}_e\|_{C^6}. \tag{4.38}$$

Such a result will be used in the consistency analysis below.

5. Construction of approximate solutions.

In this section we perform consistency analysis of the numerical difference scheme. The goal is to construct approximate velocity and density profiles and show that they satisfy the numerical scheme (3.2) up to an $O(h^2)$ error.

5.1. **Leading order consistency analysis.** As mentioned in the introduction, the flow motion governed by the PEs formulated in mean vorticity is determined by the combination of the mean velocity field and the vertical derivative of the horizontal velocity field. Accordingly, the approximate velocity field is constructed by using a similar methodology. We extend the exact mean stream function $\overline{\psi}_e$ smoothly to $[-\delta, 1+\delta]^2$ and let $\overline{\Psi}^0_{i,j} = \overline{\psi}_e(x_i, y_j)$ for $-1 \le i, j \le N+1$. The approximate profiles $\overline{U}^0$, $\overline{V}^0$, $\overline{\Omega}^0$ are constructed through the finite difference of $\overline{\Psi}^0$ to maintain the consistency, especially near the boundary,

$$\overline{U}^0_{i,j} = -\tilde{D}_y \overline{\Psi}^0 \,, \quad \overline{V}^0_{i,j} = \tilde{D}_x \overline{\Psi}^0 \,, \quad \overline{\Omega}_{i,j} = \triangle_h \overline{\Psi}^0 \,, \qquad \text{for} \quad 0 \le i, j \le N \,. \tag{5.1}$$

The profile for the vertical derivative of the horizontal velocity (ξ^0, ζ^0) is taken as the exact solution (ξ_e, ζ_e). The approximate profile for total velocity field in horizontal direction U^0, V^0 is given by the solution of the following discrete system at each fixed horizontal grid point (i, j):

$$\begin{cases} \tilde{D}_z U^0 = \xi_e \,, \qquad \tilde{D}_z V^0 = \zeta_e \,, \qquad \text{at} \qquad (i,j,k) \,, \quad 1 \le k \le N-1 \,, \\ \overline{\overline{U}}^0_{i,j} = \dfrac{1}{H_0} \displaystyle\sum_{k=0}^{N_z-1} \Big(\frac{1}{2} \triangle z \, (U^0_{i,j,k} + U^0_{i,j,k+1}) \Big) = \overline{U}^0_{i,j} \,, \\ \overline{\overline{V}}^0_{i,j} = \dfrac{1}{H_0} \displaystyle\sum_{k=0}^{N_z-1} \Big(\frac{1}{2} \triangle z \, (V^0_{i,j,k} + V^0_{i,j,k+1}) \Big) = \overline{V}^0_{i,j} \,, \\ (\tilde{D}_z \boldsymbol{V}^0)_{i,j,0} = (\tilde{D}_z \boldsymbol{V}^0)_{i,j,N} = 0 \,, \\ \text{i.e.,} \qquad \boldsymbol{V}^0_{i,j,-1} = \boldsymbol{V}^0_{i,j,1} \,, \ \boldsymbol{V}^0_{i,j,N+1} = \boldsymbol{V}^0_{i,j,N-1} \,, \end{cases} \tag{5.2}$$

which is analogous to (3.2c). The solver for the above system is outlined in section 3.3. The approximate vertical velocity W^0 is given by the following discrete equations analogous to (3.2d):

$$\begin{cases} D_z^2 W^0 = -\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e \,, \qquad \text{at} \qquad (i,j,k) \,, \quad 1 \le k \le N_z - 1 \,, \\ W^0_{i,j,0} = W^0_{i,j,N} = 0 \,. \end{cases} \tag{5.3}$$

For the density field, we choose the leading order approximation as

$$\Theta^0 = \rho_e + h^2 \Theta^* \,, \tag{5.4}$$

in which ρ_e denotes the exact density function. The reason for the addition of an $O(h^2)$ correction terms $h^2\Theta^*$ in the expansion (5.4) is due to the higher order consistency of the approximate profile Θ with the boundary condition given in the numerical scheme (3.2e). The correction function Θ^* turns out to be the solution of the Poisson equation

$$\triangle \Theta^* = C^1 \,, \tag{5.5a}$$

with the Neumann boundary condition

$$\begin{aligned} &\partial_z \Theta^*(x,y,-H_0) = -\frac{1}{6}\, \partial_z^3 \rho_e(x,y,-H_0) \,, \quad \partial_z \Theta^*(x,y,0) = -\frac{1}{6}\, \partial_z^3 \rho_e(x,y,0) \,, \\ &\partial_x \Theta^*(0,y,z) = -\frac{1}{6}\, \partial_x^3 \rho_e(0,y,z) \,, \quad \partial_x \Theta^1(1,y,z) = -\frac{1}{6}\, \partial_x^3 \rho_e(1,y,z) \,, \\ &\partial_y \Theta^*(x,0,z) = -\frac{1}{6}\, \partial_y^3 \rho_e(x,0,z) \,, \quad \partial_y \Theta^1(x,1,z) = -\frac{1}{6}\, \partial_y^3 \rho_e(x,1,z) \,. \end{aligned} \tag{5.5b}$$

The number C^1 (a function in time t) is chosen as $\int_{\mathcal{M}} C^1 \, d\boldsymbol{x} \, dz = \int_{\partial\mathcal{M}} \frac{\partial \Theta^*}{\partial \boldsymbol{n}} \, d\boldsymbol{n}$ to maintain the consistency that follows from the Neumann boundary condition, i.e.,

$$\begin{aligned} C^1 = \; & \frac{1}{|\mathcal{M}|}\Big(\frac{1}{6}\int_{\mathcal{M}_0} \partial_z^3 \rho_e(x,y,-H_0) - \partial_z^3 \rho_e(x,y,0) \, d\boldsymbol{x} \\ & + \frac{1}{6}\int_{-H_0}^0 \int_0^1 \partial_x^3 \rho_e(0,y,z) \, dy \, dz - \frac{1}{6}\int_{-H_0}^0 \int_0^1 \partial_x^3 \rho_e(1,y,z) \, dy \, dz \\ & + \frac{1}{6}\int_{-H_0}^0 \int_0^1 \partial_y^3 \rho_e(x,0,z) - \partial_y^3 \rho_e(x,1,z) \, dx \, dz\Big) \, . \end{aligned} \tag{5.6}$$

The approximate solutions $\boldsymbol{V}^0$, W^0, $\overline{\Omega}^0$, $\overline{\Psi}^0$, Θ^0 are proven to satisfy the following estimates

$$\left\{ \begin{aligned} & \partial_t \overline{\Omega}^0 + \tilde{D}_x \tilde{D}_y (\overline{\overline{V^0 V^0}} - \overline{\overline{U^0 U^0}}) + (D_x^2 - D_y^2) \overline{\overline{U^0 V^0}} \\ & \qquad + \frac{\beta}{Ro} \overline{V^0} = \nu_1 \triangle_h \overline{\Omega}^0 + h^2 \boldsymbol{f}_{\overline{\omega}}^0 \, , \\ & \triangle_h \overline{\Psi}^0 = \overline{\Omega}^0 \, , \qquad \overline{\Psi}^0 \mid_{\partial \mathcal{M}_0} = 0 \, , \\ & \overline{U}^0 = -\tilde{D}_y \overline{\Psi}^0 \, , \qquad \overline{V}^0 = \tilde{D}_x \overline{\Psi}^0 \, , \\ & \overline{\Omega}^0_{i,0} = \frac{1}{h^2}(4 \overline{\Psi}^0_{i,1} - \frac{1}{2} \overline{\Psi}^0_{i,2}) + O(h^2) \|\boldsymbol{v}_e\|_{C^3} \, , \end{aligned} \right. \tag{5.7a}$$

$$\left\{ \begin{aligned} & \partial_t \xi_e + U^0 \tilde{D}_x \xi_e + V^0 \tilde{D}_y \xi_e + W^0 \tilde{D}_z \xi_e - (\tilde{D}_y V^0) \xi_e + (\tilde{D}_y U^0) \zeta_e \\ & \quad - \frac{f}{Ro} \zeta_e - \frac{1}{Ro} \tilde{D}_x \Theta^0 = (\nu_1 \triangle_h + \nu_2 D_z^2) \xi_e + h^2 \boldsymbol{f}_\xi^0 \, , \\ & \partial_t \zeta_e + U^0 \tilde{D}_x \zeta_e + V^0 \tilde{D}_y \zeta_e + W^0 \tilde{D}_z \zeta_e - (\tilde{D}_x U^0) \zeta_e + (\tilde{D}_x V^0) \xi_e \\ & \quad + \frac{f}{Ro} \xi_e - \frac{1}{Ro} \tilde{D}_y \Theta^0 = (\nu_1 \triangle_h + \nu_2 D_z^2) \zeta_e + h^2 \boldsymbol{f}_\zeta^0 \, , \\ & (\xi_e)_{i,j,0} = 0 \, , \quad (\zeta_e)_{i,j,0} = 0 \, , \quad (\xi_e)_{i,j,N} = 0 \, , \quad (\zeta_e)_{i,j,N} = 0 \, , \\ & (\xi_e)_{i,0,k} = 0 \, , \quad (\zeta_e)_{i,0,k} = 0 \, , \quad (\xi_e)_{0,j,k} = 0 \, , \quad (\zeta_e)_{0,j,k} = 0 \, , \end{aligned} \right. \tag{5.7b}$$

