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Error analysis of a mixed finite element method for a Cahn–Hilliard–Hele–Shaw system

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Abstract We present and analyze a mixed finite element numerical scheme for the Cahn–Hilliard–Hele–Shaw equation, a modified Cahn–Hilliard equation coupled with the Darcy flow law. This numerical scheme was first reported in Feng and Wise (SIAM J Numer Anal 50:1320–1343, 2012), with the weak convergence to a weak solution proven. In this article, we provide an optimal rate error analysis. A convex splitting approach is taken in the temporal discretization, which in turn leads to the unique solvability and unconditional energy stability. Instead of the more standard $\ell^\infty(0,T;L^2)\cap\ell^2(0,T;H^2)$ error estimate, we perform a discrete $\ell^\infty(0,T;H^1)\cap\ell^2(0,T;H^3)$ error estimate for the phase variable, through an L^2 inner product with the numerical error function associated with the chemical potential. As a result, an unconditional convergence (for the time step τ in terms of the spatial resolution h) is derived. The nonlinear analysis is accomplished with the help of a discrete Gagliardo–Nirenberg type inequality in the finite element space, gotten by introducing a discrete Laplacian Δ_h of the numerical solution, such that $\Delta_h\phi\in S_h$, for every $\phi\in S_h$, where S_h is the finite element space.

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1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open, bounded and convex polygonal or polyhedral domain. We consider the following Cahn–Hilliard–Hele–Shaw problem with natural and no-flux/no-flow boundary conditions:

$$\partial_t \phi = \varepsilon \Delta \mu - \nabla \cdot (\phi \mathbf{u}), \qquad \text{in } \Omega_T := \Omega \times (0, T), \tag{1.1a}$$

$$\mu = \varepsilon^{-1} \left(\phi^3 - \phi \right) - \varepsilon \Delta \phi, \qquad \text{in } \Omega_T, \tag{1.1b}$$

$$\mathbf{u} + \nabla p = -\gamma \phi \nabla \mu, \qquad \text{in } \Omega_T, \tag{1.1c}$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \text{in } \Omega_T, \tag{1.1d}$$

$$\partial_n \phi = \partial_n \mu = 0, \ \mathbf{u} \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega \times (0, T), \tag{1.1e}$$

with initial data $\phi_0(\,\cdot\,) = \phi(0, \,\cdot\,) \in H^1(\Omega)$. We assume that the model parameters satisfy $\varepsilon, \gamma > 0$.

We can reformulate the model by eliminating the velocity:

$$\partial_t \phi = \varepsilon \Delta \mu + \nabla \cdot (\phi (\nabla p + \gamma \phi \nabla \mu)), \qquad \text{in } \Omega_T, \tag{1.2a}$$

$$\mu = \varepsilon^{-1} \left(\phi^3 - \phi \right) - \varepsilon \Delta \phi, \qquad \text{in } \Omega_T, \tag{1.2b}$$

$$-\Delta p = \gamma \nabla \cdot (\phi \nabla \mu), \qquad \text{in } \Omega_T, \tag{1.2c}$$

$$\partial_n \phi = \partial_n \mu = \partial_n p = 0, \qquad \text{on } \partial\Omega \times (0, T). \tag{1.2d}$$

If needed, the velocity may be back-calculated as $\mathbf{u} = -(\nabla p + \gamma \phi \nabla \mu)$. A weak formulation of the problem may be expressed as

$$(\partial_t \phi, \nu) + \varepsilon (\nabla \mu, \nabla \nu) + (\nabla p + \gamma \phi \nabla \mu, \phi \nabla \nu) = 0, \qquad \forall \nu \in H^1(\Omega), \tag{1.3a}$$

$$\varepsilon^{-1} \left(\phi^3 - \phi, \psi \right) + \varepsilon (\nabla \phi, \nabla \psi) - (\mu, \psi) = 0, \qquad \forall \psi \in H^1(\Omega), \tag{1.3b}$$

$$(\nabla p + \gamma \phi \nabla \mu, \nabla q) = 0, \qquad \forall q \in H^1(\Omega), \tag{1.3c}$$

for almost every $t \in (0, T)$. We will also consider a weak formulation that keeps the velocity as separate variable:

$$(\partial_t \phi, \nu) + \varepsilon a (\mu, \nu) - b (\phi, \mathbf{u}, \mu) = 0, \qquad \forall \nu \in H^1(\Omega), \tag{1.4a}$$

