Fourth Order Convergence of Compact Finite Difference Solver for 2D Incompressible Flow

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ABSTRACT: We study a fourth order finite difference method for the unsteady incompressible Navier-Stokes equations in vorticity formulation. The scheme is essentially compact and can be implemented very efficiently. Either Briley's formula, or a new higher order formula, which will be derived in this paper, can be chosen as the vorticity boundary condition. By formal Taylor expansion, the new formula for the vorticity on the boundary gives 4th order accuracy; while Briley's formula provides only 3rd order accuracy. However, the use of either formula results in a stable method and achieves full 4th order accuracy. The convergence analysis of the scheme with our new formula will be given in this paper, while that with Briley's formula has been established in earlier literature. The consistency analysis is easier than that of Briley's formula, no Strang type analysis is needed. In the stability analysis part, we adopt the technique of controlling some local terms by the diffusion term via discrete elliptic regularity. Physical no-slip boundary conditions are used throughout.

AMS(MOS) Subject Classification. 65M06, 76M20.

1. INTRODUCTION

We start with the 2-D Navier-Stokes equations in vorticity-stream function formulation:

$$
\begin{cases}
\partial_t \omega + \nabla \cdot (\boldsymbol{u}\omega) = \nu \Delta \omega \,, \\
\Delta \psi = \omega \,, \\
u = -\partial_y \psi \,, \qquad v = \partial_x \psi \,,
\end{cases}
\tag{1.1}
$$

with the no-slip boundary condition written in terms of the stream function ψ:

$$
\psi = 0, \quad \frac{\partial \psi}{\partial \mathbf{n}} = 0 \,. \tag{1.2}
$$

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Here $\boldsymbol{u} = (u, v)$ denotes the velocity field, ω denotes the vorticity.

The subject of fourth order schemes for (1.1) and (1.2) has attracted considerable attentions recently. For example, E and Liu proposed an Essentially Compact Fourth Order Scheme (EC4) in [4], and proved the fourth order convergence of the method. Their analysis resorts to high order expansion of Strang type. A technical assumption of one-sided physical, one-sided periodic boundary condition was also imposed.

The purpose of this paper is to thoroughly analyze the fourth order scheme proposed by E and Liu. The boundary condition for vorticity will also be analyzed in detail. Briley's formula, which was derived in [2], was used in [4]. We will derive a new formula in this paper, which gives higher order accuracy for the vorticity on the boundary by formal Taylor expansion.

Then we treat the full Navier-Stokes equations in 2-D with a $[0,1]^2$ box as the domain, with the physical boundary condition (1.2) applied to all boundaries. The convergence proof for the analogous 4th order scheme (EC4) is then presented, with our new 4-th order vorticity boundary condition. The use of this new boundary condition results in no Strang type expansion being needed, thus simplifies the consistency analysis.

The procedure of our convergence proof is standard: consistency analysis and error estimate. The style of consistency analysis is similar to that in [15]. Yet there are still some differences since our fourth order scheme involves an intermediate variable for vorticity. We construct the approximate intermediate vorticity variable via the finite difference of the exact stream function, and recover the approximate vorticity by solving a linear system, whose eigenvalues are controlled, through the approximate intermediate vorticity variable with suitable boundary conditions. To maintain a higher order consistency for vorticity, which will be needed when we compute its finite difference, we add an $O(h^4)$ correction term to the exact vorticity on the boundary when we set our boundary condition for the approximate vorticity. The approximate velocity will be constructed via finite differences of the exact stream function. Then it can be shown that the constructed profiles satisfy the numerical scheme up to an $O(h^4)$ truncation error, including the vorticity on the boundary. Next, we perform stability analysis and error estimate. A technique similar to that used in [4] by E and Liu is adopted. The basic strategy is to use energy estimates, with special care taken at the boundary. Standard local estimates do not work for the boundary terms, due to the interior points of stream function involved in the boundary vorticity formula, so we have to apply elliptic regularity at the discrete level, and then control these local terms by global terms.

In section 2 we outline the main idea of the EC4 scheme and present the derivation of both boundary conditions. The rigorous convergence proof of the method with the new formula as vorticity boundary condition will be presented in section 3, where the consistency analysis is explained in detail. The detailed stability and error estimates of the diffusion term and the convection term are given in section 4 and section 5, respectively.

2. DESCRIPTION OF THE SCHEME

Essentially compact fourth order scheme (EC4) for 2-D Navier-Stokes equations was proposed by E and Liu in [4]. The starting point of the scheme is the fact that Laplacian operator Δ can be approximated with the fourth order by

$$\Delta = \frac{\Delta_h + \frac{h^2}{6} D_x^2 D_y^2}{1 + \frac{h^2}{12}\Delta_h} + O(h^4) . \tag{2.1}$$

Multiplying the denominator difference operator $1+\dfrac{h^2}{12}\Delta_h$ to the momentum equation in (1.1) gives

$$(1 + \frac{h^2}{12}\Delta_h)\partial_t \omega + (1 + \frac{h^2}{12}\Delta_h)\nabla\cdot(\boldsymbol{u}\omega) = \nu\Big(\Delta_h + \frac{h^2}{6} D_x^2 D_y^2\Big)\omega , \tag{2.2}$$

and multiplying the same operator to the kinematic equation leads to

$$\Big(\Delta_h + \frac{h^2}{6} D_x^2 D_y^2\Big)\psi = (1 + \frac{h^2}{12}\Delta_h)\omega . \tag{2.3}$$

As in [4], the corresponding nonlinear convection term in the vorticity dynamic equation can be estimated as

$$\begin{aligned} (1 + \frac{h^2}{12}\Delta_h)(\boldsymbol{u}\cdot\nabla\omega) = \quad & \widetilde{D}_x\Big(1 + \frac{h^2}{6} D_y^2\Big)(u\omega) + \widetilde{D}_y\Big(1 + \frac{h^2}{6} D_x^2\Big)(v\omega) \\ - \quad & \frac{h^2}{12}\Delta_h\Big(u\widetilde{D}_x\omega + v\widetilde{D}_y\omega\Big) + O(h^4) . \end{aligned} \tag{2.4}$$

The first and the second terms in (2.4) are compact. The third term is not compact, yet it does not cause any trouble in practical computations since $u^n\widetilde{D}_x\omega^n + v^n\widetilde{D}_y\omega^n$ can be taken as 0 on the boundary. The case of boundary condition with slip can be treated similarly, as discussed in [4].

Thus, by the introduction of an intermediate variable $\overline{\omega}$,

$$\overline{\omega} = (1 + \frac{h^2}{12}\Delta_h)\omega , \tag{2.5}$$

NSE can be approximated by

$$\begin{cases} \partial_t\overline{\omega} = (\Delta_h + \dfrac{h^2}{6} D_x^2 D_y^2)\omega - \mathcal{NL} , \\ (\Delta_h + \dfrac{h^2}{6} D_x^2 D_y^2)\psi = \overline{\omega} , \qquad \psi\mid_\Gamma = 0 , \\ (1 + \dfrac{h^2}{12}\Delta_h)\omega = \overline{\omega} , \end{cases} \tag{2.6}$$

where the approximate nonlinear term $\mathcal{NL}$ is given by

$$\mathcal{NL} = \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)(u\omega) + \widetilde{D}_y(1+\frac{h^2}{6}D_x^2)(v\omega) - \frac{h^2}{12}\Delta_h(u\widetilde{D}_x\omega + v\widetilde{D}_y\omega)\,. \tag{2.7}$$

Note that the implementation of the third term of (2.7) needs the boundary value of $u\widetilde{D}_x\omega + v\widetilde{D}_y\omega$, which is set to be 0 since the velocity $\boldsymbol{u}$ vanishes on the boundary.

The velocity $\boldsymbol{u} = (-\partial_y\psi, \partial_x\psi)$ can be estimated by using the standard long-stencil 4-th order formulas:

$$u = -\widetilde{D}_y(1-\frac{h^2}{6}\,D_y^2)\psi\,, \quad v = \widetilde{D}_x(1-\frac{h^2}{6}\,D_x^2)\psi\,. \tag{2.8}$$

Note that the implementation of (2.8) near the boundary needs "ghost point" values of ψ, which will be discussed in the next subsection.

The vorticity is determined by $\overline{\omega}$ via (2.5), whose implementation needs the boundary condition for ω, which is the main issue in the next subsection.