$$\left\{ \begin{aligned} & \tilde{D}_z U^0 = \xi_e \, , \qquad \tilde{D}_z V^0 = \zeta_e \, , \qquad \text{at} \quad (i,j,k) \, , \quad 1 \le k \le N-1 \, , \\ & \overline{\overline{U}}^0_{i,j} = \overline{U}^0_{i,j} \, , \quad \overline{\overline{V}}^0_{i,j} = \overline{V}^0_{i,j} \, , \\ & (\tilde{D}_z \boldsymbol{V}^0)_{i,j,0} = (\tilde{D}_z \boldsymbol{V}^0)_{i,j,N} = 0 \, , \end{aligned} \right. \tag{5.7c}$$

$$\left\{ \begin{aligned} & D_z^2 W^0 = -\tilde{D}_x \xi_e - \tilde{D}_y \zeta_e \, , \qquad \text{at} \quad (i,j,k) \, , \quad 1 \le k \le N_z - 1 \, , \\ & W^0_{i,j,0} = W^0_{i,j,N} = 0 \, , \end{aligned} \right. \tag{5.7d}$$

$$\left\{ \begin{aligned} & \partial_t \Theta^0 + U^0 \tilde{D}_x \Theta^0 + V^0 \tilde{D}_y \Theta^0 + W^0 \tilde{D}_z \Theta^0 = \Big(\kappa_1 \triangle_h + \kappa_2 D_z^2\Big) \Theta^0 + h^2 \boldsymbol{f}_\rho^0 \, , \\ & \Theta^0_{i,j,-1} = \Theta^0_{i,j,1} + h^5 \boldsymbol{e}_{\rho b} \, , \quad \Theta^0_{-1,j,k} = \Theta^0_{1,j,k} + h^5 \boldsymbol{e}_{\rho l} \, , \\ & \Theta^0_{i,-1,k} = \Theta^0_{i,1,k} + h^5 \boldsymbol{e}_{\rho d} \, , \end{aligned} \right. \tag{5.7e}$$

with the local error terms

$$
\begin{aligned}
&|\boldsymbol{f}^0_{\overline{\omega}}| \le C(\|\boldsymbol{v}_e\|_{C^6} + \|\boldsymbol{v}_e\|^2_{C^6}) \,, \quad |\boldsymbol{f}^0_{\rho}| \le C\Big(\|\boldsymbol{u}_e\|_{C^5} \|\rho_e\|_{C^5} + \|\rho_e\|_{C^{4,\alpha}} \Big) \,, \\
&|\boldsymbol{f}^0_{\xi}|, |\boldsymbol{f}^0_{\zeta}| \le C\Big(\|\boldsymbol{v}_e\|_{C^6} (\|\boldsymbol{v}_e\| + 1) + \|\rho_e\|_{C^5} \Big) \,, \quad |\boldsymbol{e}_{\rho b}| \,, |\boldsymbol{e}_{\rho l}| \,, |\boldsymbol{e}_{\rho d}| \le C \|\rho_e\|_{C^5} \,.
\end{aligned} \tag{5.8}
$$

5.2. **Local truncation error analysis for the leading order expansion $\boldsymbol{V}^0$, W^0, Θ^0.** We first look at the mean profile $\overline{\boldsymbol{V}}^0$, $\overline{\Omega}^0$. A direct Taylor expansion for $\overline{\psi}_e$ up to the boundary shows that at grid points (x_i, y_j), $0 \le i, j \le N$,

$$
\begin{aligned}
\overline{U}^0 &= \overline{u}_e - \frac{h^2}{6} \partial_y^3 \overline{\psi}_e - \frac{h^4}{120} \partial_y^5 \overline{\psi}_e + O(h^5) \|\overline{\psi}_e\|_{C^6} \,, \\
\overline{V}^0 &= \overline{v}_e + \frac{h^2}{6} \partial_x^3 \overline{\psi}_e + \frac{h^4}{120} \partial_x^5 \overline{\psi}_e + O(h^5) \|\overline{\psi}_e\|_{C^6} \,, \\
\overline{\Omega}^0 &= \overline{\omega}_e + \frac{h^2}{12} (\partial_x^4 + \partial_y^4) \overline{\psi}_e + O(h^4) \|\overline{\psi}_e\|_{C^6} \,.
\end{aligned} \tag{5.9}
$$

As a result, we have

$$
\|\overline{U}^0 - \overline{u}_e\|_{W^{2,\infty}(\mathcal{M}_0)} + \|\overline{V}^0 - \overline{v}_e\|_{W^{2,\infty}(\mathcal{M}_0)} + \|\overline{\Omega}^0 - \overline{\omega}_e\|_{W^{2,\infty}(\mathcal{M}_0)} \le C h^2 \|\overline{\psi}_e\|_{C^6} \,. \tag{5.10}
$$

According to the analysis in section 4.2, the second order accuracy is valid between the constructed solution and the exact solution for the horizontal velocity, provided that the exact mean velocity field and exact profile (ξ_e, ζ_e) is used in the numerical procedure (3.2c). See the result in (4.38). Moreover, it is straightforward to prove that a second order approximation is preserved if the exact mean velocity used in the numerical solver is replaced by the approximation $\overline{\boldsymbol{V}}^0$, i.e.,

$$
\|\boldsymbol{V}^0 - \boldsymbol{v}_e\|_{W^{2,\infty}(\mathcal{M}_0)} \le C h^2 \|\boldsymbol{v}_e\|_{C^6} + \|\overline{\boldsymbol{V}}^0 - \overline{\boldsymbol{v}}_e\|_{W^{2,\infty}(\mathcal{M}_0)} \,. \tag{5.11}
$$

The combination of (5.11) and (5.10) leads to

$$
\|\boldsymbol{V}^0 - \boldsymbol{v}_e\|_{W^{2,\infty}(\mathcal{M}_0)} \le C h^2 \|\boldsymbol{v}_e\|_{C^6} \,. \tag{5.12}
$$

For the vertical velocity, the analysis in section 4.1 gives us

$$
\|W^0 - w_e\|_{L^\infty(\mathcal{M})} + \|\nabla_h (W^0 - w_e)\|_{L^\infty(\mathcal{M})} \le C h^2 \|\boldsymbol{v}_e\|_{C^6} \,, \tag{5.13}
$$

since the exact profile of (ξ_e, ζ_e) was used in the numerical solver for W^0 in (5.3). See (4.12).

For the density field, it is obvious that an application of the Schauder estimate to the Poisson equation (5.5) results in

$$
\|\Theta^*\|_{C^{m,\alpha}} \le C \|\rho_e\|_{C^{m+2,\alpha}} \,, \quad \text{for } \ m \ge 2 \,. \tag{5.14}
$$

Consequently, we have

$$
\|\Theta^*\|_{W^{2,\infty}(\mathcal{M})} \le C \|\Theta^*\|_{C^{2,\alpha}} \le C \|\rho_e\|_{C^{4,\alpha}} \,, \tag{5.15}
$$

whose insertion into the expansion (5.4) results in

$$
\|\Theta^0 - \rho_e\|_{W^{2,\infty}(\mathcal{M})} \le C h^2 \|\rho_e\|_{C^{4,\alpha}} \,. \tag{5.16}
$$

5.2.1. *Truncation error for the mean vorticity equation.* The estimate (5.10) indicates that

$$\triangle_h \overline{\Omega}^0 = \triangle_h \overline{\omega}_e + O(h^2)\|\overline{\psi}_e\|_{C^6} \,. \tag{5.17}$$

Meanwhile, the Taylor expansion of ω_e reads

$$\triangle_h \overline{\omega}_e = \triangle \overline{\omega}_e + O(h^2)\|\overline{\omega}_e\|_{C^4} = \triangle \overline{\omega}_e + O(h^2)\|\overline{\psi}_e\|_{C^6} \,. \tag{5.18}$$

The combination of (5.17) and (5.18) leads to the estimate of the diffusion term

$$\triangle_h \overline{\Omega}^0 = \triangle \overline{\omega}_e + O(h^2)\|\overline{\psi}_e\|_{C^6} \,, \tag{5.19}$$

at grid points (x_i, y_j), $1 \le i, j \le N-1$.

The nonlinear convection terms can be dealt with in a similar fashion. The estimate (5.12) implies that

$$\tilde{D}_x \tilde{D}_y (V^0 V^0) = \tilde{D}_x \tilde{D}_y (v_e v_e) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,, \tag{5.20a}$$

which in term shows

$$\tilde{D}_x \tilde{D}_y (\overline{\overline{V^0 V^0}}) = \tilde{D}_x \tilde{D}_y (\overline{v_e v_e}) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,. \tag{5.20b}$$

Meanwhile, the Taylor expansion for $\boldsymbol{v}_e$ reads

$$\tilde{D}_x \tilde{D}_y (\overline{v_e v_e}) = \partial_x \partial_y (\overline{v_e v_e}) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,. \tag{5.21}$$

The combination of (5.21) and (5.20b) gives

$$\tilde{D}_x \tilde{D}_y (\overline{\overline{V^0 V^0}}) = \partial_x \partial_y (\overline{v_e v_e}) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,. \tag{5.22}$$

Using the same argument, we obtain

$$\begin{aligned} \tilde{D}_x \tilde{D}_y (\overline{\overline{U^0 U^0}}) &= \partial_x \partial_y (\overline{u_e u_e}) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,, \\ (D_x^2 - D_y^2)(\overline{\overline{U^0 V^0}}) &= (\partial_x^2 - \partial_y^2)(\overline{u_e v_e}) + O(h^2)\|\boldsymbol{v}_e\|^2_{C^6} \,. \end{aligned} \tag{5.23}$$