$$\varepsilon^{-1} \left(\phi^3 - \phi, \psi \right) + \varepsilon a (\phi, \psi) - (\mu, \psi) = 0, \qquad \forall \psi \in H^1(\Omega), \tag{1.4b}$$

$$(\mathbf{u}, \mathbf{v}) + c (\mathbf{v}, p) - \gamma b (\phi, \mathbf{v}, \mu) = 0, \qquad \forall \mathbf{v} \in \mathbf{L}^2(\Omega), \tag{1.4c}$$

$$c (\mathbf{u}, q) = 0, \qquad \forall q \in H^1(\Omega), \tag{1.4d}$$

where

$$a(u, v) := (\nabla u, \nabla v), \quad b(\psi, \mathbf{v}, v) := (\psi \mathbf{v}, \nabla v), \quad c(\mathbf{v}, q) := (\mathbf{v}, \nabla q). \tag{1.5}$$

We consider

$$\begin{aligned} E(\phi) &= \frac{1}{4\varepsilon} \left\| \phi^2 - 1 \right\|^2 + \frac{\varepsilon}{2} \|\nabla \phi\|^2 \\ &= \frac{1}{4\varepsilon} \|\phi\|_{L^4}^4 - \frac{1}{2\varepsilon} \|\phi\|^2 + \frac{|\Omega|}{4\varepsilon} + \frac{\varepsilon}{2} \|\nabla \phi\|^2, \end{aligned} \tag{1.6}$$

which is defined for all $\phi \in \mathcal{H} := \left\{\phi \in H^1(\Omega) \mid \left(\phi - \overline{\phi}_0, 1\right) = 0\right\}$, where $\overline{\phi}_0 = \frac{1}{|\Omega|} \int_\Omega \phi_0(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$. From now on, we denote by $\|\cdot\|$ the standard L^2 norm, provided there is no ambiguity. Clearly, $E(\phi) \geq 0$ for all $\phi \in \mathcal{H}$. It is straightforward to show that weak solutions dissipate the energy (1.6). In other words, (1.1a)–(1.1e) is a conserved gradient flow with respect to the energy (1.6). Precisely, for any $t \in [0, T]$, we have the energy law

$$E(\phi(t)) + \int_0^t \left(\frac{1}{\gamma} \|\mathbf{u}(s)\|^2 + \varepsilon \|\nabla \mu(s)\|^2 \right) ds = E(\phi_0), \tag{1.7}$$

and, in addition, the following mass conservation law: $(\phi(t, \cdot), 1) = (\phi_0, 1) = \overline{\phi}_0 \cdot |\Omega|$. Formally, one can also easily demonstrate that μ in (1.1b) is the variational derivative of E with respect to ϕ. In symbols, $\mu = \delta_\phi E$.

Definition 1.1 Define

$$\mathbf{W} := \left\{ \mathbf{u} \in \mathbf{L}^2(\Omega) \mid (\mathbf{u}, \nabla q) = 0, \ \forall q \in H^1(\Omega) \right\}. \tag{1.8}$$

The projection $\mathcal{P} : \mathbf{L}^2(\Omega) \to \mathbf{W}$ is defined via

$$\mathcal{P}(\mathbf{w}) = \nabla p + \mathbf{w}, \tag{1.9}$$

where $p \in \mathring{H}^1(\Omega) := \left\{\phi \in H^1(\Omega) \mid (\phi, 1) = 0\right\}$ is the unique solution to

$$(\nabla p + \mathbf{w}, \nabla q) = 0, \quad \forall q \in H^1(\Omega). \tag{1.10}$$

Clearly $\mathcal{P}(\mathbf{w}) \in \mathbf{W}$ for any $\mathbf{w} \in \mathbf{L}^2(\Omega)$. Furthermore, we have

Lemma 1.2 *$\mathcal{P}$ is linear, and, given $\mathbf{w} \in \mathbf{L}^2(\Omega)$, it follows that*

$$(\mathcal{P}(\mathbf{w}) - \mathbf{w}, \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{W}. \tag{1.11}$$

In particular, since $\mathcal{P}(\mathbf{w}) \in \mathbf{W}$,

$$(\mathcal{P}(\mathbf{w}) - \mathbf{w}, \mathcal{P}(\mathbf{w})) = 0, \tag{1.12}$$

and, consequently,

$$\|\mathcal{P}(\mathbf{w})\| \le \|\mathbf{w}\| \,, \tag{1.13}$$

for all $\mathbf{w} \in \mathbf{L}^2$.

