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POSITIVITY PROPERTY OF SECOND-ORDER FLUX-SPLITTING SCHEMES FOR THE COMPRESSIBLE EULER EQUATIONS

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Abstract. A class of upwind flux splitting methods in the Euler equations of compressible flow is considered in this paper. Using the property that Euler flux $F(U)$ is a homogeneous function of degree one in U, we reformulate the splitting fluxes with $F^+ = A^+U$, $F^- = A^-U$, and the corresponding matrices are either symmetric or symmetrizable and keep only non-negative and non-positive eigenvalues. That leads to the conclusion that the first order schemes are positive in the sense of Lax-Liu [18], which implies that it is L^2-stable in some suitable sense. Moreover, the second order scheme is a stable perturbation of the first order scheme, so that the positivity of the second order schemes is also established, under a CFL-like condition. In addition, these splitting methods preserve the positivity of density and energy.

1. **Introduction.** The general form of a system of hyperbolic conservation laws can be written as

$$\partial_t U + \partial_x F(U) + \partial_y G(U) + \partial_z H(U) = 0\,, \qquad U \in R^n\,, \tag{1.1}$$

such as the governing equations for compressible flow, MHD, etc. For example, the 1-D Euler equations for gas dynamics is given by

$$\partial_t U + \partial_x F(U) = 0, \qquad U = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \qquad F = \begin{bmatrix} m \\ \rho v^2 + p \\ v(E+p) \end{bmatrix}, \tag{1.2}$$

in which $\rho, v, m = \rho v$ and E stand for density, velocity, momentum and total energy, respectively, with the state equation for the pressure $p = (\gamma - 1)(E - \frac{1}{2}\rho v^2)$ (see [13]). The solutions to these equations, as well as the physical phenomenon, are very complicated. One of the distinguished futures is the appearance of discontinuous solutions such as shock waves. This imposes a great difficulty in the design of

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numerical schemes for these systems and also for the mathematical analysis. A scalar equation

$$u_t + f(u)_x = 0\,, \tag{1.3}$$

is usually used as a simple model to give the guideline in the corresponding algorithm design and analysis. The concept of Total-variation-diminishing (TVD) has played a crucial role in the development of modern shock capturing schemes, although the concept of TVD is only valid in the analysis of the scalar equation. Nevertheless, direct extension to general systems has been highly successful in the computation of many complicated physical systems. See the relevant references, such as the introduction of "Monotonic Upstream-centered Scheme for Conservation laws" (MUSCL) by Van Leer in [34, 35, 36, 37, 38], the corresponding analysis in [16, 25], the discussion of the "essentially nonoscillatory" (ENO) scheme in [8, 20, 24, 31], etc. A concept of positivity property, which was recently proposed by X.-D. Liu and P. Lax in [18], is a natural extension of TV-stable property. In the scalar case, any consistent scheme in which the numerical solution u^{n+1} at the time step t^{n+1} can be written as convex (positive) combination of u^n was proven to be TVD. In the case of a hyperbolic system, the extension of a positive coefficient (combination) to a positive symmetric matrix gives a positive scheme.

Let's have a review of the TVD scheme and its extension to a general system by positivity scheme, in the context of flux splitting. The idea can be easily explained by the upwind scheme

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + a_j^{+,n}\frac{u_j^n - u_{j-1}^n}{\triangle x} + a_j^{-,n}\frac{u_{j+1}^n - u_j^n}{\triangle x} = 0, \tag{1.4}$$

applied for the scalar linear advect equation,

$$u_t + a u_x = 0\,, \tag{1.5}$$

with the decomposition $a = a^+ + a^-$ and $a^+ \geq 0$, $a^- \leq 0$. The upwind scheme (1.4) can be reformulated so that u_j^{n+1} is a convex combination of the profile u^n, namely,

$$u_j^{n+1} = a_j^{+,n}\frac{\triangle t}{\triangle x}u_{j-1}^n + \Big(1 - a_j^{+,n}\frac{\triangle t}{\triangle x} + a_j^{-,n}\frac{\triangle t}{\triangle x}\Big)u_j^n - a_j^{-,n}\frac{\triangle t}{\triangle x}u_{j+1}^n\,. \tag{1.6}$$

As a result, the stability property of the scheme (1.4) follows directly from a CFL-like condition

$$\Big(a_j^{+,n} - a_j^{-,n}\Big)\frac{\triangle t}{\triangle x} \leq 1\,. \tag{1.7}$$

For scalar nonlinear conservation laws

$$u_t + f(u)_x = 0\,, \tag{1.8}$$

the flux can be decomposed as

$$f = f^+ + f^-\,, \qquad \text{in which} \qquad {f^+}' \geq 0\,, \quad {f^-}' \leq 0\,. \tag{1.9}$$

The upwind scheme can be applied to (1.8) by naturally using flux splitting

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + \frac{f^+(u_j^n) - f^+(u_{j-1}^n)}{\triangle x} + \frac{f^-(u_{j+1}^n) - f^-(u_j^n)}{\triangle x} = 0\,. \tag{1.10}$$

It is obvious that the scheme (1.10) can be recast in the form of (1.6) with

$$
\begin{aligned}
a_j^{+,n} &= \frac{f^+(u_j^n) - f^+(u_{j-1}^n)}{u_j^n - u_{j-1}^n} = f^{+'}(\xi_j^n) \ge 0, \\
a_j^{-,n} &= \frac{f^-(u_{j+1}^n) - f^-(u_j^n)}{u_{j+1}^n - u_j^n} = f^{-'}(\eta_j^n) \le 0\,,
\end{aligned}
\tag{1.11}
$$

in which ξ_j^n lies between u_{j-1} and u_j, η_j^n between u_j and u_{j+1}, respectively. Moreover, we have

$$
1 - a_j^{+,n}\frac{\triangle t}{\triangle x} + a_j^{-,n}\frac{\triangle t}{\triangle x} \ge 0\,, \tag{1.12}
$$

under the following assumption

$$
\frac{\triangle t}{\triangle x}\Big(\max|f^{+'}| + \max|f^{-'}|\Big) \le 1\,. \tag{1.13}
$$

Therefore, the combination of (1.11), (1.12) and (1.6) shows that the first-order flux splitting method (1.10) for scalar conservation law is TVD by using the argument in [6], provided that the CFL-like condition (1.13) is satisfied.

In a second-order approximation, the positive and negative fluxes can be approximated by piecewise linear function

$$
f^{+,n}(x) = f_j^{+,n} + s_j^{+,n}(x - x_j)\,, \qquad f^{-,n}(x) = f_j^{-,n} + s_j^{-,n}(x - x_j)\,, \tag{1.14}
$$

for $x_{j-1/2} < x < x_{j+1/2}$, where the notation $f_j^{+,n} = f^+(u_j^n)$, $f_j^{-,n} = f^-(u_j^n)$ is used. The slopes are determined by a limiter function

$$
s_j^{\pm,n} = \frac{f_j^{\pm,n} - f_{j-1}^{\pm,n}}{\triangle x}\,\phi^0\Big(\frac{f_{j+1}^{\pm,n} - f_j^{\pm,n}}{f_j^{\pm,n} - f_{j-1}^{\pm,n}}\Big)\,. \tag{1.15}
$$

In this paper, we only consider *minmod* limiter

$$
\phi^0(\theta) = max\Big(0, min(1,\theta)\Big)\,, \tag{1.16}
$$

or *Van Leer* limiter (see [1])

$$
\phi^0(\theta) = \frac{|\theta| + \theta}{1 + \theta}\,. \tag{1.17}
$$

Both limiters are symmetric, i.e., $a\phi^0(\frac{b}{a}) = b\phi^0(\frac{a}{b})$, so that we can write

$$
\phi^0(a,b) = a\phi^0(\frac{b}{a}) = b\phi^0(\frac{a}{b})\,, \tag{1.18}
$$

without ambiguity. Consequently, the second-order flux splitting scheme is given by

$$
\begin{aligned}
\frac{u_j^{n+1} - u_j^n}{\triangle t} + &\quad \frac{f^{+,n}(x_{j+1/2}-) - f^{+,n}(x_{j-1/2}-)}{\triangle x} \\
+ &\quad \frac{f^{-,n}(x_{j+1/2}+) - f^{+,n}(x_{j-1/2}+)}{\triangle x} = 0\,.
\end{aligned}
\tag{1.19}
$$

Using the notations in (1.14)-(1.18), we can rewrite the above scheme as

$$
\frac{u_j^{n+1} - u_j^n}{\triangle t} + \psi_j^{+n}\frac{f_j^{+,n} - f_{j-1}^{+,n}}{\triangle x} + \psi_j^{-n}\frac{f_{j+1}^{-,n} - f_j^{-,n}}{\triangle x} = 0\,, \tag{1.20a}
$$

with the two coefficients

$$\psi_j^{+n} = 1 + \frac{1}{2}\phi^0\Big(\frac{f_{j+1}^{+,n} - f_j^{+,n}}{f_j^{+,n} - f_{j-1}^{+,n}}\Big) - \frac{1}{2}\phi^0\Big(\frac{f_{j-1}^{+,n} - f_{j-2}^{+,n}}{f_j^{+,n} - f_{j-1}^{+,n}}\Big),$$
$$\psi_j^{-n} = 1 + \frac{1}{2}\phi^0\Big(\frac{f_{j+2}^{-,n} - f_{j+1}^{-,n}}{f_{j+1}^{-,n} - f_j^{-,n}}\Big) - \frac{1}{2}\phi^0\Big(\frac{f_j^{-,n} - f_{j-1}^{-,n}}{f_{j+1}^{-,n} - f_j^{-,n}}\Big). \tag{1.20b}$$

As in (1.11), by applying the intermediate value theorem for f^+, f^-, the second-order upwind scheme for the nonlinear problem can still be recast in the form of (1.6), namely

$$a_j^{+,n} = \psi_j^{+n} f^{+'}(\xi_j^n), \qquad a_j^{-,n} = \psi_j^{-n} f^{-'}(\eta_j^n), \tag{1.21}$$

in which ξ_j^n lies between u_{j-1} and u_j, and η_j^n between u_j and u_{j+1}, respectively. Both the *minmod* and *Van Leer* limiter functions satisfy

$$0 \le \frac{\phi^0(\theta)}{\theta} \le 2 \qquad \text{and} \qquad 0 \le \phi^0(\theta) \le 2, \tag{1.22}$$

so that the two coefficients ψ_j^{+n} and ψ_j^{-n} are between 0 and 2, which results in the positivity of $a_j^{+,n}$ and $-a_j^{-,n}$. Furthermore, (1.12) is satisfied under the following CFL-like condition

$$2\frac{\triangle t}{\triangle x}\Big(\max|f^{+'}| + \max|f^{-'}|\Big) \le 1, \tag{1.23}$$

due to the second property of ϕ^0 in (1.22). Therefore, the stability and TVD property of the scheme (1.20) follows directly from Harten's argument [6]. As a result, the numerical scheme converges to a weak solution. It can be observed that the CFL-like assumption (1.23) is more strict than the first-order version (1.13), due to the usage of a limiter function. In other words, the second-order scheme (1.20) can be viewed as a stable perturbation of the first-order one (1.10), due to the fact that both ψ_j^{+n} and ψ_j^{-n} are positive coefficients bounded by 2.

This simple stability argument no longer holds for a general system of conservation laws

$$U_t + F(U)_x = 0, \tag{1.24}$$

since no TV-stable property can be automatically applied to the analysis of a nonlinear system. Nevertheless, a direct extension of the above scheme enjoys a great success. The flux vector splitting (FVS) method reconstructs the flux F as

$$F(U) = F^+(U) + F^-(U), \tag{1.25}$$

where the Jacobians of F^+, F^- have only non-negative and non-positive eigenvalues, respectively. To achieve high order accuracy, we apply the second-order flux vector splitting scheme to each component

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle x}(F_{j+1/2}^{+n} - F_{j-1/2}^{+n}) - \frac{\triangle t}{\triangle x}(F_{j+1/2}^{-n} - F_{j-1/2}^{-n}), \tag{1.26a}$$

where the $p-$th component of $F^{\pm}_{j+1/2}$, $F^{\pm(p)}_{j+1/2}$ is given by

$$\begin{aligned} F^{+n(p)}_{j+1/2} &= F^{+n(p)}_{j} + \frac{1}{2}(F^{+n(p)}_{j} - F^{+n(p)}_{j-1})\phi^0\Big(\frac{F^{+n(p)}_{j+1} - F^{+n(p)}_{j}}{F^{+n(p)}_{j} - F^{+n(p)}_{j-1}}\Big)\,, \\ F^{-n(p)}_{j+1/2} &= F^{-n(p)}_{j+1} + \frac{1}{2}(F^{-n(p)}_{j+1} - F^{-n(p)}_{j})\phi^0\Big(\frac{F^{-n(p)}_{j+2} - F^{-n(p)}_{j+1}}{F^{-n(p)}_{j+1} - F^{-n(p)}_{j}}\Big)\,, \end{aligned} \tag{1.26b}$$

and the limiter function ϕ^0 is chosen as either (1.16) or (1.17). The approach of the FVS scheme goes back to Van Leer [1]. See also [3]. The second order version (1.26) is in fact the same as the convex ENO scheme discussed by Liu and Osher in [19].