The difference between the time marching term $\partial_t \overline{\Omega}^0$ and the exact value $\partial_t \overline{\omega}_e$ can be controlled by $O(h^2)$ of $\|\partial_t \overline{\psi}_e\|_{C^4}$:

$$\partial_t \overline{\Omega}^0 - \partial_t \overline{\omega}_e = \triangle_h \partial_t \overline{\psi}_e - \triangle \partial_t \overline{\psi}_e = (\triangle_h - \triangle)\partial_t \overline{\psi}_e = O(h^2)\|\partial_t \overline{\psi}_e\|_{C^4} \,. \tag{5.24}$$

Moreover, applying the Schauder estimate to the following Poisson equation

$$\begin{cases} \triangle(\partial_t \overline{\psi}_e) = \partial_t \overline{\omega}_e \,, & \text{in} \quad \mathcal{M}_0 \,, \\ \partial_t \overline{\psi}_e = 0 \,, & \text{on} \quad \partial\mathcal{M}_0 \,, \end{cases} \tag{5.25}$$

shows that

$$\|\partial_t \overline{\psi}_e\|_{C^{4,\alpha}} \le C\|\partial_t \overline{\omega}_e\|_{C^{2,\alpha}} \le C(\|\overline{\psi}_e\|_{C^{6,\alpha}} + \|\boldsymbol{v}_e\|^2_{C^{4,\alpha}}) \le C(\|\boldsymbol{v}_e\|_{C^{5,\alpha}} + \|\boldsymbol{v}_e\|^2_{C^{4,\alpha}}) \,. \tag{5.26}$$

Note that the mean vorticity equation was applied in the second step. Consequently, the following estimate holds

$$\partial_t \overline{\Omega}^0 - \partial_t \overline{\omega}_e = O(h^2)(\|\boldsymbol{v}_e\|_{C^{5,\alpha}} + \|\boldsymbol{v}_e\|^2_{C^{4,\alpha}}) \,. \tag{5.27}$$

Combining (5.10), (5.19), (5.22), (5.23) and (5.27), and applying the original PDE of the exact mean vorticity equation, we obtain the first equation in (5.7a), which shows the second order consistency for the mean vorticity equation.

In addition, regarding the boundary condition for the mean vorticity, we need to show that $\overline{\Omega}^0$ satisfies Wilkes' formula applied to $\overline{\Psi}^0$ up to an $O(h^2)$ error.

For simplicity of presentation, only one boundary section $\Gamma_x \in \partial\mathcal{M}_0$, $j=0$, is considered. The other three boundary sections can be dealt with in the same way. One-sided Taylor expansion for $\overline{\Psi}^0$ in the y-th direction near $y=0$ gives

$$\frac{1}{h^2}(4\overline{\Psi}^0_{i,1}-\frac{1}{2}\overline{\Psi}^0_{i,2}) = \partial_y^2\overline{\psi}_e(x_i,0)+O(h^2)\|\overline{\psi}_e\|_{C^4} = \overline{\omega}_e(x_i,0)+O(h^2)\|\boldsymbol{v}_e\|_{C^3}\,. \quad (5.28)$$

Meanwhile, (5.9) shows that the difference between $\overline{\Omega}^0_{i,0}$ and $\overline{\omega}_e(x_i,0)$ on Γ_x is also of order $O(h^2)\|\overline{\psi}_e\|_{C^4}$. Then we get the boundary estimate for the mean vorticity in (5.7a).

5.2.2. *Truncation error for the evolution equations of* (ξ,ζ). Using similar arguments, we have the following estimates

$$\begin{aligned}
&U^0\tilde{D}_x\xi_e - u_e\partial_x\xi_e\,,\ V^0\tilde{D}_y\xi_e - v_e\partial_y\xi_e = O(h^2)\|\boldsymbol{v}_e\|_{C^5}\|\boldsymbol{v}_e\|_{C^4}\,,\\
&W^0\tilde{D}_z\xi_e - w_e\partial_z\xi_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^4}\,,\\
&(\tilde{D}_yV^0)\xi_e - (\partial_y v_e)\xi_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^1}\,,\\
&(\tilde{D}_yU^0)\zeta_e - (\partial_y u_e)\zeta_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^1}\,,\quad \tilde{D}_x\Theta^0-\partial_x\rho_e = O(h^2)\|\rho_e\|_{C^3}\,,
\end{aligned}\quad (5.29)$$

$$\begin{aligned}
&U^0\tilde{D}_x\zeta_e - u_e\partial_x\zeta_e\,,\ V^0\tilde{D}_y\zeta_e - v_e\partial_y\zeta_e = O(h^2)\|\boldsymbol{v}_e\|_{C^5}\|\boldsymbol{v}_e\|_{C^4}\,,\\
&W^0\tilde{D}_z\zeta_e - w_e\partial_z\zeta_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^4}\,,\\
&(\tilde{D}_xU^0)\zeta_e - (\partial_x u_e)\zeta_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^1}\,,\\
&(\tilde{D}_xV^0)\xi_e - (\partial_x v_e)\xi_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^1}\,,\quad \tilde{D}_y\Theta^0-\partial_y\rho_e = O(h^2)\|\rho_e\|_{C^5}\,,
\end{aligned}\quad (5.30)$$

$$\begin{aligned}
&(\nu_1\triangle_h+\nu_2D_z^2)\xi_e - (\nu_1\triangle+\nu_2\partial_z^2)\xi_e = O(h^2)\|\boldsymbol{v}_e\|_{C^5}\,,\\
&(\nu_1\triangle_h+\nu_2D_z^2)\zeta_e - (\nu_1\triangle+\nu_2\partial_z^2)\zeta_e = O(h^2)\|\boldsymbol{v}_e\|_{C^5}\,,
\end{aligned}\quad (5.31)$$

Consequently, the exact profile (ξ_e,ζ_e) combined with the constructed velocity $\boldsymbol{V}^0$, W^0 and the density Θ^0 satisfies the numerical difference scheme with an $O(h^2)$ error, as shown in (5.7b), and the homogeneous Dirichlet boundary condition for (ξ_e,ζ_e) is exactly satisfied.

5.2.3. *Truncation error for the density transport equation.* Applying the estimates for the constructed velocity and density field as shown in (5.10), (5.11) and (5.14), we can deal with the terms of the density transport equation in a similar way

$$\partial_t\Theta^0-\partial_t\rho_e = O(h^2)\|\partial_t\rho_e\|_{C^{2,\alpha}} = O(h^2)\Big(\|\boldsymbol{u}_e\|_{C^{2,\alpha}}\|\rho_e\|_{C^{3,\alpha}}+\|\rho_e\|_{C^{4,\alpha}}\Big)\,, \quad (5.33)$$

$$\begin{aligned}
&U^0\tilde{D}_x\Theta^0 - u_e\partial_x\rho_e\,,\ V^0\tilde{D}_y\Theta^0 - v_e\partial_y\rho_e = O(h^2)\|\boldsymbol{v}_e\|_{C^5}\|\rho_e\|_{C^5}\,,\\
&W^0\tilde{D}_z\Theta^0 - w_e\partial_z\rho_e = O(h^2)\|\boldsymbol{v}_e\|_{C^6}\|\boldsymbol{v}_e\|_{C^5}\,,\\
&(\kappa_1\triangle_h+\kappa_2D_z^2)\Theta^0 - (\nu_1\triangle+\nu_2\partial_z^2)\rho_e = O(h^2)\|\rho_e\|_{C^{4,\alpha}}\,.
\end{aligned}\quad (5.34)$$

The combination of (5.33) and (5.34) gives (5.7e).

Moreover, the approximated density Θ^0 satisfies the discrete boundary condition in (3.29e) up to an $O(h^5)$ order, due to the choice of the boundary condition for Θ^* in (5.5b). Local Taylor expansion for the exact density field ρ_e around the boundary gives

$$
\begin{aligned}
(\rho_e)_{i,j,-1} &= (\rho_e)_{i,j,1} - \frac{\triangle z^3}{3}\,\partial_z^3\rho_e(x_i,y_j,-H_0) + O(h^5)\|\rho_e\|_{C^5}\,,\\
(\rho_e)_{-1,j,k} &= (\rho_e)_{1,j,k} - \frac{\triangle x^3}{3}\,\partial_x^3\rho_e(0,y_j,z_k) + O(h^5)\|\rho_e\|_{C^5}\,,\\
(\rho_e)_{i,-1,k} &= (\rho_e)_{i,1,k} - \frac{\triangle y^3}{3}\,\partial_y^3\rho_e(x_i,0,z_k) + O(h^5)\|\rho_e\|_{C^5}\,,
\end{aligned}
\tag{5.35}
$$

by using the no-flux boundary condition for ρ_e on all the six boundary sections. Performing Taylor expansion of Θ^* gives

$$
\begin{aligned}
\Theta^*_{i,j,-1} &= \Theta^*_{i,j,1} + \frac{\triangle z}{3}\,\partial_z^3\rho_e(x_i,y_j,-H_0) + O(h^3)\|\rho_e\|_{C^{5,\alpha}}\,,\\
\Theta^*_{-1,j,k} &= \Theta^*_{1,j,k} + \frac{\triangle x}{3}\,\partial_x^3\rho_e(0,y_j,z_k) + O(h^3)\|\rho_e\|_{C^{5,\alpha}}\,,\\
\Theta^*_{i,-1,k} &= \Theta^*_{i,1,k} + \frac{\triangle y}{3}\,\partial_y^3\rho_e(x_i,0,z_k) + O(h^3)\|\rho_e\|_{C^{5,\alpha}}\,,
\end{aligned}
\tag{5.36}
$$

where the boundary condition (5.5b) and the Schauder estimate $\|\Theta^*\|_{C^3} \le C\|\rho_e\|_{C^{5,\alpha}}$ given by (5.12) are used. The combination of (5.35) and (5.36) results in

$$
\begin{aligned}
&\Theta^0_{i,j,-1} = \Theta^0_{i,j,1} + O(h^5)\|\rho_e\|_{C^{5,\alpha}}\,, \quad \Theta^0_{-1,j,k} = \Theta^0_{1,j,k} + O(h^5)\|\rho_e\|_{C^{5,\alpha}}\,,\\
&\Theta^0_{i,-1,k} = \Theta^0_{i,1,k} + O(h^5)\|\rho_e\|_{C^{5,\alpha}}\,.
\end{aligned}
\tag{5.37}
$$

Similar results can be obtained at the other boundary section $z = 0$, $x = 1$, $y = 1$. This finishes the consistency analysis for the leading order approximate profile.