2.1 Vorticity Boundary Condition

As pointed out in [3], [4], there are two boundary conditions for ψ. The Dirichlet boundary condition $\psi\,|_\Gamma = 0$ can be implemented to solve the stream function. Yet the normal boundary condition $\dfrac{\partial\psi}{\partial\boldsymbol{n}} = 0$, which cannot be enforced directly, could be converted into the boundary condition for the vorticity. For example, Briley's formula

$$\omega_{i,0} = \frac{1}{h^2}(6\psi_{i,1} - \frac{3}{2}\psi_{i,2} + \frac{2}{9}\psi_{i,3}) - \frac{11}{3h}\left(\frac{\partial\psi}{\partial y}\right)_{i,0}\,, \tag{2.9}$$

was used in the EC4 scheme (see [2], [4]). It should be noted that Briley's formula is only third order accurate for the vorticity on the boundary by formal local Taylor expansion. Yet it still preserves 4th order accuracy, as was first proved in [4] and argued by a 1-D model in [15]. Next, we derive our new 4th order vorticity boundary condition. First, a 4th order approximation of $\omega = (\partial_x^2 + \partial_y^2)\psi$ can be applied on the boundary

$$\omega_{i,0} = \partial_y^2\psi_{i,0} = \frac{1}{12h^2}\Big(16(\psi_{i,-1}+\psi_{i,1}) - (\psi_{i,-2}+\psi_{i,2})\Big) + O(h^4)\,, \tag{2.10}$$

where in the first step we used the fact that ψ vanishes on the boundary, and $(i,-1)$, $(i,-2)$ refer to the "ghost" grid points outside the computational domain. Note that we need five points of ψ to obtain fourth order accuracy for ω, which is different from the second order case, where we only need three points of ψ, as discussed in [15]. Then the values for the "ghost" points of ψ are prescribed by using the no-slip boundary condition $\dfrac{\partial\psi}{\partial y} = 0$, on $y = 0$, along with a 6-th order one-sided approximations for ψ:

$$\psi_{i,-1} = 10\psi_{i,1} - 5\psi_{i,2} + \frac{5}{3}\psi_{i,3} - \frac{1}{4}\psi_{i,4} - 5h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O(h^6)\,, \tag{2.11}$$

and

$$\psi_{i,-2} = 80\psi_{i,1} - 45\psi_{i,2} + 16\psi_{i,3} - \frac{5}{2}\psi_{i,4} - 30h\left(\frac{\partial\psi}{\partial y}\right)_{i,0} + O(h^6)\,. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we obtain

$$\omega_{i,0} = \frac{1}{h^2}(8\psi_{i,1} - 3\psi_{i,2} + \frac{8}{9}\psi_{i,3} - \frac{1}{8}\psi_{i,4}) - \frac{25}{6h}\left(\frac{\partial\psi}{\partial x}\right)_{i,0}\,. \tag{2.13}$$

The last terms in (2.11), (2.12) and (2.13) vanish if no-slip boundary condition for velocity is assumed. This new formula is used to perform our analysis. The system (2.6) along with the boundary condition (2.9) or (2.13) can be implemented very efficiently via an explicit time-stepping procedure introduced by E and Liu in [4].

The following is the main theorem in this paper.

Theorem 2.1. *Let $\boldsymbol{u}_e \in L^\infty([0,T]; C^{7,\alpha}(\overline{\Omega}))$ be the solution of the Navier-Stokes equations (1.1)-(1.2) and $\boldsymbol{u}_h$ be the approximate solution of EC4, then we have*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^\infty([0,T],L^2)} \le Ch^4\|\boldsymbol{u}_e\|_{L^\infty([0,T],C^{7,\alpha})}(1 + \|\boldsymbol{u}_e\|_{L^\infty([0,T],C^5)})\exp\left\{\frac{CC^*T}{\nu}\right\}\,, \tag{2.14}$$

where $C^* = (1 + \|\boldsymbol{u}_e\|_{L^\infty([0,T],C^5)})^2$.

Here are some notations which will be used later.

Notation. We will use the discrete L^2-norm and the discrete L^2-inner product

$$\|u\| = \langle u\,,\, u\rangle^{1/2}\,, \qquad \text{where} \quad \langle u\,,\, v\rangle = \sum_{1\le i,j\le N-1} u_{i,j}\, v_{i,j}\, h^2\,. \tag{2.15}$$

For $u\,|_\Gamma = 0$, we introduce the notation $\|\nabla_h u\|$ by defining

$$\|\nabla_h u\|^2 = \sum_{0\le i,j\le N-1} \left\{(D_x^+ u_{i,j})^2 + (D_y^+ u_{i,j})^2\right\}\ h^2\,, \tag{2.16}$$

where $D_x^+ u_{i,j} = \dfrac{u_{i+1,j} - u_{i,j}}{h}$ and $D_y^+ u_{i,j} = \dfrac{u_{i,j+1} - u_{i,j}}{h}$.

3. CONVERGENCE PROOF

The convergence analysis follows from the consistency analysis, stability and error estimate. As can be seen, direct truncation error analysis gives us fourth order accuracy for both the momentum equation and the vorticity on the boundary if the new formula is used. The methodology in the consistency analysis is to construct approximate velocity and vorticity via the exact stream function to satisfy NSE up to $O(h^4)$ order. Yet, the construction of the approximate vorticity needs some technique: first, an approximate intermediate vorticity variable will be constructed via the finite difference of the exact stream function; then, the approximate vorticity field will be constructed by solving a linear system using the approximate intermediate vorticity variable. The eigenvalues corresponding to the linear system are controlled. To maintain higher order consistency for the approximate vorticity, we add an $O(h^4)$ correction term to the exact vorticity on the boundary when we set its boundary condition, which makes it easier when its finite differences are computed.

3.1. Consistency Analysis

We denote $\psi_e, \boldsymbol{u}_e, \omega_e$ as the exact solutions of NSE, and extend ψ_e smoothly to $[-\delta, 1+\delta]^2$. Let $\Psi_{i,j} = \psi_e(x_i, y_j)$ for $-2 \le i, j \le N+2$, and construct U, V via the finite differences of Ψ

$$U_{i,j} = -\widetilde{D}_y(1 - \frac{h^2}{6} D_y^2)\Psi \,, \qquad V_{i,j} = \widetilde{D}_x(1 - \frac{h^2}{6} D_x^2)\Psi \,, \qquad \text{for } 0 \le i, j \le N \,. \tag{3.1}$$

The construction of the approximate vorticity is quite tricky. First we define

$$\overline{\Omega}_{i,j} = (\Delta_h + \frac{h^2}{6} D_x^2 D_y^2)\Psi \,, \qquad \text{for } 1 \le i, j \le N-1 \,, \tag{3.2}$$

and then recover Ω by solving the following system

$$(1 + \frac{h^2}{12}\Delta_h)\Omega_{i,j} = \overline{\Omega}_{i,j} \,. \tag{3.3}$$

It should be mentioned that (3.3) always has a solution since the eigenvalues of the matrix corresponding to $(1 + \frac{h^2}{12}\Delta_h)$ are all non-zero. On the other hand, the implementation of (3.3) requires the boundary value for Ω. To maintain the higher order consistency needed in the truncation error estimate below for the discrete derivatives of the constructed vorticity, we introduce

$$\widehat{\omega} = (-\frac{1}{240}\partial_x^4 - \frac{1}{240}\partial_y^4 + \frac{1}{90}\partial_x^2\partial_y^2)\omega_e \,, \tag{3.4}$$

and set boundary condition for Ω (say on Γ_x, $j = 0$) to be

$$\Omega_{i,0} = (\omega_e)_{i,0} + h^4 \widehat{\omega}_{i,0} \,, \qquad 0 \le i \le N \,. \tag{3.5}$$

It should be noted here that $h^4\widehat{\omega}$ is just the $O(h^4)$ truncation error of $(\Delta_h + \dfrac{h^2}{6} D_x^2 D_y^2)\psi_e - (1 +$

$\frac{h^2}{12}\Delta_h)\omega_e$. The purpose of the introduction of $h^4\widehat{\omega}$ is to maintain the higher order consistency needed in the truncation error estimate below for the discrete derivatives of the constructed vorticity, as can be seen in the following lemma.

Lemma 3.1 *We have at grid points* $0 \le i, j \le N$,

$$\Omega = \omega_e + h^4\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8} \,. \tag{3.6}$$

Proof. Our construction of Ω and Ψ and Taylor expansion of ψ_e and ω_e indicate that at grid points (x_i, y_j), $1 \le i, j \le N-1$,

$$(1 + \frac{h^2}{12}\Delta_h)\Omega = (\Delta_h + \frac{h^2}{6}D_x^2 D_y^2)\psi_e = (1 + \frac{h^2}{12}\Delta_h)\omega_e + h^4\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8} \,, \tag{3.7}$$

where $\widehat{\omega}$ was introduced in (3.4). (3.7) gives us

$$(1 + \frac{h^2}{12}\Delta_h)(\Omega - \omega_e - h^4\widehat{\omega}) = -\frac{h^6}{12}\Delta_h\widehat{\omega} + O(h^6)\|\psi_e\|_{C^8} = O(h^6)\|\psi_e\|_{C^8} \,, \tag{3.8}$$

since the second order differences of $\widehat{\omega}$ is bounded by $\|\psi_e\|_{C^8}$. (3.8) along with (3.5), the boundary condition for Ω, and the property that the matrix $I + \frac{h^2}{12}\Delta_h$ is uniformly diagonally dominant, lead to (3.6). Therefore Lemma 3.1 was proved.