Meanwhile, a framework to provide some theoretical guides for the general systems of conservation laws has been proposed by P. Lax and X.-D. Liu in [18] with the concept of positive scheme, which is motivated by the fact that the only functional known to be bounded for solutions of linear hyperbolic equation is energy, as proven by Friedrichs in [4]. A conservative scheme of the form

$$U^{n+1}_j = U_j + \frac{\triangle t}{\triangle x}\Big(F_{j+1/2} - F_{j-1/2}\Big)\,, \tag{1.27}$$

where F is a consistent numerical flux, is called positive, if U^{n+1} can be written as

$$U^{n+1}_j = \sum_K C_K U^n_{j+K}\,, \tag{1.28}$$

so that the coefficient matrices C_K, which themselves depend on all the U_{j+K} that occur in (1.27), have the following properties:

$$\text{each } C_K \text{ is symmetric and positive}\,, \qquad \text{i.e.,} \qquad C_K \geq 0\,; \tag{1.29a}$$

$$\sum_K C_K = I\,, \tag{1.29b}$$

$$C_K = 0\,, \qquad \text{except for a finite set of K}\;. \tag{1.29c}$$

Then they argued that, for positive schemes, the numerical solution U^n is L^2 stable in some suitable sense.

Note that for the case of a scalar equation (1.8), the positivity (1.29) is reduced to the convex combination form (1.6), while the combination coefficients c_{-1}, c_1, c_0 correspond to $a^{+,n}_j$, $a^{-,n}_j$, $1 - a^{+,n}_j\frac{\triangle t}{\triangle x} + a^{-,n}_j\frac{\triangle t}{\triangle x}$, respectively. In the second-order flux vector splitting scheme, the general form can be written as (3.15) in Section 3, yet the form of the combination matrices C_K depends on the concrete splitting.

It should be noted that the symmetry of the coefficient matrices C_K plays an important role in the L^2-stability of the positivity scheme [18]. However, a symmetric Jacobian matrix can hardly be found for a system of hyperbolic conservation laws in terms of physical variables. Instead, a symmetrizable system is taken into consideration in [18]

$$B_0(U)\partial_t U + B(U)\partial_x U = 0\,, \tag{1.30}$$

in which B is symmetric, B_0 is positive and its symmetric square root is denoted as S, i.e., $B_0 = S^2$. Multiplying (1.30) by B_0^{-1} gives

$$\partial_t U + A(U)\partial_x U = 0\,, \qquad \text{with} \qquad A = B_0^{-1}B\,. \tag{1.31}$$

Moreover, it turns out that SAS^{-1} is symmetric. For such a symmetrizable system, the numerical scheme is defined as positive if it can be recast in the form of (1.28) and the matrices

$$D_K(J) = S(J)C_K(J)S^{-1}(J)\,, \tag{1.32}$$

satisfy the following conditions

$$D_K \text{ differs by } O(\triangle t) \text{ from a symmetric matrix}\,, \tag{1.33a}$$

$$\text{the symmetric part of } D_K \text{ has non-negative eigenvalues}\,, \tag{1.33b}$$

$$\sum_K D_K = I\,, \tag{1.33c}$$

$$D_K = 0\,, \qquad \text{except for a finite set of K}\ . \tag{1.33d}$$

The L^2-stability of the positive scheme satisfying (1.33) was established in [18]. Due to the fact that the compressible Euler equations are not symmetric but symmetrizable, they constructed a second-order positive scheme in the article by using Roe matrix decomposition, which was proven to be very efficient.

In this paper we show the positivity and weak stability of the second-order limiter schemes with some well-known FVS in the computation of the compressible Euler equation, including Steger-Warming and Van Leer splittings proposed in [32] and [1]. We recall that the gas equation is symmetrizable only in terms of the velocity, pressure and entropy variables. Its conservative form, which is comprised of the dynamic equations for the density, momentum and the total energy, is not. Yet, we have to keep the shock-capturing scheme in the conservative form (1.27) consistent with the conservation laws to get correct shock speed, in view of Lax-Wendroff theorem. The key point in this article is the reformulation of the positive and negative fluxes in the Steger-Warming and Van Leer splitting so that the corresponding matrices are either symmetric or symmetrizable and keep non-negative and non-positive eigenvalues, respectively. This fact is reported in Proposition 2.1 in Section 2 below. The basic idea is to use the property of Euler flux that it is a homogeneous function of degree one in the variable. A direct consequence of it is the positivity of the first order scheme, which comes from the rewritten forms of the splitting fluxes F^+, F^-. It is shown in Section 3 that the second order limiter scheme is a stable perturbation of the first order one. The corresponding second order scheme using Steger-Warming or Van Leer splitting is proven to be positive under some CFL-like conditions in Section 4 and 5, due to the representation formula that the numerical U^{n+1} is a positive combination of U^n.

Furthermore, both splitting schemes preserve the positivity of density and energy variables. To achieve this, the fluxes F^+ and F^- need to be formulated in another way so that the corresponding matrices A^+, A^- are diagonal with respect to the first and third components. The diagonal elements are also bounded by fluid velocity and sound speeds. Then the density and energy at time step t^{n+1} is shown to be a positive combination of their values at time step t^n, which indicates the positivity preserving property of the scheme under a CFL-like condition. A variety of formulations of F^+ and F^- are possible because of the nonlinearity of Euler flux.

Moreover, we note that condition (1.33b) is in fact a consistency requirement. Since the system is nonlinear, we can write (1.27) in several different forms. To simplify our presentation, by observing that all the flux splitting schemes we consider here are in a consistent conservative form, we modify the condition (1.33c) by

$$\sum_K C_K = I + O(\triangle t)\,, \qquad \text{if} \quad U^n \text{ is Lipschitz continuous at time step } n\,. \tag{1.34}$$

Note that in the scalar case, condition (1.29a) plus

$$\sum_K C_K = 1 + O(\triangle t)\,, \tag{1.35}$$

can ensure the TV-stable property of the numerical scheme. In this article, the requirement of positivity is set to be

$$\begin{aligned}&\text{each } C_K \text{ is either symmetric or symmetrizable and} \quad C_K \geq 0\,,\\ &\text{up to } O(\triangle t) \text{ difference}\,;\end{aligned} \tag{1.36a}$$

$$\sum_K C_K = I + O(\triangle t)\,, \quad \text{if} \quad U^n \text{ is Lipschitz continuous at time step } n\,; \tag{1.36b}$$

$$C_K = 0\,, \qquad \text{except for a finite set of K}\,. \tag{1.36c}$$

2. **The compressible Euler equations and symmetrizable flux splitting.** The 1-D system of the Euler equations is given by

$$\partial_t U + \partial_x F(U) = 0, \qquad U = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix}, \qquad F = \begin{bmatrix} m \\ \rho v^2 + p \\ v(E+p) \end{bmatrix}, \tag{2.1}$$

where $\rho, v, m = \rho v$ and E are density, velocity, momentum and total energy, respectively, and the state equation for the pressure is given by $p = (\gamma - 1)(E - \frac{1}{2}\rho v^2)$. The eigenvalues of the Jacobian matrix of $F(U)$ are

$$\lambda_1 = v - c\,, \qquad \lambda_2 = v\,, \qquad \lambda_3 = v + c\,, \tag{2.2}$$

with the sound speed $c = \sqrt{\frac{\gamma p}{\rho}}$. The constant γ depends on the gas and usually ranges from 1 to 3. For example, such constant can be taken as $\gamma = 1.4$ for the air. Our analysis below is valid for any $1 < \gamma < 3$.

2.1. **Steger-Warming flux splitting.** Through the similarity transformation on the flux vector using the property, $F(U) = A(U)U$, $A(U) = F'(U)$, due to the fact that the flux vector is a homogeneous function of degree one, the Steger-Warming splitting of F in [32] is given by

$$F(U) = F^+(U) + F^-(U) = Q^{-1}\Gamma^+ QU + Q^{-1}\Gamma^- QU\,, \tag{2.3}$$

where $A = Q^{-1}\Gamma Q$ and $\Gamma^\pm = diag(\lambda_1^\pm, \lambda_2^\pm, \lambda_3^\pm)$, $\lambda_i^\pm = (\lambda_i \pm |\lambda_i|)/2, i = 1, 2, 3$.

The corresponding F^+ and F^- for the subsonic case $0 \leq v \leq c$ read

$$\begin{aligned} F^+_{SWS} &= \frac{\rho}{2\gamma}\begin{bmatrix} 2\gamma v + c - v \\ 2(\gamma-1)v^2 + (v+c)^2 \\ (\gamma-1)v^3 + \frac{1}{2}(v+c)^3 + \frac{3-\gamma}{2(\gamma-1)}(v+c)c^2 \end{bmatrix}, \\ F^-_{SWS} &= \frac{\rho}{2\gamma}\begin{bmatrix} v - c \\ (v-c)^2 \\ \frac{1}{2}(v-c)^3 + \frac{3-\gamma}{2(\gamma-1)}(v-c)c^2 \end{bmatrix}. \end{aligned} \tag{2.4a}$$

In the supersonic case $v > c$,

$$F^+ = F\,, \qquad F^- = 0\,. \tag{2.4b}$$

The subvectors F^+, F^- for other cases, including the subsonic case $-c \le v \le 0$ and supersonic case $v < -c$, can be obtained in a similar way.

2.2. **Van Leer Splitting (VLS).** In [1] Van Leer developed the splitting which is differentiable even at sonic points in terms of the local Mach number M, $M = u/c$. For $M \ge 1$, the eigenvalues of Jacobian of F are all positive and thus $F^+_{VLS} = F, F^-_{VLS} = 0$. Similarly, $F^+_{VLS} = 0$ and $F^-_{VLS} = F$ for $M \le -1$. The Van Leer splitting is given as follows for $-1 < M < 1$,

$$F^{\pm}_{VLS} = \begin{bmatrix} f_1^{\pm} \\ f_1^{\pm}((\gamma-1)v \pm 2c)/\gamma \\ f_1^{\pm}((\gamma-1)v \pm 2c)^2/2(\gamma^2-1) \end{bmatrix}, \tag{2.5}$$

where $f_1^{\pm} = \pm\rho c(M \pm 1)^2/4$. The Jacobian of the positive flux, $(F^+_{VLS})'$ has two positive and one zero eigenvalues, while $(F^-_{VLS})'$ has two negative and one zero eigenvalues.

2.3. **Symmetrizable reformulation of the splitting fluxes F^+ and F^-.** Note that the positive and negative matrices $Q^{-1}\Gamma^+Q$, $Q^{-1}\Gamma^-Q$ of the Steger-Warming splitting are not symmetrizable in its original form. The same is for the Van Leer splitting. The key observation in this section is that the representation form of such matrices is not unique due to the nonlinearity of the fluxes. The matrices can be reformulated to be symmetrizable and keep positive and negative eigenvalues, respectively. The following proposition is crucial to the positivity argument of both flux splittings presented in later sections.

Proposition 2.1. *The fluxes F^+, F^- in either Steger-Warming splitting (2.4) or Van Leer splitting (2.5) can be represented as*

$$F^+ = A^+U\,, \qquad F^- = A^-U\,, \tag{2.6a}$$

in which A^+, A^- are either symmetric or symmetrizable and

$$A^+ \ge 0\,, \qquad A^- \le 0\,. \tag{2.6b}$$

Moreover, $A^+ - A^-$ is a diagonal matrix with non-negative elements.

A direct consequence of Proposition 2.1 is the positivity of the first order method using either flux splitting.

Corollary 2.1. *The first order flux splitting method*

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle x}(F_j^{+n} - F_{j-1}^{+n}) - \frac{\triangle t}{\triangle x}(F_{j+1}^{-n} - F_j^{-n})\,, \tag{2.7}$$

with the positive and negative fluxes F^+, F^- given by either Steger Warming (2.4) or Van Leer (2.5) is positive, under some CFL-like conditions, namely (2.21)-(2.22) or (2.29) below.