5.3. **Higher order expansion of the numerical scheme.** However, the consistency analysis (5.7) is not enough to recover the L^∞ a-priori assumption for the numerical value of the vertical velocity field in the full nonlinear system of the PEs. We make the following expansion for the approximate solution corresponding to the numerical scheme (3.2)

$$
\begin{aligned}
&\overline{\Psi} = \overline{\Psi}^0 + h^2\overline{\Psi}^1\,, \quad \overline{\Omega} = \overline{\Omega}^0 + h^2\overline{\Omega}^1\,, \quad \boldsymbol{V} = \boldsymbol{V}^0 + h^2\boldsymbol{V}^1\,, \quad \boldsymbol{W} = \boldsymbol{W}^0 + h^2\boldsymbol{W}^1\,,\\
&\Theta = \Theta^0 + h^2\Theta^1\,, \quad \mathcal{S} = \xi_e + h^2\xi^1\,, \quad \Phi = \zeta_e + h^2\zeta^1\,.
\end{aligned}
\tag{5.38}
$$

The expanded profile satisfy the numerical scheme on the regular grid up to order $O(h^4)$:

$$
\begin{cases}
\partial_t\overline{\Omega} + \tilde{D}_x\tilde{D}_y(\overline{\overline{VV}} - \overline{\overline{UU}}) + (D_x^2 - D_y^2)\overline{\overline{UV}} + \dfrac{\beta}{Ro}\overline{V} = \nu_1\triangle_h\overline{\Omega} + h^4\boldsymbol{f}_{\overline{\omega}}\,,\\
\triangle_h\overline{\Psi} = \overline{\Omega}\,, \qquad \overline{\Psi}\,|_{\partial\mathcal{M}_0} = 0\,,\\
\overline{U} = -\tilde{D}_y\overline{\Psi}\,, \qquad \overline{V} = \tilde{D}_x\overline{\Psi}\,,\\
\overline{\Omega}_{i,0} = \dfrac{1}{h^2}(4\overline{\Psi}_{i,1} - \dfrac{1}{2}\overline{\Psi}_{i,2}) + h^3(\boldsymbol{e}_{\overline{\omega}})_i\,,
\end{cases}
\tag{5.39a}
$$

$$\left\{\begin{array}{l}\partial_t\mathcal{S}+U\tilde{D}_x\mathcal{S}+V\tilde{D}_y\mathcal{S}+W\tilde{D}_z\mathcal{S}-(\tilde{D}_yV)\mathcal{S}+(\tilde{D}_yU)\Phi \\ \qquad -\dfrac{f}{Ro}\Phi-\dfrac{1}{Ro}\tilde{D}_x\Theta=(\nu_1\triangle_h+\nu_2D_z^2)\mathcal{S}+h^4\boldsymbol{f}_\xi\,, \\ \partial_t\Phi+U\tilde{D}_x\Phi+V\tilde{D}_y\Phi+W\tilde{D}_z\Phi-(\tilde{D}_xU)\Phi+(\tilde{D}_xV)\mathcal{S} \\ \qquad +\dfrac{f}{Ro}\mathcal{S}-\dfrac{1}{Ro}\tilde{D}_y\Theta=(\nu_1\triangle_h+\nu_2D_z^2)\Phi+h^4\boldsymbol{f}_\zeta\,, \\ \mathcal{S}_{i,j,0}=0\,,\quad \Phi_{i,j,0}=0\,,\quad \mathcal{S}_{i,j,N}=0\,,\quad \Phi_{i,j,N}=0\,, \\ \mathcal{S}_{i,0,k}=0\,,\quad \Phi_{i,0,k}=0\,,\quad \mathcal{S}_{0,j,k}=0\,,\quad \Phi_{0,j,k}=0\,,\end{array}\right. \tag{5.39b}$$

$$\left\{\begin{array}{l}\tilde{D}_zU=\mathcal{S}\,,\qquad \tilde{D}_zV=\Phi\,,\qquad \text{at}\quad (i,j,k)\,,\quad 1\le k\le N-1\,, \\ \overline{\overline{U}}_{i,j}=\overline{U}_{i,j}\,,\quad \overline{\overline{V}}_{i,j}=\overline{V}_{i,j}\,, \\ (\tilde{D}_z\boldsymbol{V})_{i,j,0}=(\tilde{D}_z\boldsymbol{V})_{i,j,N}=0\,,\end{array}\right. \tag{5.39c}$$

$$\left\{\begin{array}{l}D_z^2W=-\tilde{D}_x\mathcal{S}-\tilde{D}_y\Phi\,,\qquad \text{at}\quad (i,j,k)\,,\quad 1\le k\le N_z-1\,, \\ W_{i,j,0}=W_{i,j,N}=0\,,\end{array}\right. \tag{5.39d}$$

$$\left\{\begin{array}{l}\partial_t\Theta+U\tilde{D}_x\Theta+V\tilde{D}_y\Theta+W\tilde{D}_z\Theta=\Big(\kappa_1\triangle_h+\kappa_2D_z^2\Big)\Theta+h^4\boldsymbol{f}_\rho\,, \\ \Theta_{i,j,-1}=\Theta_{i,j,1}+h^5\boldsymbol{e}_{\rho b}\,,\quad \Theta_{-1,j,k}=\Theta_{1,j,k}+h^5\boldsymbol{e}_{\rho l}\,, \\ \Theta_{i,-1,k}=\Theta_{i,1,k}+h^5\boldsymbol{e}_{\rho d}\,,\end{array}\right. \tag{5.39e}$$

in which the local truncation error and the boundary error terms are bounded in the L^∞ norm

$$|\boldsymbol{f}_{\overline{\omega}}|\,,|\boldsymbol{f}_\xi|\,,|\boldsymbol{f}_\zeta|\,,|\boldsymbol{f}_\rho|\,,|\boldsymbol{e}_{\overline{\omega}}|\,,|\boldsymbol{e}_{\rho b}|\,,|\boldsymbol{e}_{\rho l}|\,,|\boldsymbol{e}_{\rho d}|\le\mathcal{C}^*\,, \tag{5.40}$$

with the constant $\mathcal{C}^*$ depending on the exact solution. The consistency analysis is completed.

Remark 1. *As stated earlier, the purpose of the higher order expansion (5.38) is to obtain the L^∞ estimate of the error function via its L^2 norm in higher order accuracy by utilizing an inverse inequality in spatial discretization. Such an expansion is always possible under suitable regularity assumption of the exact solution. A detailed analysis shows that*

$$|\boldsymbol{v}_e-\boldsymbol{V}|+|w_e-\boldsymbol{W}|+|\rho_e-\Theta|\le Ch^2\,, \tag{5.41}$$

with C was introduced in Theorem 3.1. This estimate will be used later.

6. **Proof of the convergence theorem.** We denote the following error functions

$$\begin{array}{l}\tilde{\boldsymbol{v}}=(\tilde{u},\tilde{v})=\boldsymbol{V}-\boldsymbol{v}=(U-u,V-v)\,,\quad \tilde{w}=\boldsymbol{W}-w\,,\quad \tilde{\overline{\omega}}=\overline{\Omega}-\overline{\omega}\,, \\ \tilde{\overline{\psi}}=\overline{\Psi}-\overline{\psi}\,,\quad \tilde{\xi}=\mathcal{S}-\xi\,,\quad \tilde{\zeta}=\Phi-\zeta\,,\quad \tilde{\rho}=\Theta-\rho\,.\end{array} \tag{6.1}$$

Subtracting (3.2) from (5.39) gives the following system for the error functions:

$$\left\{\begin{aligned}&\partial_t\tilde{\overline{\omega}} + \tilde{D}_x\tilde{D}_y(\overline{\overline{V\tilde{v}+v\tilde{v}}} - \overline{\overline{U\tilde{u}+u\tilde{u}}}) + (D_x^2 - D_y^2)\overline{\overline{V\tilde{u}+u\tilde{v}}} + \frac{\beta}{Ro}\tilde{\overline{v}}\\&\qquad\quad = \nu_1\triangle_h\tilde{\overline{\omega}} + h^4\boldsymbol{f}_{\overline{\omega}}\,,\\&\triangle_h\tilde{\overline{\psi}} = \tilde{\overline{\omega}}\,,\qquad \tilde{\overline{\psi}}\,|_{\partial\mathcal{M}_0} = 0\,,\\&\tilde{\overline{u}} = -\tilde{D}_y\tilde{\overline{\psi}}\,,\qquad \tilde{\overline{v}} = \tilde{D}_x\tilde{\overline{\psi}}\,,\\&\tilde{\overline{\omega}}_{i,0} = \frac{1}{h^2}(4\tilde{\overline{\psi}}_{i,1} - \frac{1}{2}\tilde{\overline{\psi}}_{i,2}) + h^3(\boldsymbol{e}_{\overline{\omega}})_i\,,\end{aligned}\right. \tag{6.2a}$$