One direct consequence of (3.6) is that

$$(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2)\Omega = (\Delta_h + \frac{h^2}{6}D_x^2 D_y^2)\omega_e + O(h^4)\|\psi_e\|_{C^8} \,, \tag{3.9}$$

which together with Taylor expansion of ω_e that

$$(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2)\omega_e = (1 + \frac{h^2}{12}\Delta)\Delta\omega_e + O(h^4)\|\psi_e\|_{C^8} \,, \tag{3.10}$$

indicates the estimate of our truncation error for the diffusion term

$$(\Delta_h + \frac{h^2}{6}D_x^2 D_y^2)\Omega = (1 + \frac{h^2}{12}\Delta)\Delta\omega_e + O(h^4)\|\psi_e\|_{C^8} \,. \tag{3.11}$$

Next we look at the convection term, which is the $\mathcal{NL}$ term applied to U, V and Ω introduced in (2.7). First we estimate the difference between U, V and $\boldsymbol{u}_e$. Our definition of U, V and Taylor expansion of ψ_e shows that at the grid points (x_i, y_j), $0 \le i, j \le N$:

$$U = u_e + \frac{1}{30}h^4\partial_y^5\psi_e + O(h^5)\|\psi_e\|_{C^6} \,, \qquad V = v_e - \frac{1}{30}h^4\partial_x^5\psi_e + O(h^5)\|\psi_e\|_{C^6} \,. \tag{3.12}$$

The combination of (3.12) and (3.6) gives

$$\widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)(U\Omega) = \widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)(u_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8} \,, \tag{3.13}$$

which along with the Taylor expansion of $u_e\omega_e$

$$
\begin{aligned}
\widetilde{D}_x(1+\frac{h^2}{6}D_y^2)(u_e\omega_e) = &\quad (1+\frac{h^2}{6}\Delta)\partial_x(u_e\omega_e) + O(h^4)\|u_e\omega_e\|_{C^5} \\
= &\quad (1+\frac{h^2}{6}\Delta)\partial_x(u_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,,
\end{aligned} \tag{3.14}
$$

leads to the estimate

$$
\widetilde{D}_x(1+\frac{h^2}{6}D_y^2)(U\Omega) = (1+\frac{h^2}{6}\Delta)\partial_x(u_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,. \tag{3.15}
$$

The other convection terms can be treated similarly

$$
\widetilde{D}_y(1+\frac{h^2}{6}D_x^2)(V\Omega) = (1+\frac{h^2}{6}\Delta)\partial_y(v_e\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,, \tag{3.16}
$$

$$
\frac{h^2}{12}\Delta_h(U\widetilde{D}_x\Omega + V\widetilde{D}_y\Omega) = \frac{h^2}{12}\Delta(u_e\partial_x\omega_e + v_e\partial_y\omega_e) + O(h^4)\|\psi_e\|_{C^6}\|\psi_e\|_{C^8}\,. \tag{3.17}
$$

Now we estimate time marching term. Note that at the grid points (x_i, y_j), $1 \le i, j \le N-1$,

$$
\begin{aligned}
\partial_t(1+\frac{h^2}{12}\Delta_h)\Omega = &\quad (\Delta_h + \frac{h^2}{6}D_x^2D_y^2)\partial_t\psi_e \\
= &\quad \Big(\Delta + \frac{h^2}{12}(\partial_x^4+\partial_y^4) + \frac{h^2}{6}\partial_x^2\partial_y^2\Big)\partial_t\psi_e + O(h^4)\|\partial_t\psi_e\|_{C^6}\,,
\end{aligned} \tag{3.18}
$$

where the first term is exactly $(1+\frac{h^2}{12}\Delta)\partial_t\omega_e$. To estimate the second term, we have a look at the following Poisson equation satisfied by $\partial_t\psi_e$

$$
\begin{cases}
\Delta(\partial_t\psi_e) = \partial_t\omega_e\,, & \text{in } \Omega\,, \\
\partial_t\psi_e = 0\,, & \text{on } \Gamma\,.
\end{cases} \tag{3.19}
$$

The Schauder estimate of (3.19) gives

$$
\|\partial_t\psi_e\|_{C^{6,\alpha}} \le C\|\partial_t\omega_e\|_{C^{4,\alpha}} \le C(\|\psi_e\|_{C^{8,\alpha}} + \|\psi_e\|_{C^{7,\alpha}}\|\psi_e\|_{C^{5,\alpha}})\,, \tag{3.20}
$$

where in the second step we applied our original NSE. Then we arrive at

$$
\partial_t(1+\frac{h^2}{12}\Delta_h)\Omega = (1+\frac{h^2}{12}\Delta)\partial_t\omega_e + O(h^4)(\|\psi_e\|_{C^{8,\alpha}} + \|\psi_e\|_{C^{7,\alpha}}\|\psi_e\|_{C^{5,\alpha}})\,. \tag{3.21}
$$

Finally, the combination of (3.11), (3.15)-(3.17) and (3.21) completes the truncation error analysis for the momentum equation: at grid points (x_i, y_j), $1 \le i, j \le N-1$,

$$
\begin{aligned}
&(1+\frac{h^2}{12}\Delta_h)\partial_t\Omega + \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)(U\Omega) + \widetilde{D}_y(1+\frac{h^2}{6}D_x^2)(V\Omega) - \frac{h^2}{12}\Delta_h(U\widetilde{D}_x\Omega + V\widetilde{D}_y\Omega) \\
&\quad = \nu(\Delta_h + \frac{h^2}{6}D_x^2D_y^2)\Omega + O(h^4)(\|\boldsymbol{u}_e\|_{C^{7,\alpha}} + \|\boldsymbol{u}_e\|_{C^5}\|\boldsymbol{u}_e\|_{C^7})\,,
\end{aligned} \tag{3.22}
$$

where we applied the NSE, which implies that $(1+\frac{h^2}{12}\Delta)\Big(\partial_t\omega_e+\nabla\cdot(\boldsymbol{u}_e\omega_e)-\nu\Delta\omega_e\Big)=0$.

The estimate of Ω on the boundary is more straightforward than that of Briley's formula in [4]. As can be seen, one-sided Taylor expansion of ψ_e at the boundary shows that

$$(\omega_e)_{i,0}=\frac{1}{h^2}(8\Psi_{i,1}-3\Psi_{i,2}+\frac{8}{9}\Psi_{i,3}-\frac{1}{8}\Psi_{i,4})+O(h^4)\|\psi_e\|_{C^6}\,, \tag{3.23}$$

whose combination with our definition of $\Omega_{i,0}$ in (3.4) and the fact that $|\widehat{\omega}_{i,0}|\le C\|\psi_e\|_{C^6}$, indicates the vorticity boundary condition up to $O(h^4)$ error:

$$\Omega_{i,0}=\frac{1}{h^2}(8\Psi_{i,1}-3\Psi_{i,2}+\frac{8}{9}\Psi_{i,3}-\frac{1}{8}\Psi_{i,4})+O(h^4)\|\psi_e\|_{C^6}\,. \tag{3.24}$$

This completes our consistency analysis.

3.2. Stability and Error estimates

For $0\le i,j\le N$, we define

$$\widetilde{\psi}_{i,j}=\psi_{i,j}-\Psi_{i,j}\,,\qquad \widetilde{\omega}_{i,j}=\omega_{i,j}-\Omega_{i,j}\,,\qquad \widetilde{u}_{i,j}=u_{i,j}-U_{i,j}\,,\qquad \widetilde{v}_{i,j}=v_{i,j}-V_{i,j}\,. \tag{3.25}$$

In addition, the error function for $\overline{\omega}$ is introduced here at grid points (x_i,y_j), $1\le i,j\le N-1$

$$\widetilde{\overline{\omega}}_{i,j}=\overline{\omega}_{i,j}-\overline{\Omega}_{i,j}=(1+\frac{h^2}{12}\Delta_h)\widetilde{\omega}_{i,j}\,. \tag{3.26}$$

Our consistency analysis in **3.1** shows that

$$\begin{cases}\partial_t\widetilde{\overline{\omega}}+\mathcal{L}=\nu(\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\widetilde{\omega}+\boldsymbol{f}\,,\\ (\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\widetilde{\psi}=(1+\frac{h^2}{12}\Delta_h)\widetilde{\omega}\,,\qquad \widetilde{\psi}\mid_\Gamma=0\,,\\ \widetilde{u}=-\widetilde{D}_y(1-\frac{h^2}{6}D_y^2)\widetilde{\psi}\,,\quad \widetilde{v}=\widetilde{D}_x(1-\frac{h^2}{6}D_x^2)\widetilde{\psi}\,,\end{cases} \tag{3.27}$$

where the local truncation error $|\boldsymbol{f}|\le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5})$, and the linearized convection error $\mathcal{L}$ is represented as

$$\begin{aligned}\mathcal{L}=&\,\widetilde{D}_x\Big(1+\frac{h^2}{6}D_y^2\Big)(\widetilde{u}\Omega+u\widetilde{\omega})+\widetilde{D}_y\Big(1+\frac{h^2}{6}D_x^2\Big)(\widetilde{v}\Omega+v\widetilde{\omega})\\ &-\frac{h^2}{12}\Delta_h(u\widetilde{D}_x\widetilde{\omega}+v\widetilde{D}_y\widetilde{\omega})-\frac{h^2}{12}\Delta_h(\widetilde{u}\widetilde{D}_x\Omega+\widetilde{v}\widetilde{D}_y\Omega)\,.\end{aligned} \tag{3.28}$$

On the boundary (say near Γ_x, $j=0$), we have

$$\begin{aligned}\widetilde{\omega}_{i,0}&=\frac{1}{h^2}(8\widetilde{\psi}_{i,1}-3\widetilde{\psi}_{i,2}+\frac{8}{9}\widetilde{\psi}_{i,3}-\frac{1}{8}\widetilde{\psi}_{i,4})+h^4\boldsymbol{e}_i\,,\\ \widetilde{\psi}_{i,-1}&=10\widetilde{\psi}_{i,1}-5\widetilde{\psi}_{i,2}+\frac{5}{3}\widetilde{\psi}_{i,3}-\frac{1}{4}\widetilde{\psi}_{i,4}+O(h^5)\|\boldsymbol{u}_e\|_{C^5}\,,\end{aligned} \tag{3.29}$$

where $|\boldsymbol{e}_i| \le C\|\boldsymbol{u}_e\|_{C^5}$. The first equality in (3.29) comes from our numerical boundary condition (2.13) and our estimate (3.24); the second estimate in (3.29) comes from our numerical "ghost point" value (2.13) and Taylor expansion of ψ_e.