Proof. The insertion of (2.6a) into the first order scheme (2.7) shows that

$$\begin{aligned} U_j^{n+1} &= U_j^n - \lambda(A_j^{+n}U_j^n - A_{j-1}^{+n}U_{j-1}^n) - \lambda(A_{j+1}^{-n}U_{j+1}^n - A_j^{-n}U_j^n) \\ &= \lambda A_{j-1}^{+n}U_{j-1}^n + (-\lambda A_{j+1}^{-n})U_{j+1}^n + (I - \lambda A_j^{+n} + \lambda A_j^{-n})U_j^n\,, \end{aligned} \tag{2.8}$$

with the constant $\lambda = \frac{\triangle t}{\triangle x}$. By taking the notation

$$C_{-1} = \lambda A_{j-1}^{+n}\,, \quad C_1 = -\lambda A_{j+1}^{-n}\,, \quad C_0 = I - \lambda(A_j^{+n} - A_j^{-n})\,, \tag{2.9a}$$

we arrive at

$$U_j^{n+1} = C_{-1}U_{j-1}^n + C_0 U_j^n + C_1 U_{j+1}^n \,. \tag{2.9b}$$

It is indicated by Proposition 2.1 that both C_{-1} and C_1 are either symmetric or symmetrizable and keep non-negative eigenvalues. In addition, since $A^+ - A^-$ is a diagonal matrix with non-negative elements, C_0 is also positive symmetric if λ is bounded by the inverse of the eigenvalues, which is shown to be a CFL-like condition, as can be seen in Remarks 1, 2 below. The property that $C_{-1}+C_0+C_1 = I + O(\triangle t)$ can be verified by the smoothness of the positive and negative matrices A^+, A^- with respect to the physical variables, under the assumption that the solution is Lipschitz continuous. □

The proof of Proposition 2.1 is a constructive one and is based on the following lemma. Its proof is straightforward and we omit it.

LEMMA 2.1. *A matrix* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *is symmetrizable and keeps non-negative eigenvalues if*

$$a_{11} > 0\,, \quad a_{22} > 0\,, \quad a_{12}a_{21} > 0\,, \quad a_{11}a_{22} \geq a_{12}a_{21}\,; \tag{2.10a}$$

it is symmetrizable and keeps non-positive eigenvalues if

$$a_{11} < 0\,, \quad a_{22} < 0\,, \quad a_{12}a_{21} > 0\,, \quad a_{11}a_{22} \geq a_{12}a_{21}\,. \tag{2.10b}$$

Proof of Proposition 2.1 for the case of Steger-Warming splitting:

(a) In the subsonic case $0 \leq v < c$, using the state equation $E = \frac{1}{2}\rho v^2 + \frac{p}{\gamma-1}$, and $p = \frac{1}{\gamma}\rho c^2$, we can rewrite F^+, F^- as

$$\begin{aligned} F^+ &= \begin{bmatrix} \frac{2\gamma-1}{2\gamma}\rho v + \frac{1}{2\gamma}\rho c \\ \frac{2\gamma-1}{2\gamma}\rho v^2 + \frac{1}{\gamma}\rho v c + \frac{1}{2\gamma}\rho c^2 \\ \frac{2\gamma-1}{4\gamma}\rho v^3 + \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 + \frac{1}{2\gamma(\gamma-1)}\rho c^3 \end{bmatrix} \\ &= \begin{bmatrix} (\frac{2\gamma-1}{2\gamma}v + \frac{1}{2\gamma}c)\rho \\ (\frac{5\gamma-2-\gamma^2}{4\gamma}v + \frac{1}{\gamma}c)\rho v + \frac{\gamma-1}{2}E \\ \frac{3-\gamma}{18}(v-c)^2\rho v + a^+_{3,3}E \end{bmatrix}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} F^- &= \begin{bmatrix} \frac{1}{2\gamma}\rho v - \frac{1}{2\gamma}\rho c \\ \frac{1}{2\gamma}\rho v^2 - \frac{1}{\gamma}\rho v c + \frac{1}{2\gamma}\rho c^2 \\ \frac{1}{4\gamma}\rho v^3 - \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 - \frac{1}{2\gamma(\gamma-1)}\rho c^3 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2\gamma}(c-v)\rho \\ -\frac{1}{4\gamma}\left[4c - (2-\gamma^2+\gamma)v\right]\rho v - (-\frac{\gamma-1}{2})E \\ \frac{3-\gamma}{18}(v-c)^2\rho v + a^-_{3,3}E \end{bmatrix}, \end{aligned} \tag{2.12}$$

so that F^+, F^- can be represented as $F^+ = A^+U$, $F^- = A^-U$, with

$$A^+ = \begin{pmatrix} a^+_{1,1} & 0 & 0 \\ 0 & a^+_{2,2} & \frac{\gamma-1}{2} \\ 0 & \frac{3-\gamma}{18}(v-c)^2 & a^+_{3,3} \end{pmatrix}, \quad A^- = \begin{pmatrix} a^-_{1,1} & 0 & 0 \\ 0 & a^-_{2,2} & \frac{\gamma-1}{2} \\ 0 & \frac{3-\gamma}{18}(v-c)^2 & a^-_{3,3} \end{pmatrix}, \tag{2.13}$$

in which the coefficients have the following estimates

$$
\begin{aligned}
&a^+_{1,1} = \frac{2\gamma-1}{2\gamma}v + \frac{1}{2\gamma}c\,, \quad a^+_{2,2} = \frac{5\gamma-2-\gamma^2}{4\gamma}v + \frac{1}{\gamma}c\,,\\
&a^+_{3,3} = \frac{\frac{2\gamma-1}{4\gamma}\rho v^3 + \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 + \frac{1}{2\gamma(\gamma-1)}\rho c^3 - \frac{3-\gamma}{18}\rho v(v-c)^2}{E}\,,\\
&\frac{3-\gamma}{2\gamma}c \le a^+_{3,3} \le \frac{\gamma^2+2\gamma-1}{2\gamma}v + \frac{\gamma+3}{2\gamma}c\,,\\
&a^-_{1,1} = -\frac{1}{2\gamma}(c-v)\,, \quad a^-_{2,2} = -\frac{1}{4\gamma}\left[4c-(2-\gamma^2+\gamma)v\right] \le -\frac{1}{\gamma}(c-v)\,,\\
&a^-_{3,3} = \frac{\frac{1}{4\gamma}\rho v^3 - \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 - \frac{1}{2\gamma(\gamma-1)}\rho c^3 - \frac{3-\gamma}{18}\rho v(v-c)^2}{E}\,,\\
&\frac{1}{\gamma-1}(v-c) \le a^-_{3,3} \le \frac{3-\gamma}{\gamma+1}(v-c) < 0\,.
\end{aligned}
\tag{2.14}
$$

The verification of the above estimates is straightforward by algebraic calculation of the physical variables. The detail is omitted here. From (2.14), we have

$$
\begin{aligned}
&a^+_{2,2}a^+_{3,3} \ge \frac{3-\gamma}{2\gamma^2}c^2 \ge \frac{(\gamma-1)(3-\gamma)}{36}c^2 \ge \frac{\gamma-1}{2}\cdot\frac{3-\gamma}{18}(v-c)^2\,,\\
&a^-_{2,2}a^-_{3,3} \ge \frac{3-\gamma}{\gamma(\gamma+1)}(v-c)^2 \ge \frac{\gamma-1}{2}\cdot\frac{3-\gamma}{18}(v-c)^2\,,
\end{aligned}
\tag{2.15}
$$

for $1<\gamma<3$. The application of Lemma 2.1 shows that A^+, A^- are symmetrizable and keep non-negative and non-positive eigenvalues, since the criteria (2.10a), (2.10b) can be verified by the usage of (2.15).

It can be seen that the matrix $A^+ - A^-$ has the form

$$
A^+ - A^- = \begin{pmatrix} a^+_{1,1}-a^-_{1,1} & 0 & 0\\ 0 & a^+_{2,2}-a^-_{2,2} & 0\\ 0 & 0 & a^+_{3,3}-a^-_{3,3}\end{pmatrix}\,, \tag{2.16}
$$

which is diagonal and keeps only non-negative eigenvalues. Moreover, the estimate (2.14) leads to

$$
\begin{aligned}
&a^+_{1,1}-a^-_{1,1} = \frac{\gamma-1}{\gamma}v + \frac{1}{\gamma}c\,, \quad a^+_{2,2}-a^-_{2,2} = \frac{\gamma-1}{\gamma}v + \frac{2}{\gamma}c\,,\\
&a^+_{3,3}-a^-_{3,3} \le \frac{\gamma^2+2\gamma-1}{2\gamma}v + \frac{\gamma^2+4\gamma-3}{2\gamma(\gamma-1)}c\,,
\end{aligned}
\tag{2.17}
$$

so that the eigenvalues are bounded by the linear combination of the velocity and sound speed. This was used in the proof for the positivity of the first order method.

The other subsonic case $-c<v\le 0$ can be dealt with in the same manner. That finishes the proof of Proposition 2.1 in the subsonic case of the Steger-Warming splitting.

(b) In the supersonic region $v \ge c$, the positive and negative fluxes are given by

$$
F^+ = F = \begin{bmatrix} \rho v\\ \frac{\rho v^2+p}{\rho v}\cdot m\\ \frac{v(E+p)}{E}\cdot E\end{bmatrix}\,, \qquad F^- = 0 = A^- U\,, \tag{2.18}
$$

where $A_- = 0$. Then we can rewrite F^+, F^- in the same form as $F^+ = A^+U$, $F^- = A^-U$, with

$$A^+ = \begin{pmatrix} a^+_{1,1} & 0 & 0 \\ 0 & a^+_{2,2} & 0 \\ 0 & 0 & a^+_{3,3} \end{pmatrix}, \qquad A^- = 0\,,$$
$$a^+_{1,1} = v\,, \quad a^+_{2,2} = \frac{\rho v^2 + p}{\rho v}\,, \quad a^+_{3,3} = \frac{v(E+p)}{E}\,. \tag{2.19}$$

The following estimate for the components in F^+ is also straightforward by the algebraic calculation

$$0 \le a^+_{1,1} = v\,, \quad 0 \le a^+_{2,2} = \frac{\rho v^2 + p}{\rho v} \le v + \frac{1}{\gamma}c\,, \quad 0 \le a^+_{3,3}\frac{v(E+p)}{E} \le v + \frac{2}{\gamma}c\,. \tag{2.20}$$

In this case, $A^+ - A^-$ is exactly A^+ presented in (2.19), which is diagonal and its eigenvalue estimate is given in (2.20).

For another supersonic case $v \le -c$, $F^+ = 0$ and F^- can be similarly represented and an analogous estimate to (2.20) can be obtained. Then we finish the proof of Proposition 2.1 for the Steger-Warming splitting. □

Remark 1. *As can be seen in both the subsonic and supersonic cases, the matrix* $C_0 = I - \lambda(A^+ - A^-)$ *is diagonal and its diagonal elements are non-negative under the constraint*

$$\frac{\triangle t}{\triangle x}\Big(a^+_{i,i} - a^-_{i,i}\Big) \le 1\,, \qquad i = 1\,,2\,,3\,. \tag{2.21}$$

With the usage of the estimates (2.17), (2.20), the constraint (2.21) is satisfied provided that

$$\frac{\triangle t}{\triangle x} max\Big(\frac{\gamma^2 + 2\gamma - 1}{2\gamma}, \frac{2}{\gamma}, \frac{\gamma^2 + 4\gamma - 3}{2\gamma(\gamma-1)}\Big)(|v| + |c|) \le 1\,. \tag{2.22}$$

As a result, the first order Steger-Warming splitting method is positive under the CFL-like condition (2.22).