$$\left\{\begin{aligned}&\partial_t\tilde{\xi} + \tilde{u}\tilde{D}_x\mathcal{S} + u\tilde{D}_x\tilde{\xi} + \tilde{v}\tilde{D}_y\mathcal{S} + v\tilde{D}_y\tilde{\xi} + \tilde{w}\tilde{D}_z\mathcal{S} + w\tilde{D}_z\tilde{\xi} - \tilde{\xi}\tilde{D}_yV - \xi\tilde{D}_y\tilde{v}\\&\qquad + \tilde{\zeta}\tilde{D}_y\Phi + \zeta\tilde{D}_y\tilde{u} - \frac{f}{Ro}\tilde{\zeta} - \frac{1}{Ro}\tilde{D}_x\tilde{\rho} = (\nu_1\triangle_h + \nu_2D_z^2)\tilde{\xi} + h^4\boldsymbol{f}_{\xi}\,,\\&\partial_t\tilde{\zeta} + \tilde{u}\tilde{D}_x\Phi + u\tilde{D}_x\tilde{\zeta} + \tilde{v}\tilde{D}_y\Phi + v\tilde{D}_y\tilde{\zeta} + \tilde{w}\tilde{D}_z\Phi + w\tilde{D}_z\tilde{\zeta} - \tilde{\zeta}\tilde{D}_xU - \zeta\tilde{D}_x\tilde{u}\\&\qquad - \tilde{\xi}\tilde{D}_xV - \xi\tilde{D}_x\tilde{v} + \frac{f}{Ro}\tilde{\xi} - \frac{1}{Ro}\tilde{D}_y\tilde{\rho} = (\nu_1\triangle_h + \nu_2D_z^2)\tilde{\zeta} + h^4\boldsymbol{f}_{\zeta}\,,\\&\tilde{\xi}_{i,j,0} = 0\,,\quad \tilde{\zeta}_{i,j,0} = 0\,,\quad \tilde{\xi}_{i,j,N} = 0\,,\quad \tilde{\zeta}_{i,j,N} = 0\,,\\&\tilde{\xi}_{i,0,k} = 0\,,\quad \tilde{\zeta}_{i,0,k} = 0\,,\quad \tilde{\xi}_{0,j,k} = 0\,,\quad \tilde{\zeta}_{0,j,k} = 0\,,\end{aligned}\right. \tag{6.2b}$$

$$\left\{\begin{aligned}&\tilde{D}_z\tilde{u} = \tilde{\xi}\,,\qquad \tilde{D}_z\tilde{v} = \tilde{\zeta}\,,\qquad \text{at}\quad (i,j,k)\,,\quad 1 \le k \le N-1\,,\\&\overline{\overline{\tilde{u}}}_{i,j} = \tilde{\overline{u}}_{i,j}\,,\quad \overline{\overline{\tilde{v}}}_{i,j} = \tilde{\overline{v}}_{i,j}\,,\\&(\tilde{D}_z\tilde{\boldsymbol{v}})_{i,j,0} = (\tilde{D}_z\tilde{\boldsymbol{v}})_{i,j,N} = 0\,,\end{aligned}\right. \tag{6.2c}$$

$$\left\{\begin{aligned}&D_z^2\tilde{w} = -\tilde{D}_x\tilde{\xi} - \tilde{D}_y\tilde{\zeta}\,,\qquad \text{at}\quad (i,j,k)\,,\quad 1 \le k \le N_z - 1\,,\\&\tilde{w}_{i,j,0} = \tilde{w}_{i,j,N} = 0\,,\end{aligned}\right. \tag{6.2d}$$

$$\left\{\begin{aligned}&\partial_t\tilde{\rho} + \tilde{u}\tilde{D}_x\Theta + u\tilde{D}_x\tilde{\rho} + \tilde{v}\tilde{D}_y\Theta + v\tilde{D}_y\tilde{\rho} + \tilde{w}\tilde{D}_z\Theta + w\tilde{D}_z\tilde{\rho}\\&\qquad\quad = \Big(\kappa_1\triangle_h + \kappa_2D_z^2\Big)\tilde{\rho} + h^4\boldsymbol{f}_{\rho}\,,\\&\tilde{\rho}_{i,j,-1} = \tilde{\rho}_{i,j,1} + h^5\boldsymbol{e}_{\rho b}\,,\quad \tilde{\rho}_{-1,j,k} = \tilde{\rho}_{1,j,k} + h^5\boldsymbol{e}_{\rho l}\,,\\&\tilde{\rho}_{i,-1,k} = \tilde{\rho}_{i,1,k} + h^5\boldsymbol{e}_{\rho d}\,.\end{aligned}\right. \tag{6.2e}$$

The convergence of the numerical scheme is based on the energy estimate of the above system. The following preliminary results will be needed in the detailed analysis for the error functions. The verification is straightforward and the proof is skipped.

$$\|f\|_\infty \le \frac{C}{h^{\frac{3}{2}}}\|f\|_{L^2}\,, \tag{6.3a}$$

$$\|\tilde{u}\|_0 \le C_1(\|\tilde{\overline{u}}\|_2 + \|\tilde{\xi}\|_0)\,,\quad \|\tilde{v}\|_0 \le C_1(\|\tilde{\overline{v}}\|_2 + \|\tilde{\zeta}\|_0)\,, \tag{6.3b}$$

$$\|\tilde{w}\|_0 \le C_2(\|\tilde{D}_x\tilde{\xi}\|_0 + \|\tilde{D}_y\tilde{\zeta}\|_0)\,. \tag{6.3c}$$

Note that C_1, C_2 depends only on the geometric size of $\mathcal{M}$ and H_0.

In the analysis of the error function presented in the following sections, we assume a-priori that

$$\|\tilde{\boldsymbol{v}}\|_{L^\infty} + \|\tilde{\xi}\|_{L^\infty} + \|\tilde{\zeta}\|_{L^\infty} \le h^{\frac{5}{4}}\,. \tag{6.4}$$

Such a-priori assumption will be verified later by the inverse inequality given in (6.3a). Furthermore, the system of the difference equations (6.2c), (6.2d) shows that

$$\|\tilde{\boldsymbol{v}}\|_{L^\infty} \le C(\|\tilde{\overline{\boldsymbol{v}}}\|_{L^\infty} + \|\tilde{\xi}\|_{L^\infty} + \|\tilde{\zeta}\|_{L^\infty}) \,, \|\tilde{w}\|_{L^\infty} \le C(\|\tilde{D}_x\tilde{\xi}\|_{L^\infty} + \|\tilde{D}_y\tilde{\zeta}\|_{L^\infty}) \,. \tag{6.5}$$

Therefore, the a-priori assumption indicates the following L^∞ bound for the error functions of both the horizontal and vertical velocities

$$\|\tilde{\boldsymbol{v}}\|_{L^\infty} \le Ch^{\frac{5}{4}} \,, \qquad \|\tilde{w}\|_{L^\infty} \le Ch^{\frac{1}{4}} \,. \tag{6.6}$$

6.1. **Estimate of the mean vorticity equation.** Multiplying the error equation for the mean vorticity in (6.2a) by $-\tilde{\overline{\psi}}$ and summing over interior points (of the 2-D horizontal domain $\mathcal{M}_0$) (i,j) with $i,j = 1, ...N-1$ gives

$$\begin{aligned} -\langle\tilde{\overline{\psi}}, \partial_t\tilde{\overline{\omega}}\rangle_2 + \nu_1\langle\tilde{\overline{\psi}}, \triangle_h\tilde{\overline{\omega}}\rangle_2 = \quad & \langle\tilde{\overline{\psi}}, \tilde{D}_x\tilde{D}_y(\overline{\overline{V\tilde{v}+v\tilde{v}}} - \overline{\overline{U\tilde{u}+u\tilde{u}}})\rangle_2 \\ & +\langle\tilde{\overline{\psi}}, (D_x^2 - D_y^2)\overline{\overline{V\tilde{u}+u\tilde{v}}}\rangle_2 - h^4\langle\tilde{\overline{\psi}}, \boldsymbol{f}_{\overline{\omega}}\rangle_2 \,. \end{aligned} \tag{6.7}$$

The time marching term and the diffusion term can handled by using the homogeneous Dirichlet boundary condition for $\tilde{\overline{\psi}}$ in the discrete Poisson equation, along with summation by parts on regular numerical grid

$$-\langle\tilde{\overline{\psi}}, \partial_t\tilde{\overline{\omega}}\rangle_2 = -\langle\tilde{\overline{\psi}}, \partial_t\triangle_h\tilde{\overline{\psi}}\rangle_2 = \frac{1}{2}\frac{d}{dt}\|\nabla_h\tilde{\overline{\psi}}\|_2^2 \,, \tag{6.8}$$

$$\langle\tilde{\overline{\psi}}, \triangle_h\tilde{\overline{\omega}}\rangle_2 = \langle\tilde{\overline{\psi}}, (D_x^2 + D_y^2)\tilde{\overline{\omega}}\rangle_2 = \langle D_x^2\tilde{\overline{\psi}}, \tilde{\overline{\omega}}\rangle_2 + \langle D_y^2\tilde{\overline{\psi}}, \tilde{\overline{\omega}}\rangle_2 + \mathcal{B} = \|\tilde{\overline{\omega}}\|_2^2 + \mathcal{B}^{\overline{\omega}} \,; \tag{6.9a}$$

the boundary term $\mathcal{B}^{\overline{\omega}}$ can be decomposed into $\mathcal{B}^{\overline{\omega}} = \mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4$ as follows

$$\begin{aligned} \mathcal{B}_1 &= \sum_{i=1}^{N-1} \tilde{\overline{\psi}}_{i,1}\tilde{\overline{\omega}}_{i,0} \,, \quad \mathcal{B}_2 = \sum_{i=1}^{N-1} \tilde{\overline{\psi}}_{i,N-1}\tilde{\overline{\omega}}_{i,N} \,, \\ \mathcal{B}_3 &= \sum_{j=1}^{N-1} \tilde{\overline{\psi}}_{1,j}\tilde{\overline{\omega}}_{0,j} \,, \quad \mathcal{B}_4 = \sum_{j=1}^{N-1} \tilde{\overline{\psi}}_{N-1,j}\tilde{\overline{\omega}}_{N,j} \,. \end{aligned} \tag{6.9b}$$