Multiplying the vorticity error equation in (3.27) by $-(1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}$, using the fact that $\widetilde{\psi}$ vanishes on the boundary, we arrive at

$$-\Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,\partial_t\widetilde{\widetilde{\omega}}\Big\rangle = -\Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,(\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\partial_t\widetilde{\psi}\Big\rangle = \frac{1}{2}\frac{d\widetilde{E}}{dt}\,, \tag{3.30}$$

where $\widetilde{E}$ is denoted as

$$\widetilde{E} = \|\nabla_h\widetilde{\psi}\|^2 - \frac{h^2}{12}\|\Delta_h\widetilde{\psi}\|^2 - \frac{h^2}{6}\|D_xD_y\widetilde{\psi}\|^2 + \frac{h^4}{72}(\|D_xD_y^2\widetilde{\psi}\|^2 + \|D_yD_x^2\widetilde{\psi}\|^2)\,. \tag{3.31}$$

We should note that $\frac{1}{3}\|\nabla_h\widetilde{\psi}\|^2 \le \widetilde{E}$ since $\widetilde{E}$ vanishes on the boundary. The combination of (3.30) and (3.31), along with the application of Cauchy inequality to $\Big\langle (1+\frac{h^2}{12})\Delta_h\widetilde{\psi}, \boldsymbol{f}\Big\rangle$ results in

$$\frac{1}{2}\frac{d\widetilde{E}}{dt} + \nu\Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,(\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\widetilde{\omega}\Big\rangle - \Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,\mathcal{L}\Big\rangle \le C\|\widetilde{\psi}\|^2 + C\|\boldsymbol{f}\|^2\,. \tag{3.32}$$

The estimates of the diffusion term and the convection term are stated in Propositions 3.2 and 3.3, whose proofs will be given in section 4 and section 5, respectively.

Proposition 3.2 *For sufficiently small h, we have*

$$\Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,(\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\widetilde{\omega}\Big\rangle \ge \frac{3}{16}\|\widetilde{\widetilde{\omega}}\|^2 - h^8\,. \tag{3.33}$$

Proposition 3.3 *Assume a-prior that the error function for the velocity field satisfy*

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \le h\,, \tag{3.34}$$

then we have

$$\Big|\Big\langle (1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,\mathcal{L}\Big\rangle\Big| \le \widetilde{C}\|\nabla_h\widetilde{\psi}\|^2 + \frac{\nu}{8}\|\widetilde{\widetilde{\omega}}\|^2 + h^8\,, \tag{3.35}$$

where $\mathcal{L}$ was defined in (3.28) and $\widetilde{C} = \dfrac{32(1+\|\boldsymbol{u}_e\|_{C^0})^2}{\nu} + C(2+\|\boldsymbol{u}_e\|_{C^1})^2 + C\|\boldsymbol{u}_e\|_{C^5}$.

Then we go back to our convergence analysis. First we assume that (3.34) holds. Applying Propositions 3.2, 3.3 back into (3.32), we obtain

$$\frac{1}{2}\frac{d\widetilde{E}}{dt} \le (1+\nu)h^8 + C\|\boldsymbol{f}\|^2 + C\|\widetilde{\psi}\|^2 + \widetilde{C}\|\nabla_h\widetilde{\psi}\|^2 - \frac{\nu}{16}\|\widetilde{\widetilde{\omega}}\|^2\,. \tag{3.36}$$

As can be seen, $\|\widetilde{\psi}\|^2$ can be absorbed in the coefficient of $\|\nabla_h\widetilde{\psi}\|^2$ since the Poincare inequality for $\widetilde{\psi}$: $\|\widetilde{\psi}\|^2 \le C\|\nabla_h\widetilde{\psi}\|^2$ can be applied here. Integrating in time for (3.36) gives

$$\widetilde{E} + \frac{\nu}{8}\int_0^T \|\widetilde{\widetilde{\omega}}\|^2 dt \le C\int_0^T \|\boldsymbol{f}\|^2\, dt \;+ \widetilde{C}\int_0^T \|\nabla_h\widetilde{\psi}\|^2\, dt + CTh^8\,. \tag{3.37}$$

As be mentioned earlier, $\frac{1}{3}\|\nabla_h\widetilde{\psi}\|^2 \le \widetilde{E}$, whose application into (3.37) leads to

$$\|\nabla_h\widetilde{\psi}\|^2 + \frac{3\nu}{8}\int_0^T \|\widetilde{\widetilde{\omega}}\|^2\, dt \;\le C\int_0^T (\|\boldsymbol{f}\|^2 + h^8)\, dt \;+ \widetilde{C}\int_0^T \|\nabla_h\widetilde{\psi}\|^2\, dt\,. \tag{3.38}$$

By Gronwall inequality, we have

$$\begin{aligned} \|\nabla_h\widetilde{\psi}\|^2 \le &\quad C\exp\left\{\widetilde{C}T\right\}\int_0^T (\|\boldsymbol{f}(\cdot,s)\|^2 + h^8)\, ds \\ \le &\quad Ch^8\exp\left\{\frac{CC^*T}{\nu}\right\}\|\boldsymbol{u}_e\|^2_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5})^2\,, \end{aligned} \tag{3.39}$$

since $\widetilde{C} \le \dfrac{CC^*T}{\nu}$ where C^* was introduced after (2.14). Thus, we have proved

$$\|\boldsymbol{u}(\cdot,t) - \boldsymbol{u}(t)\|_{L^2} \le Ch^4\|\boldsymbol{u}_e\|_{C^{7,\alpha}}(1+\|\boldsymbol{u}_e\|_{C^5})\exp\left\{\frac{CC^*T}{\nu}\right\}\,, \tag{3.40}$$

which implies (2.14). Using the inverse inequality, we have

$$\|\widetilde{\boldsymbol{u}}\|_{L^\infty} \le Ch^3\,. \tag{3.41}$$

Now we can adopt a standard device which asserts that (3.34) will never be violated if h is small enough. Therefore Theorem 2.1 is proved.

4. PROOF OF PROPOSITION 3.2

Summing by parts and keeping in mind that $\widetilde{\psi}\,|_\Gamma = 0$, we have

$$\left\langle(1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\,(\Delta_h + \frac{h^2}{6}D_x^2D_y^2)\widetilde{\omega}\right\rangle = \left\langle(\Delta_h + \frac{h^2}{6}D_x^2D_y^2)\widetilde{\psi}\,,\,(1+\frac{h^2}{12}\Delta_h)\widetilde{\omega}\right\rangle + \mathcal{B}\,, \tag{4.1}$$

where the first term is exactly $\|\widetilde{\widetilde{\omega}}\|^2$ since $(\Delta_h + \frac{h^2}{6}D_x^2D_y^2)\widetilde{\psi} = (1+\frac{h^2}{12}\Delta_h)\widetilde{\omega} = \widetilde{\widetilde{\omega}}$, and the boundary term $\mathcal{B}$ can be decomposed as three parts $\mathcal{B}=\mathcal{B}_1+\mathcal{B}_2+\mathcal{B}_3$, where

$$
\begin{aligned}
\mathcal{B}_1 &= \sum_{i=1}^{N-1}\Big((1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}\cdot\widetilde{\omega}_{i,0}+(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,N-1}\cdot\widetilde{\omega}_{i,N})\Big)\\
&\quad+\sum_{j=1}^{N-1}\Big((1+\frac{h^2}{6}D_y^2)\widetilde{\psi}_{1,j}\cdot\widetilde{\omega}_{0,j}+(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}_{N-1,j}\cdot\widetilde{\omega}_{N,j})\Big)\\
\mathcal{B}_2 &= \frac{h^4}{72}\sum_{i=1}^{N-1}(D_x^2\widetilde{\psi}_{i,1}D_x^2\widetilde{\omega}_{i,0}+D_x^2\widetilde{\psi}_{i,N-1}D_x^2\widetilde{\omega}_{i,N})\\
&\quad+\frac{h^4}{72}\sum_{j=1}^{N-1}D_y^2\widetilde{\psi}_{1,j}D_y^2\widetilde{\omega}_{0,j}+D_y^2\widetilde{\psi}_{N-1,j}D_y^2\widetilde{\omega}_{N,j})\\
\mathcal{B}_3 &= \frac{1}{6}(\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0}+\widetilde{\psi}_{1,N-1}\widetilde{\omega}_{0,N}+\widetilde{\psi}_{N-1,1}\widetilde{\omega}_{N,0}+\widetilde{\psi}_{N-1,N-1}\widetilde{\omega}_{N,N})\,.
\end{aligned}
\tag{4.2}
$$

Next, we estimate the three boundary terms separately.