Proof of Proposition 2.1 for the case of Van Leer splitting:

In the subsonic case $0 \le M < 1$, F^+ and F^- can be represented as

$$
\begin{aligned}
F^+ &= \begin{bmatrix} \rho c(M^2+2M+1)/4 \\ \rho c\frac{M^2+2M+1}{4}\cdot\frac{(\gamma-1)v+2c}{\gamma} \\ \rho c\frac{M^2+2M+1}{4}\cdot\frac{[(\gamma-1)v+2c]^2}{2(\gamma^2-1)} \end{bmatrix} \\
&= \begin{bmatrix} (\frac{1}{2}v+\frac{M^2+1}{4}c)\rho \\ \frac{\gamma-1}{4\gamma}(M+2)\rho v^2+\frac{1}{2\gamma}\rho v^2+\frac{\gamma+3}{4\gamma}\rho vc+\frac{1}{2\gamma}\rho c^2 \\ \rho c\frac{M^2+2M+1}{4}\cdot\frac{[(\gamma-1)v+2c]^2}{2(\gamma^2-1)} \end{bmatrix} \\
&= \begin{bmatrix} (\frac{1}{2}v+\frac{M^2+1}{4}c)\rho \\ \left[\frac{\gamma+3}{4\gamma}c+\left(\frac{\gamma-1}{4\gamma}(M+2)+\frac{1}{2\gamma}-\frac{\gamma-1}{4}\right)v\right]\rho v+\frac{\gamma-1}{2}E \\ \frac{(3-\gamma)^2}{72}(v-c)^2+a^+_{3,3}E \end{bmatrix} \\
&= A^+U\,,
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
F^- &= \begin{bmatrix} -\rho c\frac{M^2-2M+1}{4} \\ -\rho c\frac{M^2-2M+1}{4}\frac{(\gamma-1)v-2c}{\gamma} \\ -\rho c\frac{M^2-2M+1}{4}\frac{[(\gamma-1)v-2c]^2}{2(\gamma^2-1)} \end{bmatrix} \\
&= \begin{bmatrix} \left[-\frac{c}{4}+\frac{1}{4}(2-M)v\right]\rho \\ \frac{\gamma-1}{4\gamma}(2-M)\rho v^2+\frac{1}{2\gamma}\rho v^2-\frac{\gamma+3}{4\gamma}\rho vc+\frac{1}{2\gamma}\rho c^2 \\ -\rho c\frac{M^2-2M+1}{4}\frac{[(\gamma-1)v-2c]^2}{2(\gamma^2-1)} \end{bmatrix} \\
&= \begin{bmatrix} \left[-\frac{c}{4}+\frac{1}{4}(2-M)v\right]\rho \\ -\left[\frac{\gamma+3}{4\gamma}c-\left(\frac{1}{2\gamma}+\frac{\gamma-1}{4\gamma}(2-M)-\frac{\gamma-1}{4}\right)v\right]\rho v-(-\frac{\gamma-1}{2})E \\ \frac{(3-\gamma)^2}{72}(v-c)^2\rho v+a^-_{3,3}E \end{bmatrix} \\
&= A^-U\,.
\end{aligned}
\tag{2.24}
$$

in which the matrices A_+, A_- have the form

$$
\begin{aligned}
A^+ &= \begin{pmatrix} a^+_{1,1} & 0 & 0 \\ 0 & a^+_{2,2} & \frac{\gamma-1}{2} \\ 0 & \frac{(3-\gamma)^2}{72}(v-c)^2 & a^+_{3,3} \end{pmatrix}, \\
A_- &= \begin{pmatrix} a^-_{1,1} & 0 & 0 \\ 0 & a^-_{2,2} & \frac{\gamma-1}{2} \\ 0 & \frac{(3-\gamma)^2}{72}(v-c)^2 & a^-_{3,3} \end{pmatrix},
\end{aligned}
\tag{2.25a}
$$

$$\begin{aligned}
a_{1,1}^{+} &= \frac{1}{2}v + \frac{M^2+1}{4}c\,, \qquad a_{3,3}^{+} = \frac{F^{+(3)} - \frac{(3-\gamma)^2}{72}(v-c)^2\rho v}{E}\,,\\
a_{2,2}^{+} &= \frac{\gamma+3}{4\gamma}c + \Big(\frac{\gamma-1}{4\gamma}(M+2) + \frac{1}{2\gamma} - \frac{\gamma-1}{4}\Big)v\,,\\
a_{1,1}^{-} &= -\frac{1}{4}c + \frac{1}{4}(2-M)v\,, \qquad a_{3,3}^{-} = \frac{F^{-(3)} - \frac{(3-\gamma)^2}{72}(v-c)^2\rho v}{E}\,,\\
a_{2,2}^{-} &= -\left[\frac{\gamma+3}{4\gamma}c - \Big(\frac{1}{2\gamma} + \frac{\gamma-1}{4\gamma}(2-M) - \frac{\gamma-1}{4}\Big)v\right].
\end{aligned} \tag{2.25b}$$

The following estimates can be verified by a careful calculation. The detail is omitted.

$$\begin{gathered}
0 \le a_{1,1}^{+} \le \frac{1}{2}(|v|+c)\,,\\
\frac{\gamma+3}{4\gamma}c - \frac{3-\gamma}{4}|v| \le a_{2,2}^{+} \le \frac{4\gamma-1-\gamma^2}{4\gamma}|v| + \frac{\gamma+3}{4\gamma}c\,,\\
\frac{1}{24}c \le a_{3,3}^{+} \le \frac{\gamma(\gamma^2+2\gamma-3)}{2(\gamma+1)}|v| + \frac{2\gamma}{\gamma+1}c\,, \qquad -\frac{1}{4}(c+3|v|) \le a_{1,1}^{-} \le 0\,,\\
-\Big(\frac{4\gamma-1-\gamma^2}{4\gamma}|v| + \frac{\gamma+3}{4\gamma}c\Big) \le a_{2,2}^{-} \le -\Big(\frac{\gamma+3}{4\gamma}c - \frac{3-\gamma}{4}|v|\Big)\,,\\
-\left[\frac{\gamma(\gamma^2+2\gamma-3)}{2(\gamma+1)}|v| + \Big(\frac{2\gamma}{\gamma+1} + \frac{(3-\gamma)^2}{6}\Big)c\right] \le a_{3,3}^{-} \le -\frac{(3-\gamma)^2}{4(\gamma+1)^2}c(1-M)^2\,.
\end{gathered} \tag{2.26}$$

Similarly, the estimate (2.26) results in

$$\begin{aligned}
a_{2,2}^{+}a_{3,3}^{+} &\ge \tfrac{1}{24}\Big(\tfrac{\gamma+3}{4\gamma} - \tfrac{3-\gamma}{4}\Big)c^2 \ge \frac{(3-\gamma)^2)(\gamma-1)}{144}c^2\\
&\ge \frac{\gamma-1}{2}\cdot\frac{(3-\gamma)^2(\gamma-1)}{72}(v-c)^2\,,\\
a_{2,2}^{-}a_{3,3}^{-} &\ge \frac{(3-\gamma)^2}{4(\gamma+1)^2}\Big(\frac{\gamma+3}{4\gamma} - \frac{3-\gamma}{4}\Big)c^2(1-M)^2 \ge \frac{\gamma-1}{2}\cdot\frac{(3-\gamma)^2}{72}(v-c)^2\,,
\end{aligned} \tag{2.27}$$

for $1<\gamma<3$. Moreover, by applying Lemma 2.1 and using the estimate (2.27), we conclude that A^+, A^- are symmetrizable and keep non-negative and non-positive eigenvalues, respectively.

It can be seen that the matrix $A^+ - A^-$ has the form

$$A^+ - A^- = \begin{pmatrix} a_{1,1}^{+} - a_{1,1}^{-} & 0 & 0\\ 0 & a_{2,2}^{+} - a_{2,2}^{-} & 0\\ 0 & 0 & a_{3,3}^{+} - a_{3,3}^{-} \end{pmatrix}, \tag{2.28}$$

which is diagonal and keeps only non-negative eigenvalues. The eigenvalues are bounded by the linear combination of the velocity and sound speed

$$\begin{aligned}
a_{1,1}^{+} - a_{1,1}^{-} &\le \frac{1}{4}v + \frac{3}{4}c\,, \quad a_{2,2}^{+} - a_{2,2}^{-} \le \frac{\gamma-1}{2\gamma}v + \frac{2}{\gamma}c\,,\\
a_{3,3}^{+} - a_{3,3}^{-} &\le \frac{\gamma(\gamma^2+2\gamma-3)}{\gamma+1}v + \Big(\frac{4\gamma}{\gamma+1} + \frac{(3-\gamma)^2}{3}\Big)c\,.
\end{aligned} \tag{2.29}$$

For another subsonic case $-1 < M \leq 0$, F^+ and F^- can be similarly represented and an analogous estimate to (2.25) can be obtained.

In the supersonic case, either $M \geq 1$ or $M \leq -1$, the representation and the estimates (2.18), (2.19) are also valid. Then we finish the analysis of the Van Leer splitting, thus complete the proof of Proposition 2.1. □

REMARK 2. *In both the subsonic and supersonic cases, the matrix $C_0 = I - \lambda(A^+ - A^-)$ is diagonal and its diagonal elements are non-negative under the constraint (2.21), which is assured to be satisfied if*

$$\frac{\triangle t}{\triangle x} max\Big(1, \frac{\gamma(\gamma^2+2\gamma-3)}{\gamma+1}, \frac{2}{\gamma}, \frac{4\gamma}{\gamma+1} + \frac{(3-\gamma)^2}{3}\Big)(|v|+|c|) \leq 1\,, \tag{2.30}$$

with the usage of the estimates (2.29), (2.20). As a result, the first order Van Leer splitting method is positive under the CFL-like condition (2.30).

3. **Analysis of second-order flux splitting.** In this section we give a general analysis for the second-order FVS scheme (1.26) applied to a system of conservation laws (1.24). We need to rewrite the scheme for the convenience of the positivity proof in the following sections. Substituting (1.18) into (1.26b) gives

$$\begin{aligned}\phi^0(F_{j+1}^{+(p)} - F_j^{+(p)}, F_j^{+(p)} - F_{j-1}^{+(p)}) &= (F_{j+1}^{+(p)} - F_j^{+(p)})\phi^0\Big(\frac{F_j^{+(p)} - F_{j-1}^{+(p)}}{F_{j+1}^{+(p)} - F_j^{+(p)}}\Big)\\ &= (F_j^{+(p)} - F_{j-1}^{+(p)})\phi^0\Big(\frac{F_{j+1}^{+(p)} - F_j^{+(p)}}{F_j^{+(p)} - F_{j-1}^{+(p)}}\Big)\,,\end{aligned} \tag{3.1a}$$

$$\begin{aligned}\phi^0(F_{j+1}^{-(p)} - F_j^{-(p)}, F_j^{-(p)} - F_{j-1}^{-(p)}) &= (F_{j+1}^{-(p)} - F_j^{-(p)})\phi^0\Big(\frac{F_j^{-(p)} - F_{j-1}^{-(p)}}{F_{j+1}^{-(p)} - F_j^{-(p)}}\Big)\\ &= (F_j^{-(p)} - F_{j-1}^{-(p)})\phi^0\Big(\frac{F_{j+1}^{-(p)} - F_j^{-(p)}}{F_j^{-(p)} - F_{j-1}^{-(p)}}\Big)\,,\end{aligned} \tag{3.1b}$$

in which $F^{\pm n(p)}$ denotes the p-th component of the positive (negative) flux $F^{\pm n}$. Consequently, the positive flux terms $F_{j+1/2}^{+n}$, $F_{j-1/2}^{+n}$ can be rewritten as

$$\begin{aligned}F_{j+1/2}^{+n} = &\quad F_j^{+n} + \frac{\triangle x}{2}\phi^0\Big(\frac{F_{j+1}^{+n} - F_j^{+n}}{\triangle x}, \frac{F_j^{+n} - F_{j-1}^{+n}}{\triangle x}\Big)\\ = &\quad F_j^{+n} + \frac{1}{2}\phi^0(F_{j+1}^{+n} - F_j^{+n}, F_j^{+n} - F_{j-1}^{+n})\\ = &\quad F_j^{+n} + \frac{1}{2}\Phi_{j+1/2}^{+}(F_j^{+n} - F_{j-1}^{+n})\,,\end{aligned} \tag{3.2}$$

where $\Phi_{j+1/2}^{+} = diag(\phi_{j+1/2}^{+(1)}, \phi_{j+1/2}^{+(2)}, ...\phi_{j+1/2}^{+(m)})$, and

$$\phi_{j+1/2}^{+(p)} = \phi^0\Big(\frac{F_{j+1}^{+n(p)} - F_j^{+n(p)}}{F_j^{+n(p)} - F_{j-1}^{+n(p)}}\Big)\,, \qquad \text{for} \qquad 1 \leq p \leq m\,. \tag{3.3}$$

Similarly, we can express $F^{+n}_{j-1/2}$ as

$$\begin{aligned} F^{+n}_{j-1/2} &= F^{+n}_{j-1} + \frac{1}{2}\phi^0(F^{+n}_j - F^{+n}_{j-1}, F^{+n}_{j-1} - F^{+n}_{j-2}) \\ &= F^{+n}_{j-1} + \frac{1}{2}\Phi^+_{j-1/2}(F^{+n}_j - F^{+n}_{j-1}), \end{aligned} \tag{3.4}$$

with the diagonal matrix $\Phi^+_{j-1/2} = diag(\phi^{+(1)}_{j-1/2}, \phi^{+(2)}_{j-1/2}, ...\phi^{+(m)}_{j-1/2})$, and

$$\phi^{+(p)}_{j-1/2} = \phi^0\Big(\frac{F^{+n}_{j-1} - F^{+n}_{j-2}}{F^{+n}_j - F^{+n}_{j-1}}\Big) \qquad \text{for} \qquad 1 \le p \le m. \tag{3.5}$$

It is obvious from the TVD property (1.22) that

$$0 \le \phi^{+(p)}_{j+1/2}, \phi^{+(p)}_{j-1/2} \le 2, \qquad \text{for} \qquad 1 \le p \le m. \tag{3.6}$$

Therefore, a combination of (3.2) and (3.4) gives

$$F^{+n}_{j+1/2} - F^{+n}_{j-1/2} = (F^{+n}_j - F^{+n}_{j-1}) + \frac{1}{2}\bar{\Phi}(F^{+n}_j - F^{+n}_{j-1}), \tag{3.7}$$

where $\bar{\Phi} = diag(\bar{\phi}^{+(1)}, \bar{\phi}^{+(2)}, \bar{\phi}^{+(3)})$, and

$$\bar{\phi}^{+(p)} = \phi^{+(p)}_{j+1/2} - \phi^{+(p)}_{j-1/2}, \qquad p = 1, 2, ..., m. \tag{3.8}$$

Clearly we have $-2 \le \bar{\phi}^{+(p)} \le 2$, $p = 1, 2, ..., m$. Now (3.7) becomes

$$F^{+n}_{j+1/2} - F^{+n}_{j-1/2} = (I + \frac{1}{2}\bar{\Phi}^+)(F^{+n}_j - F^{+n}_{j-1}) = \Psi^+(F^{+n}_j - F^{+n}_{j-1}), \tag{3.9}$$

where $\Psi^+ = (I + \frac{1}{2}\bar{\Phi}^+) = diag(\psi^{+(1)}, \psi^{+(2)}, \psi^{+(3)})$, and

$$\psi^{+(p)} = 1 + \frac{1}{2}\bar{\phi}^{+(p)}, \qquad p = 1, 2, ..., m. \tag{3.10}$$

Combining the results of (3.6), (3.8), (3.10) shows that $0 \le \Psi^+ \le 2I$.