An estimate to control the boundary term $\mathcal{B}^{\overline{\omega}}$ is needed to ensure the numerical stability. We only consider $\mathcal{B}_1$ for conciseness of presentation. Wilkes' boundary formula for the error function of the mean vorticity as formulated in (6.2a) can be applied to recover $\mathcal{B}_1$

$$\mathcal{B}_1 = \sum_{i=1}^{N-1} \tilde{\overline{\psi}}_{i,1}\tilde{\overline{\omega}}_{i,0} = \frac{1}{h^2}\sum_{i=1}^{N-1} \tilde{\overline{\psi}}_{i,1}(4\tilde{\overline{\psi}}_{i,1} - \frac{1}{2}\tilde{\overline{\psi}}_{i,2}) + h^3\sum_{i=1}^{N-1} \tilde{\overline{\psi}}_{i,1}(\boldsymbol{e}_{\overline{\omega}})_i \equiv I_1 + I_2 \,. \tag{6.10}$$

The term I_2 can be controlled by Cauchy inequality

$$I_2 = \sum_{i=1}^{N-1} h^3\tilde{\overline{\psi}}_{i,1}(\boldsymbol{e}_{\overline{\omega}})_i \ge -\frac{1}{2}\sum_{i=1}^{N-1}\frac{\tilde{\overline{\psi}}_{i,1}^2}{h^2} - \frac{1}{2}\sum_{i=1}^{N-1} h^8(\boldsymbol{e}_{\overline{\omega}})_i^2 \ge -\frac{1}{2}\sum_{i=1}^{N-1}\frac{\tilde{\overline{\psi}}_{i,1}^2}{h^2} - Ch^7\|\boldsymbol{u}_e\|_{C^5}^2 \,, \tag{6.11}$$

where the last step comes from the estimate that $|(\boldsymbol{e}_{\overline{\omega}})_i| \le C\|\boldsymbol{u}_e\|_{C^5}$ and the identity $h = \frac{1}{N}$. Meanwhile, a direct calculation can not control the term I_1, since two interior points $\overline{\psi}_{i,1}$ and $\overline{\psi}_{i,2}$ of mean stream function are involved in Wilkes' formula.

To overcome this difficulty, we use the property that $\overline{\psi}$ vanishes on the lateral boundary and then rewrite the term appearing in the parentheses as

$$4\tilde{\overline{\psi}}_{i,1} - \frac{1}{2}\tilde{\overline{\psi}}_{i,2} = 3\tilde{\overline{\psi}}_{i,1} - \frac{1}{2}h^2(D_y^2\tilde{\overline{\psi}})_{i,1}\,. \tag{6.12}$$

The purpose of this transformation is to control local terms by global terms as can be seen later. Such methodology can be found in [28] for the treatment of the usual 2-D Navier-Stokes equations. Plugging (6.12) back into I_1 indicates

$$I_1 = \frac{1}{h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}_{i,1}\Big(3\tilde{\overline{\psi}}_{i,1} - \frac{1}{2}h^2(D_y^2\tilde{\overline{\psi}})_{i,1}\Big) = \frac{3}{h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}^2_{i,1} - \frac{1}{2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}_{i,1}(D_y^2\tilde{\overline{\psi}})_{i,1}\,, \tag{6.13}$$

and applying Cauchy inequality to the second term shows that

$$\begin{aligned} I_1 \geq\ & \frac{3}{h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}^2_{i,1} - \frac{1}{8h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}^2_{i,1} - \frac{1}{2}\sum_{i=1}^{N-1}\Big|(D_y^2\tilde{\overline{\psi}})_{i,1}\Big|^2 h^2 \\ \geq\ & \frac{2}{h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}^2_{i,1} - \frac{1}{2}\sum_{i=1}^{N-1}\Big|D_y^2\tilde{\overline{\psi}}_{i,1}\Big|^2 h^2\,. \end{aligned} \tag{6.14}$$

Consequently, we arrive at

$$\mathcal{B}_1 \geq \frac{1}{h^2}\sum_{i=1}^{N-1}\tilde{\overline{\psi}}^2_{i,1} - \frac{1}{2}\sum_{i=1}^{N-1}\Big|D_y^2\tilde{\overline{\psi}}_{i,1}\Big|^2 h^2 - \frac{1}{4}h^6\,, \tag{6.15}$$

for sufficiently small h. The treatment of the other three boundary terms is essentially the same. Now we recover $\mathcal{B}^{\overline{\omega}}$ by global terms $\|D_x^2\tilde{\overline{\psi}}\|_2^2$ and $\|D_y^2\tilde{\overline{\psi}}\|_2^2$

$$\mathcal{B}^{\overline{\omega}} \geq -\frac{1}{2}\|D_y^2\tilde{\overline{\psi}}\|_2^2 - \frac{1}{2}\|D_x^2\tilde{\overline{\psi}}\|_2^2 - h^6\,. \tag{6.16}$$

The two global terms in (6.16) can be controlled by the diffusion term $\|\tilde{\overline{\omega}}\|_2^2$. The following lemma is necessary.

Lemma 6.1. *For any ψ such that $\psi\,|_{\partial\mathcal{M}_0} = 0$, we have*

$$\|D_x^2\psi\|_2^2 + \|D_y^2\psi\|_2^2 \leq \|(D_x^2 + D_y^2)\psi\|_2^2\,, \qquad \|\tilde{D}_x\tilde{D}_y\psi\|_2 \leq \|(D_x^2 + D_y^2)\psi\|_2\,. \tag{6.17}$$

Proof. Since $\psi_{i,j}$ vanishes on $i,j = 0, N$, the Sine transformation can be made for $\{\psi_{i,j}\}$ in both i-direction and j-direction, i.e.,

$$\psi_{i,j} = \sum_{k,\ell=1}^{N-1}\Big(\frac{2}{\sqrt{2N}}\Big)^2\hat{\psi}_{k,\ell}\,\sin(k\pi x_i)\,\sin(\ell\pi y_j)\,. \tag{6.18}$$

Subsequently, the Parseval equality reads

$$\sum_{i,j=1}^{N-1}(\psi_{i,j})^2 = \sum_{k,\ell=1}^{N-1}\Big|\hat{\psi}_{k,\ell}\Big|^2\,. \tag{6.19}$$

With the introduction of eigenvalues

$$f_k = -\frac{4}{h^2}\sin^2(\frac{k\pi h}{2})\,, \qquad g_\ell = -\frac{4}{h^2}\sin^2(\frac{\ell\pi h}{2})\,, \tag{6.20}$$

we obtain the Fourier expansion of $D_x^2\psi$ and $D_y^2\psi$

$$\begin{aligned} D_x^2\psi_{i,j} &= \sum_{k,l=1}^{N-1} \Big(\frac{2}{\sqrt{2N}}\Big)^2 f_k\, \hat{\psi}_{k,l} \sin(k\pi x_i)\sin(\ell\pi y_j)\,, \\ D_y^2\psi_{i,j} &= \sum_{k,l=1}^{N-1} \Big(\frac{2}{\sqrt{2N}}\Big)^2 g_\ell\, \hat{\psi}_{k,l} \sin(k\pi x_i)\sin(\ell\pi y_j)\,, \end{aligned} \tag{6.21}$$

which in turn implies that

$$\sum_{i,j=1}^{N-1} |(D_x^2+D_y^2)\psi_{i,j}|^2 = \sum_{i,j=1}^{N-1} |\triangle_h \psi_{i,j}|^2 = \sum_{k,\ell=1}^{N-1} |g_\ell + f_k|^2\, |\hat{\psi}_{k,\ell}|^2\,. \tag{6.22}$$

Since both f_k and g_ℓ are non-positive, we have $(f_k+g_\ell)^2 \ge f_k^2 + g_\ell^2$. As a result, we arrive at

$$\sum_{i,j=1}^{N-1} |(D_x^2+D_y^2)\psi_{i,j}|^2 \ge \sum_{k,\ell=1}^{N-1} (f_k^2+g_\ell^2)|\hat{\psi}_{k,\ell}|^2 = \sum_{i,j=1}^{N-1} (|D_x^2\psi_{i,j}|^2 + |D_y^2\psi_{i,j}|^2)\,. \tag{6.23}$$

This is exactly the first inequality in (6.17).