Lemma 4.1 *We have the estimate*

$$
\mathcal{B}_1 \ge \frac{1}{3h^2}\mathcal{B}_\psi-\frac{3}{4}(\|(1+\frac{h^2}{6}D_y^2)D_x^2\widetilde{\psi}\|^2+\|(1+\frac{h^2}{6}D_x^2)D_y^2\widetilde{\psi}\|^2)-Ch^9\,,
\tag{4.3}
$$

where $\mathcal{B}_\psi$ is introduced as

$$
\mathcal{B}_\psi=\sum_{i=1}^{N-1}(\widetilde{\psi}_{i,1}^2+\widetilde{\psi}_{i,N-1}^2)+\sum_{j=1}^{N-1}(\widetilde{\psi}_{1,j}^2+\widetilde{\psi}_{N-1,j}^2)\,.
\tag{4.4}
$$

Proof. The boundary condition for $\widetilde{\omega}$ in (3.29) implies that $\sum_{i=1}^{N-1}(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}\widetilde{\omega}_{i,0}$ includes two parts: I_1 and I_2, where

$$
\begin{aligned}
I_1 &= \frac{1}{h^2}\sum_{i=1}^{N-1}(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}(8\widetilde{\psi}_{i,1}-3\widetilde{\psi}_{i,2}+\frac{8}{9}\widetilde{\psi}_{i,3}-\frac{1}{8}\widetilde{\psi}_{i,4})\,,\\
I_2 &= h^4\sum_{i=1}^{N-1}(1+\frac{h^2}{6}D_x^2)\widetilde{\psi}_{i,1}\boldsymbol{e}_i\,.
\end{aligned}
\tag{4.5}
$$

The term I_2 can be controlled by Cauchy inequality directly: first, summing by parts gives $I_2 = h^4\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(1+\frac{h^2}{6}D_x^2)\boldsymbol{e}_{i,0}$, then we have

$$
I_2 \ge -\frac{1}{36}\sum_{i=1}^{N-1}\frac{\widetilde{\psi}_{i,1}^2}{h^2}-9\sum_{i=1}^{N-1}h^{10}\Big((1+\frac{h^2}{6}D_x^2)\boldsymbol{e}_i\Big)^2\ge-\frac{1}{36}\sum_{i=1}^{N-1}\frac{\widetilde{\psi}_{i,1}^2}{h^2}-Ch^9\,,
\tag{4.6}
$$

since $|\boldsymbol{e}_{i,0}|\le C\|\boldsymbol{u}_e\|_{C^5}$. Our main concern will be I_1. Since $\widetilde{\psi}$ vanishes on the boundary, the term $8\widetilde{\psi}_{i,1}-3\widetilde{\psi}_{i,2}+\frac{8}{9}\widetilde{\psi}_{i,3}-\frac{1}{8}\widetilde{\psi}_{i,4}$ can be rewritten as

$$
8\widetilde{\psi}_{i,1}-3\widetilde{\psi}_{i,2}+\frac{8}{9}\widetilde{\psi}_{i,3}-\frac{1}{8}\widetilde{\psi}_{i,4}=\frac{25}{6}\widetilde{\psi}_{i,1}-\frac{115}{72}h^2(D_y^2\widetilde{\psi})_{i,1}+\frac{23}{36}h^2(D_y^2\widetilde{\psi})_{i,2}-\frac{1}{8}h^2(D_y^2\widetilde{\psi})_{i,3}\,,
\tag{4.7}
$$

which in turn implies that I_1 can be expressed as

$$I_1 = \frac{25}{6h^2}\sum_{i=1}^{N-1}(\widetilde{\psi}_{i,1}^2 + \frac{h^2}{6}\widetilde{\psi}_{i,1}D_x^2\widetilde{\psi}_{i,1}) - \frac{115}{72}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(1+\frac{h^2}{6}D_x^2)(D_y^2\widetilde{\psi})_{i,1} + \frac{23}{36}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(1+\frac{h^2}{6}D_x^2)(D_y^2\widetilde{\psi})_{i,2} - \frac{1}{8}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}(1+\frac{h^2}{6}D_x^2)(D_y^2\widetilde{\psi})_{i,3}\,, \tag{4.8}$$

where we summed by parts again, since $\widetilde{\psi}\ |_\Gamma = 0$. Each term in I_1 can be estimated by Cauchy inequality

$$\begin{aligned}
&\sum_{i=1}^{N-1}(\widetilde{\psi}_{i,1}^2 + \frac{h^2}{6}\widetilde{\psi}_{i,1}D_x^2\widetilde{\psi}_{i,1}) \ge \frac{1}{3}\sum_{i=1}^{N-1}\widetilde{\psi}_{i,1}^2\,,\\
&-\frac{115}{72}\widetilde{\psi}_{i,1}\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,1} \ge -\frac{1}{3h^2}\frac{115^2}{72^2}|\widetilde{\psi}_{i,1}|^2 - \frac{3}{4}h^2\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,1}\Big|^2 h^2\\
&\frac{23}{36}\widetilde{\psi}_{i,1}\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,2} \ge -\frac{1}{3h^2}\frac{23^2}{36^2}|\widetilde{\psi}_{i,1}|^2 - \frac{3}{4}h^2\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,2}\Big|^2 h^2\,.\\
&\frac{1}{8}\widetilde{\psi}_{i,1}\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,3} \ge -\frac{1}{3h^2}\frac{1^2}{8^2}|\widetilde{\psi}_{i,1}|^2 - \frac{3}{4}h^2\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)(D_y^2\widetilde{\psi})_{i,3}\Big|^2 h^2\,.
\end{aligned} \tag{4.9}$$

Since $\frac{1}{3}\cdot\frac{25}{6} - \frac{1}{3}(\frac{115^2}{72^2} + \frac{23^2}{36^2} + \frac{1^2}{8^2}) \ge \frac{13}{36}$, we have

$$I_1 \ge \frac{13}{36h^2}\sum_{i=1}^{N-1}|\widetilde{\psi}_{i,1}|^2 - \frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1,2,3}\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}\Big|^2\,. \tag{4.10}$$

The combination of I_1 and I_2 gives us

$$\sum_{i=1}^{N-1}\Big(1+\frac{h^2}{6}D_x^2\Big)\widetilde{\psi}_{i,1}\widetilde{\omega}_{i,0} \ge \frac{1}{3h^2}\sum_{i=1}^{N-1}|\widetilde{\psi}_{i,1}|^2 - \frac{3}{4}\sum_i\sum_{j=1,2,3}\Big|\Big(1+\frac{h^2}{6}D_x^2\Big)D_y^2\widetilde{\psi}_{i,j}\Big|^2 h^2 - Ch^9\,. \tag{4.11}$$

The treatment of the other three boundary terms is exactly the same. Finally we obtain

$$\mathcal{B}_1 \ge \frac{1}{3h^2}\mathcal{B}_\psi - \frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}|(1+\frac{h^2}{6}D_y^2)D_x^2\widetilde{\psi}_{i,j}|^2 - \frac{3}{4}h^2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}|(1+\frac{h^2}{6}D_x^2)D_y^2\widetilde{\psi}_{i,j}|^2 - Ch^9\,, \tag{4.12}$$

where $\mathcal{B}_\psi$ was defined in (4.4). (4.3) is a direct consequence of (4.12). This completes the proof of Lemma 4.1.

To finish the estimate of $\mathcal{B}_1$, we need to have a good control of $\|(1+\frac{h^2}{6}D_y^2)D_x^2\widetilde{\psi}\|$ and $\|(1+\frac{h^2}{6}D_x^2)D_y^2\widetilde{\psi}\|$. However, standard local estimates do not work in this case. The methodology we will adopt is similar to that in [15]: to control the local terms by global terms via elliptic regularity.