A similar procedure can be applied to rewrite F^-, which gives

$$\begin{aligned} F^{-n}_{j+1/2} &= F^{-n}_{j+1} - \frac{1}{2}\phi^0(F^{-n}_{j+2} - F^{-n}_{j+1}, F^{-n}_{j+1} - F^{-n}_j) \\ &= F^{-n}_{j+1} - \frac{1}{2}\Phi^-_{j+1/2}(F^{-n}_{j+1} - F^{-n}_j), \end{aligned} \tag{3.11a}$$

$$\begin{aligned} F^{-n}_{j-1/2} &= F^{-n}_j - \frac{1}{2}\phi^0(F^{-n}_{j+1} - F^{-n}_j, F^{-n}_j - F^{-n}_{j-1}) \\ &= F^{-n}_j - \frac{1}{2}\Phi^-_{j-1/2}(F^{-n}_{j+1} - F^{-n}_j), \end{aligned} \tag{3.11b}$$

with

$$\begin{aligned} \Phi^-_{j+1/2} &= diag(\phi^{-(1)}_{j+1/2}, \phi^{-(2)}_{j+1/2}, ...\phi^{-(m)}_{j+1/2}), \\ \Phi^-_{j-1/2} &= diag(\phi^{-(1)}_{j-1/2}, \phi^{-(2)}_{j-1/2}, ...\phi^{-(m)}_{j-1/2}), \end{aligned} \tag{3.11c}$$

and for $p = 1, 2, ..., m$,

$$\phi^{-(p)}_{j+1/2} = \phi^0\Big(\frac{F^{-n(p)}_{j+2} - F^{-n(p)}_{j+1}}{F^{-n(p)}_{j+1} - F^{-n(p)}_j}\Big), \qquad \phi^{-(p)}_{j-1/2} = \phi^0\Big(\frac{F^{-n(p)}_j - F^{-n(p)}_{j-1}}{F^{-n(p)}_{j+1} - F^{-n(p)}_j}\Big). \tag{3.12}$$

Furthermore, a similar argument shows that $0 \leq \Phi^-_{j\pm1/2} \leq 2I$. As a result, we arrive at

$$\begin{aligned} F^{-n}_{j+1/2} - F^{-n}_{j-1/2} = & \quad F^{-n}_{j+1} - F^{-n}_{j} + \frac{1}{2}(\Phi^-_{j-1/2} - \Phi^-_{j+1/2})(F^{-n}_{j+1} - F^{-n}_{j}) \\ = & \quad \Psi^-(F^{-n}_{j+1} - F^{-n}_{j}), \end{aligned} \tag{3.13}$$

with the diagonal matrix

$$0 \leq \Psi^- = I + \frac{1}{2}(\Phi^-_{j-1/2} - \Phi^-_{j+1/2}) \leq 2I\,. \tag{3.14}$$

Finally, the combination of (1.26), (3.9), (3.14) leads to

$$\begin{aligned} U^{n+1}_j &= U^n_j - \frac{\triangle t}{\triangle x}(F^{+n}_{j+1/2} - F^{+n}_{j-1/2}) - \frac{\triangle t}{\triangle x}(F^{-n}_{j+1/2} - F^{-n}_{j-1/2}) \\ &= U^n_j - \frac{\triangle t}{\triangle x}\Psi^+(F^{+n}_{j} - F^{+n}_{j-1}) - \frac{\triangle t}{\triangle x}\Psi^-(F^{-n}_{j+1} - F^{-n}_{j}), \end{aligned} \tag{3.15}$$

where $0 \leq \Psi^\pm \leq 2I$.

The above derivation shows that the second-order FVS method (3.15) is also a stable perturbation (with the addition of $\Psi^\pm$ between 0 and $2I$) of the first-order scheme in the case of a nonlinear system. Its positivity comes from the splitting of the flux and the CFL-like conditions stated below.

4. **Positivity property of Steger-Warming splitting (SWS).** Applying the flux splitting F^+, F^- into (1.26), we get the second-order Steger-Warming splitting scheme:

$$U^{n+1}_j = U^n_j - \frac{\triangle t}{\triangle x}(F^{+n}_{j+1/2} - F^{+n}_{j-1/2}) - \frac{\triangle t}{\triangle x}(F^{-n}_{j+1/2} - F^{-n}_{j-1/2}), \tag{4.1a}$$

$$\begin{aligned} F^{+n}_{j+1/2} &= F^{+n}_{j} + \frac{\triangle x}{2}\phi^0\Big(\frac{F^{+n}_{j+1} - F^{+n}_{j}}{\triangle x}, \frac{F^{+n}_{j} - F^{+n}_{j-1}}{\triangle x}\Big), \\ F^{-n}_{j+1/2} &= F^{-n}_{j+1} + \frac{\triangle x}{2}\phi^0\Big(\frac{F^{-n}_{j+2} - F^{-n}_{j+1}}{\triangle x}, \frac{F^{-n}_{j+1} - F^{-n}_{j}}{\triangle x}\Big), \end{aligned} \tag{4.1b}$$

where F^+ and F^- are given by (2.4). Here ϕ^0 is the *minmod* limiter or *Van Leer* limiter, and $\phi^0(a,b)$ for each component.

THEOREM 4.1. *The second-order Steger-Warming splitting scheme (4.1a), (4.1b) is positive under a CFL-like condition*

$$\frac{\triangle t}{\triangle x}\max\Big(a|v^n_j| + b|c^n_j|\Big) \leq 1\,, \tag{4.2a}$$

where

$$a = \frac{\gamma^2 + 2\gamma - 1}{\gamma} + \frac{2\gamma - 2}{\gamma}, \qquad b = \frac{4}{\gamma} + \frac{\gamma^2 + 4\gamma - 3}{\gamma(\gamma - 1)}\,. \tag{4.2b}$$

Proof of Theorem 4.1 First, applying the argument in Section 3 gives

$$\begin{aligned} U^{n+1}_j &= U^n_j - \frac{\triangle t}{\triangle x}(F^{+n}_{j+1/2} - F^{+n}_{j-1/2}) - \frac{\triangle t}{\triangle x}(F^{-n}_{j+1/2} - F^{-n}_{j-1/2}) \\ &= U^n_j - \frac{\triangle t}{\triangle x}\Psi^+(F^{+n}_{j} - F^{+n}_{j-1}) - \frac{\triangle t}{\triangle x}\Psi^-(F^{-n}_{j+1} - F^{-n}_{j}), \end{aligned} \tag{4.3}$$

where $0 \le \Psi^{\pm} \le 2I$, as indicated in (3.15).

The rewritten forms of F^+ and F^- as given in Proposition 2.1 can be used to substitute into (4.3). For conciseness and without loss of generality, we only consider the three cases: (1) subsonic region with the assumption $0 \le v_i^n < c_i^n, i = j-1, j, j+1$; (2) supersonic region with the assumption $v_i^n \ge c_i^n, i = j-1, j, j+1$; (3) sonic transient region with the assumption $v_{j-1}^n \ge c_{j-1}^n$, and $0 \le v_i^n < c_i^n$ for $i = j, j+1$. The other subsonic, supersonic or transient regions can be dealt with in the same manner, due to the detailed reformulation of F^+, F^- given in Section 2.

4.1. **Subsonic region.** Substitution of (2.11)-(2.14) into (4.3) gives

$$\begin{aligned} U_j^{n+1} &= U_j^n - \lambda\Psi^+(A_j^{+n}U_j^n - A_{j-1}^{+n}U_{j-1}^n) - \lambda\Psi^-(A_{j+1}^{-n}U_{j+1}^n - A_j^{-n}U_j^n) \\ &= \lambda\Psi^+A_{j-1}^{+n}U_{j-1}^n + (-\lambda\Psi^-A_{j+1}^{-n})U_{j+1}^n + (I - \lambda\Psi^+A_j^{+n} + \lambda\Psi^-A_j^{-n})U_j^n \\ &= C_{-1}U_{j-1}^n + C_0U_j^n + C_1U_{j+1}^n\,, \end{aligned} \tag{4.4a}$$

where in the last step we denoted

$$C_{-1} = \lambda\Psi^+A_{j-1}^{+n}\,, \quad C_1 = -\lambda\Psi^-A_{j+1}^{-n}\,, \quad C_0 = I - \lambda\Psi^+A_j^{+n} + \lambda\Psi^-A_j^{-n}\,. \tag{4.4b}$$

Then the remaining work is to confirm that the matrices C_{-1}, C_0, C_1 satisfy the positivity property given by (1.36a), (1.36b), (1.36c).

It can be seen that

$$C_{-1} = \begin{pmatrix} c_{1,1}^{-1} & 0 & 0 \\ 0 & c_{2,2}^{-1} & c_{2,3}^{-1} \\ 0 & c_{3,2}^{-1} & c_{3,3}^{-1} \end{pmatrix}\,, \tag{4.5a}$$

$$\begin{aligned} &c_{1,1}^{-1} = \lambda\psi^{+(1)}(\frac{2\gamma-1}{2\gamma}v_{j-1}^n + \frac{1}{2\gamma}c_{j-1}^n)\,, \\ &c_{2,2}^{-1} = \lambda\psi^{+(2)}(\frac{5\gamma-2-\gamma^2}{4\gamma}v_{j-1}^n + \frac{1}{\gamma}c_{j-1}^n)\,, \quad c_{2,3}^{-1} = \lambda\psi^{+(2)}\frac{\gamma-1}{2}\,, \\ &c_{3,2}^{-1} = \lambda\psi^{+(3)}\frac{3-\gamma}{18}(v_{j-1}^n - c_{j-1}^n)^2\,, \quad c_{3,3}^{-1} = \lambda\psi^{+(3)}(a_{3,3}^+)_{j-1}^n\,, \end{aligned} \tag{4.5b}$$

which is symmetrizable and has only non-negative eigenvalues, by the estimates (2.14), (2.15) and Lemma 2.1.

The matrix C_1 has a similar form to that of C_{-1}

$$C_1 = \begin{pmatrix} c_{1,1}^{1} & 0 & 0 \\ 0 & c_{2,2}^{1} & c_{2,3}^{1} \\ 0 & c_{3,2}^{1} & c_{3,3}^{1} \end{pmatrix}\,, \tag{4.6a}$$

$$\begin{aligned} &c_{1,1}^{1} = \lambda\psi^{-(1)}\frac{1}{2\gamma}(c_{j+1}^n - v_{j+1}^n)\,, \\ &c_{2,2}^{1} = \lambda\psi^{-(2)}\frac{1}{4\gamma}\left[4c_{j+1}^n - (2-\gamma^2+\gamma)v_{j+1}^n\right]\,, \quad c_{2,3}^{1} = -\lambda\psi^{-(2)}\frac{\gamma-1}{2}\,, \\ &c_{3,2}^{1} = -\lambda\psi^{-(3)}\frac{3-\gamma}{18}(v_{j+1}^n - c_{j+1}^n)^2\,, \quad c_{3,3}^{1} = \lambda\psi^{-(3)}(a_{3,3}^-)_{j+1}^n\,, \end{aligned} \tag{4.6b}$$

Again, the estimates (2.14), (2.15) and Lemma 2.1 indicate that C_1 is symmetrizable and has only non-negative eigenvalues.