With the projection, we have the following alternate weak formulation:

$$(\partial_t \phi, \nu) + \varepsilon\,(\nabla \mu, \nabla \nu) + (\mathcal{P}(\gamma \phi \nabla \mu), \phi \nabla \nu) = 0, \qquad \forall \nu \in H^1(\Omega), \tag{1.14a}$$

$$\varepsilon^{-1}\left(\phi^3 - \phi, \psi\right) + \varepsilon\,(\nabla \phi, \nabla \psi) - (\mu, \psi) = 0, \qquad \forall \psi \in H^1(\Omega). \tag{1.14b}$$

Equivalently, with $\mathbf{u} = -\mathcal{P}(\gamma \phi \nabla \mu)$, we have

$$(\partial_t \phi, \nu) + \varepsilon\,(\nabla \mu, \nabla \nu) - (\mathbf{u}, \phi \nabla \nu) = 0, \qquad \forall \nu \in H^1(\Omega), \tag{1.15a}$$

$$\varepsilon^{-1}\left(\phi^3 - \phi, \psi\right) + \varepsilon\,(\nabla \phi, \nabla \psi) - (\mu, \psi) = 0, \qquad \forall \psi \in H^1(\Omega). \tag{1.15b}$$

The well-posedness of this weak form, as well as the basic regularity of the weak solution, can be found in [19]. In more detail, a convex splitting numerical scheme, which treats the terms of the variational derivative implicitly or explicitly according to whether the terms corresponding to the convex or concave parts of the energy, was formulated in [19], with a mixed finite element approximation in space. Such a numerical approach assures two mathematical properties: unique solvability and unconditional energy stability; also see the related works for various PDE systems, including the phase field crystal (PFC) equation [4,5,27,34,35,39], epitaxial thin film growth model [8,10,31,33], and others [21,22]. Moreover, for a gradient system coupled with fluid motion, the idea of convex splitting can still be applied and these distinguished mathematical properties are retained, as given by a few recent works [9,12,13,19,38]. In particular, a weak convergence of the finite element numerical approximation to a global-in-time weak solution was established in [19], using certain compactness arguments.

In addition to this weak convergence result, a convergence analysis with an associated convergence order, for these gradient flows coupled with fluid motion, has attracted a great deal of attentions in recent years. For instance, a convex splitting finite element scheme applied to Cahn–Hilliard–Stokes equation was analyzed in [13] and an optimal rate convergence analysis was provided in detail. Such a convergence result was derived by an H^1 error estimate, combined with unconditional energy stability and other higher order stability properties for certain numerical variables.

Meanwhile, a careful examination shows that, this convergence analysis relies heavily on the $\ell^2(0,T;H^1)$ stability bound of the velocity field, at the numerical level. With this stability available, the maximum norm bound of the phase variable ϕ could be derived, which leads to a great simplification in the convergence analysis. However, for the CHHS system (1.1a)–(1.1e), only an $\ell^2(0,T;L^2)$ bound for the velocity field is valid. As a result, a global-in-time L^∞ bound is not available to the phase variable; see more detailed PDE analyses in [36,37], etc. Without this estimate, an error estimate for the CHHS equation (1.1a)–(1.1e) becomes very challenging, due to the appearance of a highly nonlinear convection term; the velocity error term turns out to be a Helmholtz projection of the nonlinear error associated with $-\gamma\phi\nabla\mu$. In turn, even the highest order diffusion term in the standard Cahn–Hilliard part is not able to control the numerical error term associated with the nonlinear convection.

In this paper, we provide an optimal rate convergence analysis for the mixed finite element scheme applied to the CHHS equation (1.1a)–(1.1e), as reported in [19]. Instead of the standard $\ell^\infty(0,T;L^2)\cap\ell^2(0,T;H^2)$ error estimate for the pure Cahn–Hilliard equation [1,2,14–16,18,20,32], we perform an $\ell^\infty(0,T;H^1)\cap\ell^2(0,T;H^3)$ error estimate in an alternate way. This error estimate is necessary to make the error term associated with the nonlinear convection have a non-positive inner product with the corresponding error test function, which is crucial to the convergence analysis. In particular, we note that, although the $\ell^\infty(0,T;H^1)$ error estimates have been available for the pure Cahn–Hilliard equation in the existing literature [3,17,24,29], an $\ell^2(0,T;H^3)$ error estimate remains open for the finite element approximation applied to the related PDE systems, in the authors' knowledge.