Lemma 4.2 *For any $\widetilde{\psi}$ that vanishes on the boundary, we have*

$$\|D_x^2\widetilde{\psi}\|^2 + \|D_y^2\widetilde{\psi}\|^2 \le \frac{9}{8}\|\widetilde{\omega}\|^2\,, \tag{4.13}$$

$$\|(1+\frac{h^2}{6}D_y^2)D_x^2\widetilde{\psi}\|^2+\|(1+\frac{h^2}{6}D_x^2)D_y^2\widetilde{\psi}\|^2\le\|\widetilde{\widetilde{\omega}}\|^2\,. \tag{4.14}$$

Proof of Lemma 4.2. The boundary condition $\widetilde{\psi}_{i,j}\mid_\Gamma=0$ indicates that we can Sine transform $\{\widetilde{\psi}_{i,j}\}$ on both directions, i.e.,

$$\widetilde{\psi}_{i,j}=\sum_{k,\ell}\widehat{\widetilde{\psi}}_{k,\ell}\,\sin(k\pi x_i)\,\sin(\ell\pi y_j)\,. \tag{4.15}$$

Parserval equality gives

$$\sum_{i,j}(\widetilde{\psi}_{i,j})^2=\sum_{k,\ell}\left|\widehat{\widetilde{\psi}}_{k,\ell}\right|^2. \tag{4.16}$$

If we introduce

$$f_k=-\frac{4}{h^2}\sin^2(\frac{k\pi h}{2})\,,\qquad g_\ell=-\frac{4}{h^2}\sin^2(\frac{\ell\pi h}{2})\,, \tag{4.17}$$

we obtain the Fourier expansions of $D_x^2\widetilde{\psi}$ and $D_y^2\widetilde{\psi}$

$$D_x^2\widetilde{\psi}_{i,j}=\sum_{k,l}f_k\,\widehat{\widetilde{\psi}}_{k,l}\,,\qquad D_y^2\widetilde{\psi}_{i,j}=\sum_{k,l}g_\ell\,\widehat{\widetilde{\psi}}_{k,l}\,, \tag{4.18}$$

which in turn implies that

$$\sum_{i,j}|\widetilde{\widetilde{\omega}}_{i,j}|^2=\sum_{i,j}|(\Delta_h+\frac{h^2}{6}D_x^2D_y^2)\widetilde{\psi}_{i,j}|^2=\sum_{k,\ell}(g_\ell+f_k+\frac{h^2}{6}f_kg_\ell)^2\left|\widehat{\widetilde{\psi}}_{k,\ell}\right|^2. \tag{4.19}$$

Similarly, we have

$$\begin{aligned}\sum_{i,j}|(1+\frac{h^2}{6}D_y^2)D_x^2\widetilde{\psi}_{i,j}|^2&=\sum_{k,\ell}|(1+\frac{h^2}{6}g_\ell)f_k|^2\left|\widehat{\widetilde{\psi}}_{k,\ell}\right|^2,\\ \sum_{i,j}|(1+\frac{h^2}{6}D_x^2)D_y^2\widetilde{\psi}_{i,j}|^2&=\sum_{k,\ell}|(1+\frac{h^2}{6}f_k)g_\ell|^2\left|\widehat{\widetilde{\psi}}_{k,\ell}\right|^2.\end{aligned} \tag{4.20}$$

On the other hand, the fact that $-\frac{4}{h^2}\le f_k,g_\ell\le 0$ shows that

$$|g_\ell+f_k+\frac{h^2}{6}f_kg_\ell|^2\ge|(1+\frac{h^2}{6}g_\ell)f_k|^2+|(1+\frac{h^2}{6}f_k)g_\ell|^2\,, \tag{4.21}$$

$$|g_\ell+f_k+\frac{h^2}{6}f_kg_\ell|^2\ge(f_k^2+g_\ell^2-\frac{2}{9}f_kg_\ell)\ge\frac{8}{9}(f_k^2+g_\ell^2)\,, \tag{4.22}$$

by direct calculations. The combination of (4.19), (4.20) and (4.21) indicates (4.14). (4.13) can be argued in a similar fashion. Lemma 4.2 is proved.

The combination of Lemma 4.1 and 4.2 results in the estimate of $\mathcal{B}_1$:

$$\mathcal{B}_1 \geq \frac{1}{3h^2}\mathcal{B}_\psi - \frac{3}{4}\|\widetilde{\overline{\omega}}\|^2 - Ch^9 \,. \tag{4.23}$$

$\mathcal{B}_2$ can be treated in a similar fashion. We still only look at the term $\sum_i D_x^2\widetilde{\psi}_{i,1}D_x^2\widetilde{\omega}_{i,0}$ here. Once again, (3.29), the boundary condition for $\widetilde{\omega}$ indicates that $\sum_i D_x^2\widetilde{\psi}_{i,1}D_x^2\widetilde{\omega}_{i,0}$ includes two parts: I_3 and I_4, which are denoted as

$$I_3 = \frac{1}{h^2}\sum_{i=1}^{N-1} D_x^2\widetilde{\psi}_{i,1}\Big(8D_x^2\widetilde{\psi}_{i,1} - 3D_x^2\widetilde{\psi}_{i,2} + \frac{8}{9}D_x^2\widetilde{\psi}_{i,3} - \frac{1}{8}D_x^2\widetilde{\psi}_{i,4}\Big)\,, \quad I_4 = h^4\sum_{i=1}^{N-1} D_x^2\widetilde{\psi}_{i,1}D_x^2\boldsymbol{e}_i \,. \tag{4.24}$$

The estimate of I_3 and I_4 is similar to that of I_1 and I_2, respectively. Repeating the arguments in the proof of Lemma 4.1, we arrive at (the detail is omitted here)

$$\mathcal{B}_2 \geq -\frac{1}{144}h^4\|D_y^2D_x^2\widetilde{\psi}\|^2 - Ch^9 \,. \tag{4.25}$$

On the other hand, the fact that $\|D_y^2D_x^2\widetilde{\psi}\| \leq \frac{4}{h^2}\|D_x^2\widetilde{\psi}\|$ and $\|D_y^2D_x^2\widetilde{\psi}\| \leq \frac{4}{h^2}\|D_y^2\widetilde{\psi}\|$ implies

$$\|D_y^2D_x^2\widetilde{\psi}\|^2 = \frac{1}{2}(\|D_y^2D_x^2\widetilde{\psi}\|^2 + \|D_y^2D_x^2\widetilde{\psi}\|^2) \leq \frac{8}{h^4}\|D_x^2\widetilde{\psi}\|^2 + \frac{8}{h^4}\|D_y^2\widetilde{\psi}\|^2 \leq \frac{9}{h^4}\|\widetilde{\overline{\omega}}\|^2 \,, \tag{4.26}$$

where in the last step we applied (4.13) in Lemma 4.2. Substituting (4.26) into (4.25), we arrive at

$$\mathcal{B}_2 \geq -\frac{1}{16}\|\widetilde{\overline{\omega}}\|^2 - Ch^9 \,. \tag{4.27}$$

Finally, $\mathcal{B}_3$ can be controlled by standard Cauchy inequality (still, we only look at the term $\frac{1}{6}\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0}$)

$$\frac{1}{6}\widetilde{\psi}_{1,1}\widetilde{\omega}_{0,0} \geq -\frac{1}{12}\frac{\widetilde{\psi}_{1,1}^2}{h^2} - \frac{1}{12}h^2\widetilde{\omega}_{0,0}^2 \geq -\frac{1}{12}\frac{\widetilde{\psi}_{1,1}^2}{h^2} - Ch^{10}\|\psi_e\|_{C^8}^2 \,, \tag{4.28}$$

where in the last step we used the fact that $|\widetilde{\omega}_{0,0}| \leq Ch^4\|\psi_e\|_{C^8}$ by our numerical ω and our construction of Ω in §3.1. As can be seen, the first term appearing on the right hand side of (4.28) can be absorbed in the $\mathcal{B}_\psi$ term, then we have

$$\mathcal{B}_3 \geq -\frac{1}{12h^2}\mathcal{B}_\psi - Ch^9 \,, \tag{4.29}$$

Finally, the combination of (4.27), (4.29) and Lemma 4.1 shows that $\mathcal{B} \geq -\frac{13}{16}\|\widetilde{\overline{\omega}}\|^2 - h^8$, whose substitution into (4.1) is exactly (3.33). This completes the proof of Proposition 3.2.

5. PROOF OF PROPOSITION 3.3

For the convenience of the analysis below, a new notation $\|\widetilde{\omega}\|_W$ is defined by

$$\|\widetilde{\omega}\|_W^2 = \sum_{0\le i,j\le N} \widetilde{\omega}_{i,j}^2 h^2\,. \tag{5.1}$$

Note that the difference between $\|\widetilde{\omega}\|_W$ and $\|\widetilde{\omega}\|$ is that $\|\widetilde{\omega}\|_W$ involves the boundary values of $\widetilde{\omega}$. The following lemma gives the estimate of $\|\widetilde{\omega}\|_W^2$ before the proof of Proposition 3.3 is carried out.