The matrix C_0 can be represented as

$$C_0 = \begin{pmatrix} c^0_{1,1} & 0 & 0 \\ 0 & c^0_{2,2} & c^0_{2,3} \\ 0 & c^0_{3,2} & c^0_{3,3} \end{pmatrix}, \tag{4.7a}$$

where

$$\begin{aligned} c^0_{1,1} &= 1 - \lambda\psi^{+(1)}\Big(\frac{2\gamma-1}{2\gamma}v^n_j + \frac{1}{2\gamma}c^n_j\Big) - \lambda\psi^{-(1)}\frac{1}{2\gamma}(c^n_j - v^n_j)\,, \\ c^0_{2,2} &= 1 - \lambda\psi^{+(2)}\Big(\frac{5\gamma-2-\gamma^2}{4\gamma}v^n_j + \frac{1}{\gamma}c^n_j\Big) - \lambda\psi^{-(2)}\frac{1}{4\gamma}\Big(4c^n_j - (2-\gamma^2+\gamma)v^n_j\Big)\,, \\ c^0_{2,3} &= \lambda\frac{\gamma-1}{2}\Big(-\psi^{+(2)} + \psi^{-(2)}\Big), \quad c^0_{3,2} = \lambda\frac{3-\gamma}{18}(v^n_j - c^n_j)^2\Big(-\psi^{+(3)} + \psi^{-(3)}\Big)\,, \\ c^0_{3,3} &= 1 - \lambda\psi^{+(3)}(a^+_{3,3})^n_j - \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,. \end{aligned} \tag{4.7b}$$

The three diagonal elements of C_0: $c^0_{1,1}$, $c^0_{2,2}$, $c^0_{3,3}$ can be controlled by the following argument with the usage of the preliminary estimate (2.17)

$$\begin{aligned} c^0_{1,1} &\geq 0\,, \qquad \text{if} \qquad \max_{j,n}\lambda\Big(\frac{2(\gamma-1)}{\gamma}v^n_j + \frac{2}{\gamma}c^n_j\Big) \leq 1\,, \\ c^0_{2,2} &\geq 0\,, \qquad \text{if} \qquad \max_{j,n}\lambda\Big(\frac{2(\gamma-1)}{\gamma}v^n_j + \frac{4}{\gamma}c^n_j\Big) \leq 1\,, \\ c^0_{3,3} &\geq 0\,, \qquad \text{if} \qquad \max_{j,n}\lambda\Big(\frac{\gamma^2+2\gamma-1}{\gamma}v^n_j + \frac{\gamma^2+4\gamma-3}{\gamma(\gamma-1)}c^n_j\Big) \leq 1\,. \end{aligned} \tag{4.8}$$

In addition, we note that both $\psi^{-(3)} - \psi^{+(3)}$ and $\psi^{-(2)} - \psi^{+(2)}$ are $O(\triangle x)$ so that $c^0_{2,3}$, $c^0_{3,2}$ are $O(\triangle t)$, if some suitable continuity assumption for the numerical solution is satisfied. Then we conclude that C_0 differs $O(\triangle t)$ from a diagonal positive matrix under such condition. In the general case, we can still get the positivity and symmetrizable property of C_0 by adjusting the coefficients in $c^0_{3,2}$. The technical detail is omitted.

Next we verify (1.36b). The direct calculation shows that

$$C_{-1} + C_0 + C_1 = \begin{pmatrix} s_{1,1} & 0 & 0 \\ 0 & s_{2,2} & 0 \\ 0 & s_{3,2} & s_{3,3} \end{pmatrix}, \tag{4.9a}$$

in which the diagonal elements have the following form

$$
\begin{aligned}
s_{1,1} &= 1 + \lambda\psi^{+(1)}\Big(\frac{F_{j-1}^{+n(1)}}{\rho_{j-1}^n} - \frac{F_j^{+n(1)}}{\rho_j^n}\Big) + \lambda\psi^{-(1)}\Big(\frac{F_j^{-n(1)}}{\rho_j^n} - \frac{F_{j+1}^{-n(1)}}{\rho_{j+1}^n}\Big)\,,\\
s_{2,2} &= 1 + \lambda\psi^{+(2)}\Big((a_{2,2}^+)_{j-1}^n - (a_{2,2}^+)_j^n\Big) + \lambda\psi^{-(2)}\Big((a_{2,2}^-)_j^n - (a_{2,2}^-)_{j+1}^n\Big)\,,\\
s_{3,2} &= \lambda\frac{3-\gamma}{18}\psi^{+(3)}\Big((v_{j-1}^n - c_{j-1}^n)^2 - (v_j^n - c_j^n)^2\Big)\,,\\
&\quad +\lambda\frac{3-\gamma}{18}\psi^{-(3)}\Big((v_j^n - c_j^n)^2 - (v_{j+1}^n - c_{j+1}^n)^2\Big)\,,\\
s_{3,3} &= 1 + \lambda\psi^{+(3)}\Big(\frac{F_{j-1}^{+n(3)}}{E_{j-1}^n} - \frac{F_j^{+n(3)}}{E_j^n}\Big) + \lambda\psi^{-(3)}\Big(\frac{F_j^{-n(3)}}{E_j^n} - \frac{F_{j+1}^{-n(3)}}{E_{j+1}^n}\Big)\,.
\end{aligned}
\tag{4.9b}
$$

Since F^+, F^- are smooth functions of the fluid variables ρ, m, E, it can be concluded that $C_{-1} + C_0 + C_1 = I + O(\triangle t)$ if the numerical solution is Lipschitz continuous at time t^n, thus (1.36b) is satisfied. In addition, $C_k = 0$ except for $K = -1, 0, 1$, which gives that the scheme with the flux splitting (3.1) is positive in the subsonic region.

4.2. **Supersonic case.** Substitution of (2.18), (2.19) into (4.3) gives the formula (4.4) with the same notation of C_{-1}, C_0, C_1 except that the form of $F^\pm, A^\pm$ has been changed. Since $F^{-n} = 0, A^{-n} = 0$, we actually have

$$U_j^{n+1} = C_{-1}U_{j-1}^n + C_0 U_j^n\,, \tag{4.10a}$$

where

$$C_{-1} = \lambda\Psi^+ A_{j-1}^{+n}\,, \qquad C_0 = I - \lambda\Psi^- A_j^{+n}\,. \tag{4.10b}$$

In more detail,

$$C_{-1} = \begin{pmatrix} \lambda\psi^{+(1)}v_{j-1}^n & 0 & 0\\ 0 & \lambda\psi^{+(2)}\frac{F_{j-1}^{+n(2)}}{\rho_{j-1}^n v_{j-1}^n} & 0\\ 0 & 0 & \lambda\psi^{+(3)}(a_{3,3}^+)_{j-1}^n \end{pmatrix}, \tag{4.11}$$

$$C_0 = \begin{pmatrix} 1-\lambda\psi^{+(1)}v_j^n & 0 & 0\\ 0 & 1-\lambda\psi^{+(2)}\frac{F_j^{+n(2)}}{\rho_j^n v_j^n} & 0\\ 0 & 0 & 1-\lambda\psi^{+(3)}(a_{3,3}^-)_j^n \end{pmatrix}. \tag{4.12}$$

The estimate (2.20) and the fact that $0 \le \Psi^\pm \le 2I$ ensure the positivity of C_{-1} and C_0 if

$$\lambda \max_{j,n}\Big(2v_j^n + \frac{4}{\gamma}c_j^n\Big) \le 1\,, \tag{4.14}$$

which implies that the CFL-like assumption (4.2) is a sufficient condition for (1.36a).

For the verification of (1.36b), a careful computation shows that

$$C_{-1} + C_0 = \begin{pmatrix} s_{1,1} & 0 & 0\\ 0 & s_{2,2} & 0\\ 0 & 0 & s_{3,3} \end{pmatrix}, \tag{4.15a}$$

with the diagonal elements

$$
\begin{aligned}
s_{1,1} &= 1+\lambda\psi^{+(1)}\Big(\frac{F^{+n(1)}_{j-1}}{\rho^n_{j-1}}-\frac{F^{+n(1)}_{j}}{\rho^n_{j}}\Big)\,,\\
s_{2,2} &= 1+\lambda\psi^{+(2)}\Big(\frac{F^{+n(2)}_{j-1}}{\rho^n_{j-1}v^n_{j-1}}-\frac{F^{+n(2)}_{j}}{\rho^n_{j}v^n_j}\Big)\,,\\
s_{3,3} &= 1+\lambda\psi^{+(3)}\Big(\frac{F^{+n(3)}_{j-1}}{E^n_{j-1}}-\frac{F^{+n(3)}_{j}}{E^n_{j}}\Big)\,.
\end{aligned}
\tag{4.15b}
$$

Therefore, $C_{-1}+C_0 = I+O(\triangle t)$ if U^n_j is Lipschitz continuous with respect to x. Thus the condition (1.36b) is guaranteed.

The condition (1.36c) is obvious since $C_K = 0$ except for $K=-1,0$. Then we finish the proof of the positivity property in the supersonic region.

REMARK 3. *We observe that both C_{-1} and C_0 are diagonal, hence symmetric positive, in the supersonic region.*

4.3. **Sonic transient region.** We consider the case such that $v^n_{j-1}\ge c^n_{j-1}$, and $0\le v^n_i < c^n_i$ for $i=j,j+1$. The previous argument shows that

$$
\begin{aligned}
&\frac{F^{+n(1)}_{j-1}}{\rho^n_{j+1}} = v^n_{j-1}\,,\quad \frac{F^{+n(1)}_{j}}{\rho^n_{j}} = \frac{2\gamma-1}{2\gamma}v^n_j+\frac{1}{2\gamma}c^n_j\,,\\
&\frac{F^{+n(1)}_{j+1}}{\rho^n_{j+1}} = \frac{2\gamma-1}{2\gamma}v^n_{j+1}+\frac{1}{2\gamma}c^n_{j+1}\,,\quad 0\le\frac{F^{+n(2)}_{j-1}}{m^n_{j-1}}\le v^n_{j-1}+\frac{1}{\gamma}c^n_{j-1}\,,\\
&F^{+n(2)}_{j} = \Big(\frac{5\gamma-2-\gamma^2}{4\gamma}v^n_j+\frac{1}{\gamma}c^n_j\Big)m^n_j+\frac{\gamma-1}{2}E^n_j\,,\\
&F^{+n(2)}_{j+1} = \Big(\frac{5\gamma-2-\gamma^2}{4\gamma}v^n_j+\frac{1}{\gamma}c^n_j\Big)m^n_{j+1}+\frac{\gamma-1}{2}E^n_{j+1}\,,\\
&F^{+n(3)}_{j} = \frac{3-\gamma}{18}(v^n_j-c^n_j)^2+(a^+_{3,3})^n_jE^n_j\,,\\
&F^{+n(3)}_{j+1} = \frac{3-\gamma}{18}(v^n_{j+1}-c^n_{j+1})^2+(a^+_{3,3})^n_{j+1}E^n_{j+1}\,,\\
&0\le\frac{F^{+n(3)}_{j-1}}{E^n_{j-1}}\le v^n_{j-1}+\frac{2}{\gamma}c^n_{j-1}\,,\quad \frac{3-\gamma}{2\gamma}c^n_j\le(a^+_{3,3})^n_j\le\frac{\gamma^2+2\gamma-1}{2\gamma}v^n_j+\frac{\gamma+3}{2\gamma}c^n_j\,,\\
&\frac{3-\gamma}{2\gamma}c^n_{j+1}\le(a^+_{3,3})^n_{j+1}\le\frac{\gamma^2+2\gamma-1}{2\gamma}v^n_{j+1}+\frac{\gamma+3}{2\gamma}c^n_{j+1}\,,
\end{aligned}
\tag{4.16a}
$$

$$
\begin{aligned}
&F_j^{-n(1)} = -\frac{1}{2\gamma}(c_j^n - v_j^n)\rho_j^n\,, \qquad F_{j+1}^{-n(1)} = -\frac{1}{2\gamma}(c_{j+1}^n - v_{j+1}^n)\rho_{j+1}^n\,,\\
&F_j^{-n(2)} = -\frac{1}{4\gamma}\left[4c_j^n - (2-\gamma^2+\gamma)v_j^n\right] m_j^n + \frac{\gamma-1}{2}E_j^n\,,\\
&F_{j+1}^{-n(2)} = -\frac{1}{4\gamma}\left[4c_{j+1}^n - (2-\gamma^2+\gamma)v_{j+1}^n\right] m_{j+1}^n + \frac{\gamma-1}{2}E_{j+1}^n\,,\\
&F_j^{-n(3)} = \frac{3-\gamma}{18}(v_j^n - c_j^n)^2 + (a_{3,3}^-)_j^n E_j^n\,,\\
&F_{j+1}^{-n(3)} = \frac{3-\gamma}{18}(v_{j+1}^n - c_{j+1}^n)^2 + (a_{3,3}^-)_{j+1}^n E_{j+1}^n\,,\\
&\frac{1}{\gamma-1}(v_j^n - c_j^n) \le (a_{3,3}^-)_j^n \le \frac{3-\gamma}{\gamma+1}(v_j^n - c_j^n) < 0\,,\\
&\frac{1}{\gamma-1}(v_{j+1}^n - c_{j+1}^n) \le (a_{3,3}^-)_{j+1}^n \le \frac{3-\gamma}{\gamma+1}(v_{j+1}^n - c_{j+1}^n) < 0\,,\\
&F_{j-1}^{-n} = 0\,.
\end{aligned} \tag{4.16b}
$$