To overcome the difficulty associated with the lack of regularity for the velocity field in the Darcy law, a discrete Gagliardo–Nirenberg inequality is needed in the finite element analysis, in both 2-D and 3-D cases. Meanwhile, such an inequality is involved with an H^3 norm of the numerical solution, which is beyond its regularity in the standard finite element space. In this paper, we establish the desired inequality in a modified version, which plays a key role in the nonlinear error estimate. First, a discrete Laplacian operator, Δ_h, is introduced for any H^1 function in the finite element space. Subsequently, by applying various Sobolev inequalities for continuous function, combined with a few error bounds in the finite element space, the maximum norm bound of the numerical solution could be established in terms of a discrete Gagliardo–Nirenberg inequality.

Another key point of the analysis presented in this paper is that, the $\ell^\infty(0,T;H^1)$ error estimate is performed through an L^2 inner product with the numerical error associated with the chemical potential term. Such an inner product yields an $L^2(0,T;H^1)$ stability of the chemical potential error term, which contains certain nonlinear parts. These nonlinear errors are analyzed via appropriate Sobolev inequalities, so that its growth is always controlled. Furthermore, by applying a subtle $W_3^{1^*}$ estimate for the temporal derivative of the numerical solution (at a discrete level), we could convert all the nonlinear error terms at the current time step into the ones at the previous one. With this approach, an $\ell^8(0,T;L^\infty)$ estimate of the numerical solution (for the phase variable ϕ) could be applied so that an unconditional convergence (for the time step τ in terms of the spatial resolution h) is available, and a constraint for both τ and h turns out to be very mild.

The rest of the paper is organized as follows. The fully discrete finite element scheme is reviewed in Sect. 2. Therein we recall an unconditional energy stability and a few other refined stability estimates, and a discrete Gagliardo–Nirenberg inequality is established in the finite element space. Subsequently, the detailed convergence analysis is given by Sect. 3, which results in an optimal rate error estimate. Finally, a useful discrete Gronwall inequality is restated in Appendix 1.

2 Some mixed finite element convex splitting schemes

2.1 Definitions of the schemes

Let M be a positive integer and $0 = t_0 < t_1 < \cdots < t_M = T$ be a uniform partition of $[0, T]$, with $\tau = t_i - t_{i-1}$, $i = 1, \dots, M$. Suppose $\mathcal{T}_h = \{K\}$ is a conforming, shape-regular, globally quasi-uniform family of triangulations of Ω. For $r \in \mathbb{Z}^+$, define $\mathcal{M}_r^h := \left\{v \in C^0(\Omega) \mid v|_K \in \mathcal{P}_r(K),\ \forall\ K \in \mathcal{T}_h\right\} \subset H^1(\Omega)$. Define $L_0^2(\Omega) := \left\{\phi \in L^2(\Omega) \mid (\phi, 1) = 0\right\}$. We set $S_h := \mathcal{M}_q^h$ and $\mathring{S}_h := S_h \cap L_0^2(\Omega)$, where q is a positive integer. The mixed convex-splitting scheme is defined as follows [19]: for any $1 \le m \le M$, given $\phi_h^{m-1} \in S_h$, find $\phi_h^m, \mu_h^m \in S_h$ and $p_h^m \in \mathring{S}_h$ such that

$$\left(\delta_\tau \phi_h^m, \nu\right) + \varepsilon\, a\left(\mu_h^m, \nu\right) + \left(\phi_h^{m-1}\left(\nabla p_h^m + \gamma \phi_h^{m-1} \nabla \mu_h^m\right), \nabla \nu\right) = 0, \quad \forall\, \nu \in S_h, \tag{2.1a}$$

$$\varepsilon^{-1}\left(\left(\phi_h^m\right)^3 - \phi_h^{m-1}, \psi\right) + \varepsilon\, a\left(\phi_h^m, \psi\right) - \left(\mu_h^m, \psi\right) = 0, \quad \forall\, \psi \in S_h, \tag{2.1b}$$

$$\left(\nabla p_h^m + \gamma \phi_h^m \nabla \mu_h^m, \nabla \zeta\right) = 0, \quad \forall\, \zeta \in S_h, \tag{2.1c}$$

where

$$\delta_\tau \phi_h^m := \frac{\phi_h^m - \phi_h^{m-1}}{\tau}, \quad \phi_h^0 := R_h \phi_0. \tag{2.2}$$