Lemma 5.1 *We have*

$$\|\widetilde{\omega}\|_W^2 \le C\|\widetilde{\overline{\omega}}\|^2 + \frac{C}{h^2}\|\nabla_h\widetilde{\psi}\|^2 + Ch^9\,. \tag{5.2}$$

Proof. Step 1. Establish a bound for $\|\widetilde{\omega}\|^2$

We follow the pattern of analysis in the proof of Lemma 4.2. A decomposition of $\widetilde{\omega}$ is needed since it does not vanish on the boundary: let $\widetilde{\omega}^0$ and $\widetilde{\omega}^b$ be the interior part and boundary part of $\widetilde{\omega}$, respectively, i.e. $\widetilde{\omega} = \widetilde{\omega}^0 + \widetilde{\omega}^b$, such that

$$\begin{aligned} &\widetilde{\omega}_{i,j}^0 = \widetilde{\omega}_{i,j}\,, \qquad \widetilde{\omega}_{i,j}^b = 0\,, \qquad \text{for } 1\le i,j\le N-1\,,\\ &\widetilde{\omega}_{i,j}^0 = 0\,, \qquad \widetilde{\omega}_{i,j}^b = \widetilde{\omega}_{i,j}\,, \qquad \text{on } \Gamma\,, \end{aligned} \tag{5.3}$$

and define $\widetilde{\overline{\omega}}^0 = (1+\frac{1}{12}\Delta_h)\widetilde{\omega}^0$. Since $\widetilde{\omega}^0$ vanishes on the boundary, we can Sine transform $\{\widetilde{\omega}_{i,j}^0\}$ both in the i-direction and j-direction, i.e.

$$\widetilde{\omega}_{i,j}^0 = \sum_{k,\ell} \widehat{\widetilde{\omega}}_{k,\ell}^0 \sin(k\pi x_i)\,\sin(\ell\pi y_j)\,. \tag{5.4}$$

Then, Parserval equality applied to $\widetilde{\omega}^0$ gives

$$\sum_{i,j} (\widetilde{\omega}_{i,j}^0)^2 = \sum_{k,\ell} \left|\widehat{\widetilde{\omega}}_{k,\ell}^0\right|^2 . \tag{5.5}$$

Similarly, we have $D_x^2\widetilde{\omega}_{i,j}^0 = \sum_{k,l} f_k\,\widehat{\widetilde{\omega}}_{k,l}^0$, and $D_y^2\widetilde{\omega}_{i,j}^0 = \sum_{k,l} g_\ell\,\widehat{\widetilde{\omega}}_{k,l}^0$, as in (4.17) and (4.18), which in turn indicates that

$$\sum_{i,j} \left|\widetilde{\overline{\omega}}_{i,j}^0\right|^2 = \sum_{i,j} \left|(1+\frac{h^2}{12}\Delta_h)\widetilde{\omega}_{i,j}^0\right|^2 = \sum_{k,\ell} \left|1+\frac{h^2}{12}(f_k+g_\ell)\right|^2 \left|\widehat{\widetilde{\omega}}_{k,\ell}^0\right|^2 . \tag{5.6}$$

The combination of (5.5), (5.6), along with the fact that $\frac{1}{3} \le 1+\frac{h^2}{12}(f_k+g_\ell) \le 1$, shows that

$$\|\widetilde{\overline{\omega}}^0\|^2 \ge \frac{1}{9}\|\widetilde{\omega}\|^2\,. \tag{5.7}$$

On the other hand, we note that $\widetilde{\widetilde{\omega}}^0_{i,j} = \widetilde{\widetilde{\omega}}_{i,j}$ for $2 \le i, j \le N-2$, and near the boundary (say at $j = 1$),

$$|\widetilde{\widetilde{\omega}}^0_{i,1}|^2 = (\widetilde{\widetilde{\omega}}_{i,1} - \frac{1}{12}\widetilde{\omega}_{i,0})^2 \le \frac{13}{12}\widetilde{\widetilde{\omega}}^2_{i,1} + \frac{13}{144}\widetilde{\omega}^2_{i,0}\,, \tag{5.8}$$

and near the corner (say at $i = 1$, $j = 1$)

$$|\widetilde{\widetilde{\omega}}^0_{1,1}|^2 = (\widetilde{\widetilde{\omega}}_{1,1} - \frac{1}{12}\widetilde{\omega}_{1,0} - \frac{1}{12}\widetilde{\omega}_{0,1})^2 \le \frac{7}{6}\widetilde{\widetilde{\omega}}^2_{1,1} + \frac{7}{72}(\widetilde{\omega}^2_{1,0} + \widetilde{\omega}^2_{0,1})\,, \tag{5.9}$$

which indicates that

$$\|\widetilde{\widetilde{\omega}}^0\|^2 \le \frac{7}{6}\|\widetilde{\widetilde{\omega}}\|^2 + \frac{7}{72}\Big(\sum_{i=1}^{N-1} h^2(|\widetilde{\omega}_{i,0}|^2 + |\widetilde{\omega}_{i,N}|^2) + \sum_{j=1}^{N-1} h^2(|\widetilde{\omega}_{0,j}|^2 + |\widetilde{\omega}_{N,j}|^2)\Big)\,. \tag{5.10}$$

The combination of (5.10) and (5.7) gives

$$\|\widetilde{\omega}\|^2 \le \frac{21}{2}\|\widetilde{\widetilde{\omega}}\|^2 + \frac{7}{8}h^2\mathcal{B}_\omega\,, \tag{5.11}$$

where $\mathcal{B}_\omega$ is the total sum of boundary terms for $\widetilde{\omega}$:

$$\mathcal{B}_\omega = \sum_{i=1}^{N-1}(|\widetilde{\omega}_{i,0}|^2 + |\widetilde{\omega}_{i,N}|^2) + \sum_{j=1}^{N-1}(|\widetilde{\omega}_{0,j}|^2 + |\widetilde{\omega}_{N,j}|^2)\,. \tag{5.12}$$

Step 2. (5.2) is a direct consequence of (5.11) and our boundary condition for vorticity error function as in (3.29):

By our definition of $\|\widetilde{\omega}\|_W$ in (5.1), we have $\|\widetilde{\omega}\|^2_W = \|\widetilde{\omega}\|^2 + h^2\mathcal{B}_\omega$, where $\mathcal{B}_\omega$ was defined in (5.12), whose substitution into (5.11) leads to $\|\widetilde{\omega}\|^2_W \le \frac{21}{2}\|\widetilde{\widetilde{\omega}}\|^2 + 2h^2\mathcal{B}_\omega$. Then the remaining task is to control $\mathcal{B}_\omega$. For simplicity we just look at the first term $\sum_{i=1}^{N-1}\widetilde{\omega}^2_{i,0}$. Similar to the proof of the diffusion term in Proposition 3.2, we can apply (3.29), the boundary condition for $\widetilde{\omega}$, to recover $\mathcal{B}_\omega$. The term appearing in the parenthesis in (3.29) can be rewritten as (4.7), whose substitution into (3.29) shows that

$$\widetilde{\omega}^2_{i,0} \le \frac{4}{h^4}\Big(\frac{25^2}{6^2}\widetilde{\psi}^2_{i,1} + \frac{115^2}{72^2}h^4(D_y^2\widetilde{\psi})^2_{i,1} + \frac{23^2}{36^2}h^4(D_y^2\widetilde{\psi})^2_{i,2} + \frac{1^2}{8^2}h^4(D_y^2\widetilde{\psi})^2_{i,3}\Big) + 2h^8\boldsymbol{e}^2_{i,0}\,. \tag{5.13}$$

Similar results are available for $\widetilde{\omega}^2_{i,N}$, $\widetilde{\omega}^2_{0,j}$, and $\widetilde{\omega}^2_{N,j}$. Then we arrive at

$$\begin{aligned}\mathcal{B}_\omega \le\ & \frac{C}{h^4}\mathcal{B}_\psi + C\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(D_x^2\widetilde{\psi}_{i,j})^2 + C\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(D_y^2\widetilde{\psi}_{i,j})^2 + 2h^8\mathcal{E} \\ \le\ & \frac{C}{h^4}\|\nabla_h\widetilde{\psi}\|^2 + \frac{C}{h^2}(\|D_x^2\widetilde{\psi}\|^2 + \|D_y^2\widetilde{\psi}\|^2) + 2h^8\mathcal{E}\,,\end{aligned} \tag{5.14}$$

where $\mathcal{B}_\psi$ was defined in (4.4) and $\mathcal{E}$ is introduced as

$$\mathcal{E} = \sum_{i=1}^{N-1} (\boldsymbol{e}_{i,0}^2 + \boldsymbol{e}_{i,N}^2) + \sum_{j=1}^{N-1} (\boldsymbol{e}_{0,j}^2 + \boldsymbol{e}_{N,j}^2)\,. \tag{5.15}$$

In the second step of (5.14), we absorbed all the terms of $\mathcal{B}_\psi$ appearing in (4.4) into $\|\nabla_h \widetilde{\psi}\|^2$, since $h^2(\frac{\widetilde{\psi}_{i,1} - \widetilde{\psi}_{i,0}}{h})^2 = \widetilde{\psi}_{i,1}^2$ (by the fact that $\widetilde{\psi}_{i,0} = 0$) is included in $\|\nabla_h \widetilde{\psi}\|^2$. Moreover, the second term appearing on the right hand side of (5.14) can be controlled by the order of $\|\widetilde{\overline{\omega}}\|^2$ via (4.13) in Lemma 4.2. In addition, we can see that $2h^8 \mathcal{E} \le C h^7 \|\boldsymbol{u}_e\|_{C^5}^2$ since $|\boldsymbol{e}_{i,0}| \le C\|\boldsymbol{u}_e\|_{C^5}$. Plugging these estimates back into (5.14), along with our previous argument that $\|\widetilde{\omega}\|^2 \le \frac{21}{2}\|\widetilde{\overline{\omega}}\|^2 + 2h^2\mathcal{B}_\omega$, we obtain (5.2).

Now we can continue our proof of Proposition 3.3.