The formula for U_j^{n+1} in (4.4a) involves F_{j-1}^{+n}, F_j^{+n}, F_j^{-n} and F_{j+1}^{-n}. At the supersonic point $j-1$ and the subsonic point $j+1$, F^+ and F^- can be expressed as

$$
F_{j-1}^{+n} = A_{j-1}^{+n}U_{j-1}^n\,, \qquad F_{j+1}^{-n} = A_{j+1}^{-n}U_{j+1}^n\,, \tag{4.17}
$$

in which A_{j-1}^{+n}, A_{j+1}^{+n} have the same form as in (2.19), (2.13), respectively, so that the estimates in (4.16) can be applied. At the transient point j, we assume that $1-\epsilon \le \frac{v_j^n}{c_j^n} \le 1$ if the solution is Lipschitz continuous. To facilitate the positivity argument below, we express F_j^{-n} in a similar form as that of F_{j+1}^{-n}, F_j^{+n} in a similar form as that of F_{j-1}^{+n}, i.e.,

$$
F_j^{+n} = A_j^{+n}U_j^n\,, \qquad F_j^{-n} = A_j^{-n}U_j^n\,, \tag{4.18}
$$

in which A_{+j}^n, A_{-j}^n take the form of

$$
\begin{aligned}
A_{+j}^n &= \begin{pmatrix} (a_{1,1}^+)_j^n & & 0 \\ 0 & (a_{2,2}^+)_j^n & 0 \\ 0 & 0 & (a_{3,3}^+)_j^n \end{pmatrix},\\
A_{-j}^n &= \begin{pmatrix} (a_{1,1}^-)_j^n & 0 & 0 \\ 0 & (a_{2,2}^-)_j^n & \frac{\gamma-1}{2} \\ 0 & \frac{3-\gamma}{18}(v_j^n - c_j^n)^2 & (a_{3,3}^-)_j^n \end{pmatrix},
\end{aligned} \tag{4.19a}
$$

$$
\begin{aligned}
&(a_{1,1}^+)_j^n = \frac{2\gamma-1}{2\gamma}v_j^n + \frac{1}{2\gamma}c_j^n\,, \quad (a_{2,2}^+)_j^n = \frac{F_j^{+n(2)}}{m_j^n}\,, \quad (a_{3,3}^+)_j^n = \frac{F_j^{+n(3)}}{E_j^n}\,,\\
&(a_{1,1}^-)_j^n = -\frac{1}{2\gamma}(c_j^n - v_j^n)\,, \quad (a_{2,2}^-)_j^n = -\frac{1}{4\gamma}\left[4c_j^n - (2-\gamma^2+\gamma)v_j^n\right]\,,\\
&\frac{1}{\gamma-1}(v_j^n - c_j^n) \le (a_{3,3}^-)_j^n \le \frac{3-\gamma}{\gamma+1}(v_j^n - c_j^n) < 0\,.
\end{aligned} \tag{4.19b}
$$

The estimates in (4.16) are valid to control the above terms except for $(a^+_{2,2})^n_j$. The assumption of Lipschitz continuity of the solution shows that

$$\frac{F^{+n(2)}_j}{m^n_j} = \frac{F^{+n(2)}_{j-1}}{m^n_{j-1}} + O(h)\,, \quad \text{which leads to} \quad 0 \le \frac{F^{+n(2)}_j}{m^n_j} \le v^n_j + \frac{1}{\gamma}c^n_j + O(h)\,. \tag{4.20}$$

Similar to the subsonic and supersonic cases, we have

$$U^{n+1}_j = C_{-1}U^n_{j-1} + C_0 U^n_j + C_1 U^n_{j+1}\,, \tag{4.21}$$

if we denote

$$C_{-1} = \lambda\Psi^+ A^{+n}_{j-1)}\,, \quad C_1 = -\lambda\Psi^- A^{-n}_{j+1}\,, \quad C_0 = I - \lambda\Psi^+ A^{+n}_j + \lambda\Psi^- A^{-n}_j\,. \tag{4.22}$$

In more detail, the matrices C_{-1}, C_0, C_1 take the following forms

$$C_{-1} = \begin{pmatrix} \lambda\psi^{+(1)} v^n_{j-1} & 0 & 0 \\ 0 & \lambda\psi^{+(2)} \frac{F^{+n(2)}_{j-1}}{\rho^n_{j-1} v^n_{j-1}} & 0 \\ 0 & 0 & \lambda\psi^{+(3)} (a^+_{3,3})^n_{j-1} \end{pmatrix}, \tag{4.23}$$

$$C_1 = \begin{pmatrix} c^1_{1,1} & 0 & 0 \\ 0 & c^1_{2,2} & c^1_{2,3} \\ 0 & c^1_{3,2} & c^1_{3,3} \end{pmatrix}, \tag{4.24a}$$

$$\begin{aligned}
c^1_{1,1} &= \lambda\psi^{-(1)} \frac{1}{2\gamma}(c^n_{j+1} - v^n_{j+1})\,, \\
c^1_{2,2} &= \lambda\psi^{-(2)} \frac{1}{4\gamma}\left[4c^n_{j+1} - (2 - \gamma^2 + \gamma)v^n_{j+1}\right]\,, \quad c^1_{2,3} = -\lambda\psi^{-(2)}\frac{\gamma - 1}{2}\,, \\
c^1_{3,2} &= -\lambda\psi^{-(3)}\frac{3-\gamma}{18}(v^n_{j+1} - c^n_{j+1})^2\,, \quad c^1_{3,3} = \lambda\psi^{-(3)}(a^-_{3,3})^n_{j+1}\,,
\end{aligned} \tag{4.24b}$$

$$C_0 = \begin{pmatrix} c^0_{1,1} & 0 & 0 \\ 0 & c^0_{2,2} & c^0_{2,3} \\ 0 & c^0_{3,2} & c^0_{3,3} \end{pmatrix}, \tag{4.25a}$$

with

$$\begin{aligned}
c^0_{1,1} &= 1 - \lambda\psi^{+(1)}\Big(\frac{2\gamma - 1}{2\gamma}v^n_j + \frac{1}{2\gamma}c^n_j\Big) - \lambda\psi^{-(1)}\frac{1}{2\gamma}(c^n_j - v^n_j)\,, \\
c^0_{2,2} &= 1 - \lambda\psi^{+(2)}\frac{F^{+n(2)}_j}{m^n_j} - \lambda\psi^{-(2)}\frac{1}{4\gamma}\left[4c^n_j - (2 - \gamma^2 + \gamma)v^n_j\right]\,, \\
c^0_{2,3} &= \lambda\psi^{-(2)}\frac{\gamma-1}{2}\,, \quad c^0_{3,2} = \lambda\psi^{-(3)}\frac{3-\gamma}{18}(v^n_j - c^n_j)^2\,, \\
c^0_{3,3} &= 1 - \lambda\psi^{+(3)}\frac{F^{+n(3)}_j}{E^n_j} - \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,.
\end{aligned} \tag{4.25b}$$

Using a similar argument as in the subsonic and supersonic regions, we conclude that C_1, C_0, C_1 are symmetric or symmetrizable and keep non-negative eigenvalues, and $C_{-1} + C_0 + C_1 = I + O(\triangle t)$ if U^n_j is Lipschitz continuous. Thus, the scheme (4.1) with Steger-Warming flux splitting is also positive in the transient region provided the CFL-like condition (4.2) is satisfied. Theorem 4.1 is proven. □

4.4. **Positivity of density and total energy.** In this subsection we prove the second order Steger-Warming splitting method preserves positivity of hydrodynamic variables, including density and total energy. To achieve this, we need to formulate F^+, F^- so that the corresponding matrices A^+, A^- are diagonal with respect to the first and third components. At a supersonic point, the form of A^+, A^- is still the same as (2.19) and the estimate (2.20) is valid. At a subsonic point $0 \le v < c$, the corresponding matrices can take another form

$$A^+ = \begin{pmatrix} a^+_{1,1} & 0 & 0 \\ 0 & a^+_{2,2} & \frac{\gamma-1}{2} \\ 0 & 0 & a^+_{3,3} \end{pmatrix}, \qquad A^- = \begin{pmatrix} a^-_{1,1} & 0 & 0 \\ 0 & a^-_{2,2} & \frac{\gamma-1}{2} \\ 0 & 0 & a^-_{3,3} \end{pmatrix}, \tag{4.26}$$

where $a^\pm_{1,1}$, $a^\pm_{2,2}$ are the same as in (2.14). The third diagonal element $a^\pm_{3,3}$ reads

$$\begin{aligned} a^+_{3,3} &= \frac{F^{+(3)}}{E} = \frac{\frac{2\gamma-1}{4\gamma}\rho v^3 + \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 + \frac{1}{2\gamma(\gamma-1)}\rho c^3}{E}, \\ a^-_{3,3} &= \frac{F^{-(3)}}{E} = \frac{\frac{1}{4\gamma}\rho v^3 - \frac{3}{4\gamma}\rho v^2 c + \frac{1}{2(\gamma-1)}\rho v c^2 - \frac{1}{2\gamma(\gamma-1)}\rho c^3}{E}, \end{aligned} \tag{4.27}$$

and the following estimate can be derived

$$0 \le a^+_{3,3} = \frac{F^{+(3)}}{E} \le \frac{\gamma^2+2\gamma-1}{2\gamma} v + \frac{\gamma+3}{2\gamma} c, \qquad -\frac{1}{2} c \le a^-_{3,3} = \frac{F^{-(3)}}{E} \le 0\,. \tag{4.28}$$

Then the numerical scheme can be written in the form of either (4.4), (4.10) or (4.21)-(4.22), at subsonic, supersonic or transient region, respectively. An important fact we should mention is that in all three cases, subsonic, supersonic or transient, the three matrices C_{-1}, C_0 and C_1 are diagonal with respect to the first and third components. As a result, ρ_j^{n+1} and E_j^{n+1} can be written as positive combinations of ρ_{j-1}^n, ρ_j^n, ρ_{j+1}^n, and E_{j-1}^n, E_j^n, E_{j+1}^n, respectively, i.e.

$$\rho_j^{n+1} = \lambda_{-1,1}\rho_{j-1}^n + \lambda_{0,1}\rho_j^n + \lambda_{1,1}\rho_{j+1}^n\,, \tag{4.29}$$

$$E_j^{n+1} = \lambda_{-1,3}E_{j-1}^n + \lambda_{0,3}E_j^n + \lambda_{1,3}E_{j+1}^n\,, \tag{4.30}$$

where

$$\begin{aligned} &\lambda_{-1,1} = \lambda\psi^{+(1)}(a^+_{1,1})^n_{j-1}\,, \quad \lambda_{1,1} = -\lambda\psi^{-(1)}(a^-_{1,1})^n_{j+1}\,, \\ &\lambda_{0,1} = 1 - \lambda\psi^{+(1)}(a^+_{1,1})^n_j + \lambda\psi^{-(1)}(a^-_{1,1})^n_j\,, \quad \lambda_{-1,3} = \lambda\psi^{+(3)}(a^+_{3,3})^n_{j-1}\,, \\ &\lambda_{1,3} = -\lambda\psi^{-(3)}(a^-_{3,3})^n_{j+1}\,, \quad \lambda_{0,3} = 1 - \lambda\psi^{+(3)}(a^+_{3,3})^n_j + \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,. \end{aligned} \tag{4.31}$$

It can be observed that $\lambda_{i,1}, i = -1, 0, 1$ and $\lambda_{i,3}, i = -1, 0, 1$ are non-negative provided the CFL-like condition

$$\frac{\triangle t}{\triangle x} \max\Big(a|v_j^n| + b|c_j^n|\Big) \le 1\,, \tag{4.32a}$$

$$a = \max\Big(\frac{2\gamma-1}{\gamma}, \frac{\gamma^2+2\gamma-1}{\gamma}, 2\Big)\,, \qquad b = \max\Big(\frac{4}{\gamma}, 2 + \frac{3}{\gamma}\Big)\,, \tag{4.32b}$$

with the usage of the estimates (2.14), (2.20) and (4.28). Then the positivity of ρ_j^n and E_j^n implies the positivity of ρ_j^{n+1} and E_j^{n+1}. In other words, the numerical scheme preserves the positivity of density and total energy. However, preservation of positivity for the pressure is not so obvious. It is an inferior result to [28], [5], in which the positivity of both the density and the internal energy was established.

Remark 4. *We note that a variety of formulations of F^+ and F^- are possible because of the nonlinearity of Euler flux. The matrices A^+ and A^- as shown in (4.26) in the subsonic case are not symmetrizable by Lemma 2.1 so that they cannot be used in the proof of Proposition 4.1. However, such a formulation is very useful in the proof of the positivity preserving property for the density and energy variables since the matrices are diagonal with respect to the first and third components, and the diagonal elements are also bounded by fluid velocity and sound speeds as shown in (4.27).*

5. **Positivity of Van Leer Splitting (VLS).**

The main theorem in this section is stated below.