The second inequality can be similarly proven. It is observed that

$$\tilde{D}_x\tilde{D}_y\psi_{i,j} = \sum_{k,l=1}^{N-1} q_k\, r_\ell \Big(\frac{2}{\sqrt{2N}}\Big)^2 \hat{\psi}_{k,l} \cos(k\pi x_i)\cos(\ell\pi y_j)\,, \tag{6.24}$$

with the corresponding eigenvalues

$$q_k = \frac{1}{h}\sin(k\pi h)\,, \qquad r_\ell = \frac{1}{h}\sin(\ell\pi h)\,. \tag{6.25}$$

Applying Parseval equality for the discrete Cosine transformation, we have

$$\begin{aligned} \sum_{i,j=1}^{N-1} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2 \le & \sum_{i,j=1}^{N-1} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2 + \frac{1}{4}\sum_{i=0,N}\sum_{j=0,N} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2 \\ & + \frac{1}{2}\sum_{i=1}^{N-1}\sum_{j=0,N} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2 + \frac{1}{2}\sum_{j=1}^{N-1}\sum_{i=0,N} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2 \\ = & \sum_{k,\ell=1}^{N-1} |q_k\cdot r_\ell|^2\, |\hat{\psi}_{k,\ell}|^2\,. \end{aligned} \tag{6.26}$$

Meanwhile, a careful calculation shows that

$$(q_k\cdot r_\ell)^2 \le (f_k+g_\ell)^2\,, \qquad \forall\ 1 \le k,l \le N-1\,. \tag{6.27}$$

The combined estimates of (6.22), (6.26) and (6.27) results in

$$\sum_{i,j=1}^{N-1} |(D_x^2+D_y^2)\psi_{i,j}|^2 \ge \sum_{k,\ell=1}^{N-1} (q_k\cdot r_\ell)^2|\hat{\psi}_{k,\ell}|^2 \ge \sum_{i,j=1}^{N-1} |\tilde{D}_x\tilde{D}_y\psi_{i,j}|^2\,, \tag{6.28}$$

which is equivalent to the second inequality in (6.17). Lemma 6.1 is proven. □

It is straightforward to observe that Lemma 6.1 is valid for $\tilde{\overline{\psi}}$ since it vanishes on the lateral boundary sections. Then we have

$$\|D_x^2\tilde{\overline{\psi}}\|_2^2 + \|D_y^2\tilde{\overline{\psi}}\|_2^2 \le \|(D_x^2 + D_y^2)\tilde{\overline{\psi}}\|_2^2 = \|\tilde{\overline{\omega}}\|_2^2 \,. \tag{6.29}$$

Substituting (6.29) into (6.16) and (6.9), we obtain the following estimate for the diffusion term

$$\langle\tilde{\overline{\psi}}, \triangle_h\tilde{\overline{\omega}}\rangle_2 \ge \frac{1}{2}\|\tilde{\overline{\omega}}\|_2^2 - h^6 \,. \tag{6.30}$$

The nonlinear convection terms can also be controlled by applying Lemma 6.1. We consider the term $\tilde{D}_x\tilde{D}_y(\overline{\overline{V\tilde{v} + v\tilde{v}}})$. The rest three terms can be handled in the same way. Summing by parts gives

$$\langle\tilde{\overline{\psi}}, \tilde{D}_x\tilde{D}_y(\overline{\overline{V\tilde{v} + v\tilde{v}}})\rangle_2 = \langle\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}, \overline{\overline{V\tilde{v} + v\tilde{v}}}\rangle_2 = \langle\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}, \overline{\overline{V\tilde{v}}}\rangle_2 + \langle\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}, \overline{\overline{v\tilde{v}}}\rangle_2 \,, \tag{6.31}$$

due to the vanishing boundary condition for both $\tilde{\overline{\psi}}$ and the velocity field. Cauchy inequality and the second inequality in (6.17) can be applied to both terms

$$\begin{aligned}
\left|\langle\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}, \overline{\overline{V\tilde{v}}}\rangle_2\right| &\le \frac{\nu_1}{32}\|\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}\|_2^2 + \frac{8}{\nu_1}\|\overline{\overline{V\tilde{v}}}\|_2^2 \le \frac{\nu_1}{32}\|\tilde{\overline{\omega}}\|_2^2 + \frac{8}{\nu_1}\|V\|_{L^\infty}\|\tilde{v}\|_2^2 \,, \\
\left|\langle\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}, \overline{\overline{v\tilde{v}}}\rangle_2\right| &\le \frac{\nu_1}{32}\|\tilde{D}_x\tilde{D}_y\tilde{\overline{\psi}}\|_2^2 + \frac{8}{\nu_1}\|\overline{\overline{v\tilde{v}}}\|_2^2 \le \frac{\nu_1}{32}\|\tilde{\overline{\omega}}\|_2^2 + \frac{8}{\nu_1}\|v\|_{L^\infty}\|\tilde{v}\|_2^2 \,.
\end{aligned} \tag{6.32}$$

By the a-priori bound (6.6) and estimate for the constructed V, we have

$$\|V\|_{L^\infty} \le \|\boldsymbol{v}_e\|_{C^0} + \frac{1}{2}\,,\ \|v\|_{L^\infty} \le \|V\|_{L^\infty} + \|\tilde{v}\|_{L^\infty} \le \|\boldsymbol{u}_e\|_{C^0} + 1 = \tilde{C}_1 \,, \tag{6.33}$$

provided that h is small enough. The combination of (6.31), (6.32) and (6.33) results in

$$\left|\langle\tilde{\overline{\psi}}, \tilde{D}_x\tilde{D}_y(\overline{\overline{V\tilde{v} + v\tilde{v}}})\rangle_2\right| \le \frac{\nu_1}{16}\|\tilde{\overline{\omega}}\|_2^2 + \frac{16}{\nu_1}\tilde{C}_1\|\tilde{\boldsymbol{v}}\|_2^2 \,, \tag{6.34}$$

with $\tilde{C}_1 = \|\boldsymbol{u}_e\|_{C^0} + 1$. Similar results can be obtained for the three other boundary terms

$$\left|\langle\tilde{\overline{\psi}}, \tilde{D}_x\tilde{D}_y(\overline{\overline{U\tilde{u} + u\tilde{u}}})\rangle_2\right|, \left|\langle\tilde{\overline{\psi}}, D_x^2\overline{\overline{V\tilde{u} + u\tilde{v}}}\rangle_2\right|, \left|\langle\tilde{\overline{\psi}}, D_y^2\overline{\overline{V\tilde{u} + u\tilde{v}}}\rangle_2\right| \le \frac{\nu_1}{16}\|\tilde{\overline{\omega}}\|_2^2 + \frac{16}{\nu_1}\tilde{C}_1\|\tilde{\boldsymbol{v}}\|_2^2 \,. \tag{6.35}$$

The force term in (6.7) can be easily controlled

$$h^4|\langle\tilde{\overline{\psi}}, \boldsymbol{f}_{\overline{\omega}}\rangle_2| \le \frac{1}{2}\|\tilde{\overline{\psi}}\|_2^2 + \frac{1}{2}h^8\|\boldsymbol{f}_{\overline{\omega}}\|_2^2 \le C\|\nabla_h\tilde{\overline{\psi}}\|_2^2 + \frac{1}{2}h^8\|\boldsymbol{f}_{\overline{\omega}}\|_2^2 \,, \tag{6.36}$$

in which the Poincare inequality for $\tilde{\overline{\psi}}$ was used. Substituting (6.36), (6.35), (6.34), (6.30) and (6.8) into (6.7), we get the energy estimate for the mean vorticity error function

$$\frac{1}{2}\frac{d}{dt}\|\nabla_h\tilde{\overline{\psi}}\|_2^2 + \frac{\nu_1}{4}\|\tilde{\overline{\omega}}\|_2^2 \le \frac{64}{\nu_1}\tilde{C}_1\|\tilde{\boldsymbol{v}}\|_2^2 + h^6 + \frac{1}{2}h^8\|\boldsymbol{f}_{\overline{\omega}}\|_2^2 \,. \tag{6.37}$$

6.2. **Estimate for the evolution equations of $(\tilde{\xi}, \tilde{\zeta})$.** Multiplying the first error equation in (6.2b) by $\tilde{\xi}$ and summing over interior points (of the 3-D domain $\mathcal{M}$) (i,j,k) with $i,j,k = 1, ...N-1$ give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\xi}\|_0^2 - \nu_1\langle\tilde{\xi}, \triangle_h\tilde{\xi}\rangle_0 - \nu_2\langle\tilde{\xi}, D_z^2\tilde{\xi}\rangle_0 = h^4\langle\tilde{\xi}, \boldsymbol{f}_\xi\rangle_0 - \langle\tilde{\xi}, \tilde{u}\tilde{D}_x\mathcal{S} + u\tilde{D}_x\tilde{\xi}\rangle_0 \\
&\quad -\langle\tilde{\xi}, \tilde{v}\tilde{D}_y\mathcal{S} + v\tilde{D}_y\tilde{\xi}\rangle_0 - \langle\tilde{\xi}, \tilde{w}\tilde{D}_z\mathcal{S} + w\tilde{D}_z\tilde{\xi}\rangle_0 + \langle\tilde{\xi}, \tilde{\xi}\tilde{D}_y V + \xi\tilde{D}_y\tilde{v}\rangle_0 \\
&\quad -\langle\tilde{\xi}, \tilde{\zeta}\tilde{D}_y U + \zeta\tilde{D}_y\tilde{u}\rangle_0 + \langle\tilde{\xi}, \frac{f}{Ro}\tilde{\zeta}\rangle_0 + \frac{1}{Ro}\langle\tilde{\xi}, \tilde{D}_x\tilde{\rho}\rangle_0 \,.
\end{aligned} \tag{6.38}$$

The homogeneous Dirichlet boundary condition for $\tilde{\xi}$ indicates that

$$\langle\tilde{\xi}, \triangle_h\tilde{\xi}\rangle_0 = -\|\nabla_h\tilde{\xi}\|_0^2\,, \quad \langle\tilde{\xi}, D_z^2\tilde{\xi}\rangle_0 = -\|D_z\tilde{\xi}\|_0^2\,. \tag{6.39}$$