Proof of Proposition 3.3. We decompose $\mathcal{L}$ into four parts: $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$, where

$$\begin{aligned} &\mathcal{L}_1 = \widetilde{D}_x\Big(1 + \frac{h^2}{6}D_y^2\Big)(\widetilde{u}\Omega + u\widetilde{\omega})\,, \qquad \mathcal{L}_2 = \widetilde{D}_y\Big(1 + \frac{h^2}{6}D_x^2\Big)(\widetilde{v}\Omega + v\widetilde{\omega})\,,\\ &\mathcal{L}_3 = -\frac{h^2}{12}\Delta_h\Big(u\widetilde{D}_x\widetilde{\omega} + v\widetilde{D}_x\widetilde{\omega}\Big)\,, \qquad \mathcal{L}_4 = -\frac{h^2}{12}\Delta_h\Big(\widetilde{u}\widetilde{D}_x\Omega + \widetilde{v}\widetilde{D}_x\Omega\Big)\,. \end{aligned} \tag{5.16}$$

We will show that the inner product of $(1 + \frac{h^2}{12}\Delta_h)\widetilde{\psi}$ with each term in (5.16) is bounded by the following result

$$\left|\langle(1 + \frac{h^2}{12}\Delta_h)\widetilde{\psi}\,,\, \mathcal{L}_i\rangle\right| \le \frac{1}{4}\widetilde{C}\|\nabla_h\widetilde{\psi}\|^2 + \frac{\nu}{32}\|\widetilde{\overline{\omega}}\|^2 + h^9\,, \tag{5.17}$$

for $i = 1, 2, 3, 4$, where $\widetilde{C} = \dfrac{32(1 + \|\boldsymbol{u}_e\|_{C^0})^2}{\nu} + C(2 + \|\boldsymbol{u}_e\|_{C^1})^2 + C\|\boldsymbol{u}_e\|_{C^5}$.

We only give the estimate of $\langle(1+\frac{h^2}{12}\Delta_h)\widetilde{\psi}, \mathcal{L}_1\rangle$. The other three terms can be treated in a similar fashion. Summing by parts, we have

$$\Big\langle(1 + \frac{h^2}{12}\Delta_h)\widetilde{\psi}, \widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)(\widetilde{u}\Omega + u\widetilde{\omega})\Big\rangle = -\Big\langle\widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)\widetilde{\psi}, (1 + \frac{h^2}{12}\Delta_h)(\widetilde{u}\Omega + u\widetilde{\omega})\Big\rangle + \mathcal{BL}_1\,, \tag{5.18}$$

where $\mathcal{BL}_1$ includes boundary terms

$$\begin{aligned} \mathcal{BL}_1 = \;& \frac{1}{6}h^2\sum_i(1 + \frac{h^2}{12}\Delta_h)\widetilde{\psi}_{i,1}\cdot\widetilde{D}_x(\widetilde{u}\Omega)_{i,0} + \frac{1}{6}h^2\sum_i(1 + \frac{h^2}{12}\Delta_h)\widetilde{\psi}_{i,N-1}\cdot\widetilde{D}_x(\widetilde{u}\Omega)_{i,N}\\ &+\frac{1}{12}h^2\sum_i\widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)\widetilde{\psi}_{i,1}\cdot(\widetilde{u}\Omega)_{i,0} + \frac{1}{12}h^2\sum_i\widetilde{D}_x(1 + \frac{h^2}{6}D_y^2)\widetilde{\psi}_{i,N-1}\cdot(\widetilde{u}\Omega)_{i,N}\,. \end{aligned} \tag{5.19}$$

We look at the right hand side of (5.18) term by term. By our construction of Ω as in (3.2)-(3.5) indicates that

$$\|\Omega\|_{L^\infty} \le C(\|\overline{\Omega}\|_{L^\infty} + \|\Omega\mid_\Gamma\|_{L^\infty}) \le C(\|\psi_e\|_{C^2} + \|\psi_e\|_{C^6}) \le C\|\boldsymbol{u}_e\|_{C^5}\,, \tag{5.20}$$

which in turn gives that

$$\|(1+\frac{h^2}{12}\Delta_h)(\widetilde{u}\Omega)\| \le \|\Omega\|_{L^\infty}\|(1+\frac{h^2}{12}\Delta_h)\widetilde{u}\| \le C\|\Omega\|_{L^\infty}\|\widetilde{u}\| \le C\|\boldsymbol{u}_e\|_{C^5}\|\nabla_h\widetilde{\psi}\|\,, \tag{5.21}$$

where in the last step we applied the result that $\|\widetilde{u}\| \le 2\|\nabla_h\widetilde{\psi}\|$ by the relation between $\widetilde{u}$ and $\widetilde{\psi}$. (5.21) along with the fact that

$$\|\widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}\| \le \|\widetilde{D}_x\widetilde{\psi}\| \le \|\nabla_h\widetilde{\psi}\|\,, \tag{5.22}$$

since $\widetilde{\psi}$ vanishes on the boundary, and Cauchy inequality gives

$$\left|\left\langle \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}, (1+\frac{h^2}{12}\Delta_h)(\widetilde{u}\Omega)\right\rangle\right| \le C\|\boldsymbol{u}_e\|_{C^5}\|\nabla_h\widetilde{\psi}\|^2\,. \tag{5.23}$$

Now we look at the inner product of $-\widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}$ with $(1+\frac{h^2}{12}\Delta_h)(u\widetilde{\omega})$. First, we can rewrite the latter as

$$(1+\frac{h^2}{12}\Delta_h)(u\widetilde{\omega})_{i,j} = u_{i,j}(1+\frac{h^2}{12}\Delta_h)\widetilde{\omega}_{i,j} + \mathcal{DL} = u_{i,j}\widetilde{\widetilde{\omega}}_{i,j} + \mathcal{DL}\,, \tag{5.24}$$

where $\mathcal{DL}$ includes four parts: $\frac{1}{12}(u_{i-1,j}-u_{i,j})\widetilde{\omega}_{i-1,j}$, $\frac{1}{12}(u_{i+1,j}-u_{i,j})\widetilde{\omega}_{i+1,j}$, $\frac{1}{12}(u_{i,j-1}-u_{i,j})\widetilde{\omega}_{i,j-1}$, $\frac{1}{12}(u_{i,j+1}-u_{i,j})\widetilde{\omega}_{i,j+1}$. Our construction of U in (3.1) and the a-priori assumption (3.34) gives

$$\|u\|_{L^\infty} \le \|U\|_{L^\infty} + \|\widetilde{u}\|_{L^\infty} \le \|\psi_e\|_{C^1} + h \le \|\boldsymbol{u}_e\|_{C^0} + 1 \equiv C_1\,, \tag{5.25}$$

which leads to

$$\left|\left\langle \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}, u\widetilde{\widetilde{\omega}}\right\rangle\right| \le \frac{32C_1^2}{\nu}\|\nabla_h\widetilde{\psi}\|^2 + \frac{\nu}{32}\|\widetilde{\widetilde{\omega}}\|^2\,. \tag{5.26}$$

Furthermore, we have

$$\|u_{i,j}-u_{i-1,j}\|_{L^\infty} \le \|U_{i,j}-U_{i-1,j}\|_{L^\infty} + \|\widetilde{u}_{i,j}-\widetilde{u}_{i-1,j}\|_{L^\infty} \le h\|\psi_e\|_{C^2} + 2h\,, \tag{5.27}$$

and a similar result holds for the other three neighboring points, which shows that $\|\mathcal{DL}\| \le C(\|\boldsymbol{u}_e\|_{C^1}+2)h\|\widetilde{\omega}\|_W$. Note that $\|\widetilde{\omega}\|_W$ involves the boundary values of $\widetilde{\omega}$. Then we arrive at

$$\left|\left\langle \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}, \mathcal{DL}\right\rangle\right| \le C(\|\boldsymbol{u}_e\|_{C^1}+2)^2\|\nabla_h\widetilde{\psi}\|^2 + h^2\|\widetilde{\omega}\|_W^2\,, \tag{5.28}$$

and applying Lemma 5.1, we obtain

$$\left|\left\langle \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi}, \mathcal{DL}\right\rangle\right| \le C(\|\boldsymbol{u}_e\|_{C^1}+2)^2\|\nabla_h\widetilde{\psi}\|^2 + Ch^2\|\widetilde{\widetilde{\omega}}\|^2 + Ch^{11}\,. \tag{5.29}$$

The combination of (5.24), (5.27) and (5.29) shows that

$$(5.30) \qquad \left|\left\langle \widetilde{D}_x(1+\frac{h^2}{6}D_y^2)\widetilde{\psi},(1+\frac{h^2}{12}\Delta_h)(u\widetilde{\omega})\right\rangle\right| \le C_2\|\nabla_h\widetilde{\psi}\|^2+\frac{\nu}{32}\|\widetilde{\widetilde{\omega}}\|^2+\frac{1}{2}h^9\,.$$

where $C_2=\dfrac{32C_1^2}{\nu}+C(\|\boldsymbol{u}_e\|_{C^1}+2)^2$.

Applying the similar argument to $\mathcal{BL}_1$ we can get

$$(5.31) \qquad \mathcal{BL}_1 \le C\|\nabla_h\widetilde{\psi}\|^2+\frac{1}{2}h^9\,.$$

The detail is omitted here. Finally, combining (5.23), (5.30), (5.31), we arrive at (5.17) for $i=1$. The other three terms can be estimated in a similar fashion. Thus Proposition 3.3 is proven.

ACKNOLEDGEMENT

We thank Weinan E and Hans E. Johnston for many insightful discussions. The research was supported by NSF grant DMS-9805621.

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