Theorem 5.1. *The second-order scheme (4.1a),(4.1b) using Van Leer splitting (2.5) is positive under a CFL-like condition*

$$\frac{\triangle t}{\triangle x}\max\Big(a|v_j^n| + b|c_j^n|\Big) \le 1\,, \tag{5.1a}$$

where

$$a = max\Big(1, \frac{\gamma-1}{\gamma} + \frac{\gamma(2\gamma^2+4\gamma-6)}{\gamma+1}\Big)\,, \quad b = \frac{4}{\gamma} + \frac{8\gamma}{\gamma+1} + \frac{2(3-\gamma)^2}{3}\,. \tag{5.1b}$$

Proof of Theorem 5.1 A similar analysis as in Section 4 can be carried out, using the same notation, except for the difference of F^+, F^-. The solution U_j^{n+1} is expressed as

$$U_j^{n+1} = U_j^n - \frac{\triangle t}{\triangle x}\Psi^+(F_j^{+n} - F_{j-1}^{+n}) - \frac{\triangle t}{\triangle x}\Psi^-(F_{j+1}^{-n} - F_j^{-n})\,, \tag{5.2}$$

with $0 \le \Psi^\pm \le 2I$. Denote

$$C_{-1} = \lambda\Psi^+ A_{j-1}^{+n}\,, \quad C_1 = -\lambda\Psi^- A_{j+1}^{-n}\,, \quad C_0 = I - \lambda\Psi^+ A_j^{+n} - \lambda\Psi^- A_j^{-n}\,, \tag{5.3}$$

then we have

$$U_j^{n+1} = C_{-1}U_{j-1}^n + C_0 U_j^n + C_1 U_{j+1}^n\,. \tag{5.4}$$

The positivity of the second order scheme is based on the rewritten forms of F^+, F^- as established in Proposition 2.1. Similar to the presentation in Section 4, we consider the following three cases: (1) subsonic region with $0 \le v_i^n < c_i^n, i = j-1, j, j+1$; (2) supersonic region with $v_i^n \ge c_i^n, i = j-1, j, j+1$; (3) sonic transient region with $v_{j-1}^n \ge c_{j-1}^n$, and $0 \le v_i^n < c_i^n$ for $i = j, j+1$. The other subsonic, supersonic or transient regions can be treated in the same way.

5.1. **Subsonic region.**

In this case, the matrices C_{-1}, C_0, C_1 read

$$C_{-1} = \begin{pmatrix} \lambda\psi^{+(1)}(a_{1,1}^+)_{j_1}^n & 0 & 0 \\ 0 & \lambda\psi^{+(2)}(a_{2,2}^+)_{j-1}^n & \lambda\psi^{+(2)}\frac{\gamma-1}{2} \\ 0 & \lambda\psi^{+(3)}\frac{(3-\gamma)^2}{72}(v_{j-1}^n - c_{j-1}^n)^2 & \lambda\psi^{+(3)}(a_{3,3}^+)_{j-1}^n \end{pmatrix}, \tag{5.5}$$

$$C_1 = \begin{pmatrix} \lambda\psi^{-(1)}(a_{1,1}^-)_{j+1}^n & 0 & 0 \\ 0 & \lambda\psi^{-(2)}(a_{2,2}^-)_{j+1}^n & -\lambda\psi^{-(2)}\frac{\gamma-1}{2} \\ 0 & \lambda\psi^{-(3)}\frac{(3-\gamma)^2}{72}(v_{j+1}^n - c_{j+1}^n)^2 & \lambda\psi^{-(3)}(a_{3,3}^-)_{j+1}^n \end{pmatrix}, \tag{5.6}$$

$$C_0 = \begin{pmatrix} c^0_{1,1} & 0 & 0 \\ 0 & c^0_{2,2} & c^0_{2,3} \\ 0 & c^0_{3,2} & c^0_{3,3} \end{pmatrix}, \tag{5.7a}$$

with

$$\begin{aligned} c^0_{1,1} &= 1 - \lambda\psi^{+(1)}(a^+_{1,1})^n_j - \lambda\psi^{-(1)}(a^-_{1,1})^n_j\,, \\ c^0_{2,2} &= 1 - \lambda\psi^{+(2)}(a^+_{2,2})^n_j - \lambda\psi^{-(2)}(a^-_{2,2})^n_j\,, \\ c^0_{2,3} &= -\lambda\psi^{+(2)}\frac{\gamma-1}{2} + \lambda\psi^{-(2)}\frac{\gamma-1}{2}\,, \\ c^0_{3,3} &= 1 - \lambda\psi^{+(3)}(a^+_{3,3})^n_j - \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,, \\ c^0_{3,2} &= -\lambda\frac{(3-\gamma)^2)}{72}(v^n_j - c^n_j)^2(-\psi^{+(3)} + \psi^{-(3)})\,, \end{aligned} \tag{5.7b}$$

in which $a^{\pm}_{i,i}$, $i = 1,2,3$ are given by (2.25b).

Using the same argument as in Section 4 and applying the estimate (2.26), we conclude that

$$\begin{aligned} &C_{-1}\,, C_0\,, C_1 \quad \text{are diagonal or symmetrizable and} \quad C_K \geq 0\,, \\ &\text{up to } O(\triangle t) \text{ difference}\,, \end{aligned} \tag{5.8}$$

$$C_{-1} + C_0 + C_1 = I + O(\triangle t)\,, \qquad \text{if} \qquad U^n_j \text{ is Lipschitz continuous}\,, \tag{5.9}$$

under the following conditions

$$\begin{aligned} &\max_{j,n} \lambda\Big(\frac{1}{2}|v^n_j| + \frac{3}{2}c^n_j\Big) \leq 1\,, \qquad \max_{j,n} \lambda\Big(\frac{\gamma-1}{\gamma}|v^n_j| + \frac{4}{\gamma}c^n_j\Big) \leq 1\,, \\ &\max_{j,n} \lambda\left[\frac{\gamma(2\gamma^2+4\gamma-6)}{\gamma+1}|v^n_j| + \Big(\frac{8\gamma}{\gamma+1} + \frac{2(3-\gamma)^2}{3}\Big)c^n_j\right] \leq 1\,, \qquad , \end{aligned} \tag{5.10}$$

which can be assured by the CFL-like assumption (5.1b). Thus the positivity is proven.

5.2. **Supersonic region.** In this case, the splitting flux F^+, F^- have exactly the same form as in the Steger-Warming splitting scheme. Therefore, the argument in Section 4 can be applied here. We omit the detail.

5.3. **Sonic transient region.** Without loss of generality, we assume $v^n_{j-1} \geq c^n_{j-1}$, and $0 \leq v^n_i < c^n_i$ for $i = j, j+1$. Using a similar argument as in the analysis of this case in Section 4, we can represent C_{-1}, C_0 and C_1 as the following

$$C_{-1} = \begin{pmatrix} \lambda\psi^{+(1)}v^n_{j-1} & 0 & 0 \\ 0 & \lambda\psi^{+(2)}(a^+_{2,2})^n_{j-1} & 0 \\ 0 & 0 & \lambda\psi^{+(3)}(a^+_{3,3})^n_{j-1} \end{pmatrix}, \tag{5.11}$$

$$C_1 = \begin{pmatrix} \lambda\psi^{-(1)}(a^-_{1,1})^n_{j+1} & 0 & 0 \\ 0 & \lambda\psi^{-(2)}(a^-_{2,2})^n_{j+1} & -\lambda\psi^{-(2)}\frac{\gamma-1}{2} \\ 0 & -\lambda\psi^{-(3)}\frac{(3-\gamma)^2}{72}(v^n_{j+1} - c^n_{j+1})^2 & \lambda\psi^{-(3)}(a^-_{3,3})^n_{j+1} \end{pmatrix}, \tag{5.12}$$

$$C_0 = \begin{pmatrix} c^0_{1,1} & 0 & 0 \\ 0 & c^0_{2,2} & c^0_{2,3} \\ 0 & c^0_{3,2} & c^0_{3,3} \end{pmatrix}, \tag{5.13a}$$

with

$$
\begin{aligned}
&c^0_{1,1} = 1 - \lambda\psi^{+(1)}\frac{F_j^{+n(1)}}{\rho_j^n} - \lambda\psi^{-(1)}(a^-_{1,1})^n_j\,,\\
&c^0_{2,2} = 1 - \lambda\psi^{+(2)}\frac{F_j^{+n(2)}}{m_j^n} - \lambda\psi^{-(2)}(a^-_{2,2})^n_j\,, \quad c^0_{2,3} = \lambda\psi^{-(2)}\frac{\gamma-1}{2}\,,\\
&c^0_{3,3} = 1 - \lambda\psi^{+(3)}\frac{F_j^{+n(3)}}{E_j^n} - \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,, \quad c^0_{3,2} = \lambda\psi^{-(3)}\frac{(3-\gamma)^2}{72}(v_j^n - c_j^n)^2\,.
\end{aligned}
\tag{5.13b}
$$

It is straightforward to verify that (5.8) and (5.9) are valid under the CFL-like assumption (5.1b). Therefore, Theorem 5.1 is proven. □

5.4. **Positivity of density and total energy.** Once again, we have to rewrite the fluxes F^+, F^- to make the corresponding matrices A^+, A^- are diagonal with respect to the first and third components, in the proof of the property that the second order Van Leer splitting method preserves positivity of density and total energy. At a supersonic point, the form of A^+, A^- is still the same as (2.19) and the estimate (2.20) is valid. At a subsonic point $0 \le v < c$, the corresponding matrices can take the same form as in (4.26) and we have the following estimates

$$
\begin{aligned}
&0 \le a^+_{1,1} \le \frac{1}{2}(|v| + c)\,, \qquad -\frac{1}{4}(c + 3|v|) \le a^-_{1,1} \le 0\,,\\
&0 \le a^+_{3,3} = \frac{F^{+(3)}}{E} \le \frac{\gamma(\gamma^2+2\gamma-3)}{2(\gamma+1)}|v| + \frac{2\gamma}{\gamma+1}c\,,\\
&-\Big(\frac{\gamma(\gamma^2+2\gamma-3)}{2(\gamma+1)}|v| + \frac{2\gamma}{\gamma+1}c\Big) \le a^-_{3,3} = \frac{F^{-(3)}}{E} \le 0\,.
\end{aligned}
\tag{5.14}
$$

As a result, the numerical scheme can be recast in the form of either (4.4), (4.10) or (4.21)-(4.22). We note that the three matrices C_{-1}, C_0 and C_1 are diagonal with respect to the first and third components, in all three cases: subsonic, supersonic or transient. Consequently, ρ_j^{n+1} and E_j^{n+1} can be written as positive combinations of ρ^n and E^n, respectively, i.e.,

$$
\rho_j^{n+1} = \lambda_{-1,1}\rho^n_{j-1} + \lambda_{0,1}\rho^n_j + \lambda_{1,1}\rho^n_{j+1}\,, \tag{5.15}
$$

$$
E_j^{n+1} = \lambda_{-1,3}E^n_{j-1} + \lambda_{0,3}E^n_j + \lambda_{1,3}E^n_{j+1}\,, \tag{5.16}
$$

with

$$
\begin{aligned}
&\lambda_{-1,1} = \lambda\psi^{+(1)}(a^+_{1,1})^n_{j-1}\,, \quad \lambda_{1,1} = -\lambda\psi^{-(1)}(a^-_{1,1})^n_{j+1}\,,\\
&\lambda_{0,1} = 1 - \lambda\psi^{+(1)}(a^+_{1,1})^n_j + \lambda\psi^{-(1)}(a^-_{1,1})^n_j\,, \quad \lambda_{-1,3} = \lambda\psi^{+(3)}(a^+_{3,3})^n_{j-1}\,,\\
&\lambda_{1,3} = -\lambda\psi^{-(3)}(a^-_{3,3})^n_{j+1}\,, \quad \lambda_{0,3} = 1 - \lambda\psi^{+(3)}(a^+_{3,3})^n_j + \lambda\psi^{-(3)}(a^-_{3,3})^n_j\,.
\end{aligned}
\tag{5.17}
$$

The combination coefficients $\lambda_{i,1}, i = -1, 0, 1,\ \lambda_{i,3}, i = -1, 0, 1$ are non-negative under the following CFL-like condition

$$
\frac{\triangle t}{\triangle x}\max\Big(a|v_j^n| + b|c_j^n|\Big) \le 1\,, \tag{5.18a}
$$

$$a = \max\Big(\frac{4\gamma - 1 - \gamma^2}{\gamma}, \frac{\gamma(2\gamma^2 + 4\gamma - 6)}{\gamma + 1}, \frac{5}{2}\Big), \quad b = \max\Big(\frac{4}{\gamma}, \frac{8\gamma}{\gamma + 1}, \frac{\gamma + 3}{\gamma}, \frac{3}{2}\Big), \tag{5.18b}$$

with the usage of the estimate (5.14). Then the scheme preserves the positivity of density and energy. Once again, the positivity of internal energy (henceforth the pressure field) cannot be obtained from the above derivation, thus the result reported in this section is inferior to that in [28], [5].

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