The nonlinear convection term can be handled in a similar fashion as in section 6.1. We obtain the following estimate. The details are left for interested readers.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\zeta}\|_0^2 + \frac{5}{8}\nu_1(\|D_x\tilde{\zeta}\|_0^2 + \|D_y\tilde{\zeta}\|_0^2) + \frac{7}{8}\nu_2\|D_z\tilde{\zeta}\|_0^2 \le \Big(\frac{2\tilde{C}_4^2}{\nu_1} + \tilde{C}_2\Big)\|\tilde{\boldsymbol{v}}\|_0^2 \\
&\quad +(C + \frac{1}{2}\tilde{C}_3)\|\tilde{\xi}\|_0^2 + \Big(\frac{4\tilde{C}_1^2}{\nu_1} + \frac{4\tilde{C}_2^2C_2^2}{\nu_1} + \frac{2\tilde{C}_1^2}{\nu_2} + \frac{C}{\kappa_1} + \tilde{C}_2 + \frac{3}{2}\tilde{C}_3 + C\Big)\|\tilde{\zeta}\|_0^2 \\
&\quad +\frac{1}{8}\nu_1\|D_x\tilde{\xi}\|_0^2 + \frac{1}{8}\kappa_1\|D_y\tilde{\rho}\|_0^2 + \frac{1}{2}h^8\|\boldsymbol{f}_\zeta\|_0^2\,,
\end{aligned} \tag{6.40}$$

with the constants

$$\begin{aligned}
&\tilde{C}_1 = \|\boldsymbol{u}_e\|_{C^0} + 1 \ge \|u\|_{L^\infty}, \qquad \tilde{C}_2 = 2\|\boldsymbol{u}_e\|_{C^2} + 1 \ge \|\mathcal{S}\|_{W^{1,\infty}} + \|\Phi\|_{W^{1,\infty}}\,, \\
&\tilde{C}_3 = \|\boldsymbol{u}_e\|_{C^1} + 1 \ge \|\boldsymbol{V}\|_{W^{1,\infty}}\,, \quad \tilde{C}_4 = 2\|\boldsymbol{u}_e\|_{C^1} + 1 \ge \|\xi\|_{L^\infty} + \|\zeta\|_{L^\infty}\,.
\end{aligned} \tag{6.41}$$

Similarly, multiplying the second error equation in (6.2b) by $\tilde{\zeta}$ and summing over interior points (i,j,k) with $i,j,k = 1, ...N-1$ results in

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\zeta}\|_0^2 + \frac{5}{8}\nu_1(\|D_x\tilde{\zeta}\|_0^2 + \|D_y\tilde{\zeta}\|_0^2) + \frac{7}{8}\nu_2\|D_z\tilde{\zeta}\|_0^2 \le \Big(\frac{2\tilde{C}_4^2}{\nu_1} + \tilde{C}_2\Big)\|\tilde{\boldsymbol{v}}\|_0^2 \\
&\quad +(C + \frac{1}{2}\tilde{C}_3)\|\tilde{\xi}\|_0^2 + \Big(\frac{4\tilde{C}_1^2}{\nu_1} + \frac{4\tilde{C}_2^2C_2^2}{\nu_1} + \frac{2\tilde{C}_1^2}{\nu_2} + \frac{C}{\kappa_1} + \tilde{C}_2 + \frac{3}{2}\tilde{C}_3 + C\Big)\|\tilde{\zeta}\|_0^2 \\
&\quad +\frac{1}{8}\nu_1\|D_x\tilde{\xi}\|_0^2 + \frac{1}{8}\kappa_1\|D_y\tilde{\rho}\|_0^2 + \frac{1}{2}h^8\|\boldsymbol{f}_\zeta\|_0^2\,.
\end{aligned} \tag{6.42}$$

6.3. **Estimate for the density transport equation.** Taking the $\langle\,,\,\rangle_3$ inner product of the density error equation in (6.2e) with $\tilde{\rho}$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|_3^2 - \kappa_1\langle\tilde{\rho}, \triangle_h\tilde{\rho}\rangle_3 - \kappa_2\langle\tilde{\rho}, D_z^2\tilde{\rho}\rangle_3 = h^4\langle\tilde{\rho}, \boldsymbol{f}_\rho\rangle_3 \\
&\quad -\langle\tilde{\rho}, \tilde{u}\tilde{D}_x\Theta + u\tilde{D}_x\tilde{\rho}\rangle_3 - \langle\tilde{\rho}, \tilde{v}\tilde{D}_y\Theta + v\tilde{D}_y\tilde{\rho}\rangle_3 - \langle\tilde{\xi}, \tilde{w}\tilde{D}_z\Theta + w\tilde{D}_z\tilde{\rho}\rangle_3\,.
\end{aligned} \tag{6.43}$$

The following estimate for (6.43) can be similarly obtained. We omit the detail for brevity.

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\tilde{\rho}\|_3^2 + \frac{7}{8}\kappa_1(\|D_x\tilde{\rho}\|_0^2 + \|D_y\tilde{\rho}\|_0^2) + \frac{7}{8}\kappa_2\|D_z\tilde{\rho}\|_0^2 \le \frac{1}{2}h^8\|\boldsymbol{f}_\rho\|_3^2 \\
&\quad +\Big(\tilde{C}_5 + \frac{4\tilde{C}_1^2}{\kappa_1} + \frac{4\tilde{C}_5^2C_2^2}{\nu_1} + \frac{2\tilde{C}_1^2}{\kappa_2}\Big)\|\tilde{\rho}\|_3^2 + \tilde{C}_5\|\tilde{\boldsymbol{v}}\|_0^2\,,
\end{aligned} \tag{6.44}$$

with $\tilde{C}_5 = \|\rho_e\|_{C^1} + 1 \geq \|\Theta\|_{W^{1,\infty}}$.

6.4. **Convergence result.** We conclude from (6.37), (6.41), (6.42), (6.44) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|\nabla_h\tilde{\tilde{\psi}}\|_2^2 + \|\tilde{\xi}\|_0^2 + \|\tilde{\zeta}\|_0^2 + \|\tilde{\rho}\|_3^2\Big) \\
&\leq \Big(\frac{4\tilde{C}_1^2}{\nu_1} + \frac{4\tilde{C}_2^2C_2^2}{\nu_1} + \frac{2\tilde{C}_1^2}{\nu_2} + \frac{C}{\kappa_1} + \tilde{C}_2 + 2\tilde{C}_3 + C\Big)(\|\tilde{\xi}\|_0^2 + \|\tilde{\zeta}\|_0^2) \\
&+\Big(\frac{2\tilde{C}_4^2}{\nu_1} + \tilde{C}_2 + \tilde{C}_5\Big)\|\tilde{\boldsymbol{v}}\|_0^2 + \Big(\tilde{C}_5 + \frac{4\tilde{C}_1^2}{\kappa_1} + \frac{4\tilde{C}_5^2C_2^2}{\nu_1} + \frac{2\tilde{C}_1^2}{\kappa_2}\Big)\|\tilde{\rho}\|_3^2 + \frac{64}{\nu_1}\tilde{C}_1\|\tilde{\tilde{\boldsymbol{v}}}\|_2^2 \\
&+\frac{1}{2}h^8(\|\boldsymbol{f}_{\varpi}\|_2^2 + \|\boldsymbol{f}_\xi\|_0^2 + \|\boldsymbol{f}_\zeta\|_0^2 + \|\boldsymbol{f}_\rho\|_3^2) + Ch^6 .
\end{aligned}
\tag{6.45}
$$

The application of Gronwall's inequality to (6.45) combined with the consistency analysis (5.40) lead to

$$
\|\nabla_h\tilde{\tilde{\psi}}\|_2^2 + \|\tilde{\xi}\|_0^2 + \|\tilde{\zeta}\|_0^2 + \|\tilde{\rho}\|_3^2 \leq \quad C\cdot\exp\Big(\frac{Ct}{\nu_0}\Big)(\mathcal{C}^*)^2h^8 + CTh^6 , \tag{6.46}
$$

where C is given in Theorem 3.1 and $\mathcal{C}^*$ depends only on the exact solution. Note that we used the fact

$$
\|\tilde{\tilde{u}}\|_2 = \|\tilde{D}_y\tilde{\tilde{\psi}}\|_2 \leq \|D_y\tilde{\tilde{\psi}}\|_2 , \quad \|\tilde{\tilde{v}}\|_2 = \|\tilde{D}_x\tilde{\tilde{\psi}}\|_2 \leq \|D_x\tilde{\tilde{\psi}}\|_2 , \tag{6.47}
$$

in the derivation of (6.46). Furthermore, by the preliminary result (6.3b), we conclude that the estimate (6.46) is equivalent to

$$
\|\boldsymbol{v}_h - \boldsymbol{V}\|_{L^\infty(0,T;L^2)} + \|\rho_h - \Theta\|_{L^\infty(0,T;L^2)} \leq C\mathcal{C}^*\Big(\exp\Big\{\frac{CT}{\nu_0}\Big\} + T\Big)h^3 , \tag{6.48}
$$

whose combination with the estimate (5.41) gives the convergence result (3.20). The inverse inequality in 3-D as given in (6.3a) shows that

$$
\|\tilde{\boldsymbol{v}}\|_{L^\infty} \leq C\frac{h^3}{h^{\frac{3}{2}}} \leq Ch^{\frac{3}{2}} , \qquad \|\tilde{\xi}\|_{L^\infty} + \|\tilde{\zeta}\|_{L^\infty} \leq C\frac{h^3}{h^{\frac{3}{2}}} \leq Ch^{\frac{3}{2}} , \tag{6.49}
$$

As a result, the a-priori assumption (6.4) is satisfied if h is small enough. Theorem 3.1 is proven. □

Acknowledgments. The author is grateful to the referees for their insightful comments.

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Received September 2002; revised September 2003; final version February 2004.

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