The operator $R_h : H^1(\Omega) \to S_h$ is the Ritz projection:

$$a\left(R_h \varphi - \varphi, \chi\right) = 0, \quad \forall\, \chi \in S_h, \quad \left(R_h \varphi - \varphi, 1\right) = 0. \tag{2.3}$$

The velocity may be defined from the other variables as

$$\mathbf{u}_h^m := -\nabla p_h^m - \gamma \phi_h^{m-1} \nabla \mu_h^m \in \mathbf{L}^2. \tag{2.4}$$

Now we define a discrete projection.

Definition 2.1 Define

$$\mathbf{W}_h := \left\{\mathbf{u} \in \mathbf{L}^2(\Omega) \mid (\mathbf{u}, \nabla q) = 0,\ \forall\, q \in S_h\right\}. \tag{2.5}$$

Observe that $\mathbf{W} \subset \mathbf{W}_h$. The projection $\mathcal{P}_h : \mathbf{L}^2(\Omega) \to \mathbf{W}_h$ is defined via

$$\mathcal{P}_h(\mathbf{w}) = \nabla p + \mathbf{w}, \tag{2.6}$$

where $p \in \mathring{S}_h$ is the unique solution to

$$(\nabla p + \mathbf{w}, \nabla q) = 0, \quad \forall\, q \in S_h. \tag{2.7}$$

Clearly $\mathcal{P}_h \in \mathbf{W}_h$. Furthermore, we have

Lemma 2.2 *$\mathcal{P}_h$ is linear, and given any $\mathbf{w} \in \mathbf{L}^2(\Omega)$, it follows that*

$$(\mathcal{P}_h(\mathbf{w}) - \mathbf{w}, \mathbf{v}) = 0, \quad \forall\, \mathbf{v} \in \mathbf{W}_h. \tag{2.8}$$

In particular, since $\mathcal{P}_h(\mathbf{w}) \in \mathbf{W}_h$,

$$(\mathcal{P}_h(\mathbf{w}) - \mathbf{w}, \mathcal{P}_h(\mathbf{w})) = 0, \tag{2.9}$$

and, consequently,

$$\|\mathcal{P}_h(\mathbf{w})\| \le \|\mathbf{w}\|, \tag{2.10}$$

for all $\mathbf{w} \in \mathbf{L}^2$.

There is an estimate for the difference between the projections $\mathcal{P}$ and $\mathcal{P}_h$.

Lemma 2.3 *Suppose that $\mathbf{w} \in \mathbf{H}^q(\Omega)$ with the compatible boundary conditions $\mathbf{w} \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $p \in H^{q+1}(\Omega)$ where*

$$\nabla p = \mathcal{P}(\mathbf{w}) - \mathbf{w}. \tag{2.11}$$

Then

$$\|\mathcal{P}_h(\mathbf{w}) - \mathcal{P}(\mathbf{w})\| \le C h^q\, |p|_{H^{q+1}}. \tag{2.12}$$

Proof By definition,

$$\mathcal{P}(\mathbf{w}) = \nabla p + \mathbf{w}, \tag{2.13}$$

where $p \in \mathring{H}^1(\Omega)$ is the unique solution to

$$(\nabla p + \mathbf{w}, \nabla q) = 0, \quad \forall q \in H^1(\Omega), \tag{2.14}$$

and

$$\mathcal{P}_h(\mathbf{w}) = \nabla p_h + \mathbf{w}, \tag{2.15}$$

where $p_h \in \mathring{S}_h$ is the unique solution to

$$(\nabla p_h + \mathbf{w}, \nabla q) = 0, \quad \forall q \in S_h. \tag{2.16}$$

Thus

$$\|\mathcal{P}_h(\mathbf{w}) - \mathcal{P}(\mathbf{w})\| = \|\nabla (p - p_h)\| \le C h^q\, |p|_{H^{q+1}}, \tag{2.17}$$

by a standard approximation estimate. □

We may re-express the scheme as

$$\left(\delta_\tau \phi_h^m, \nu\right) + \varepsilon\, a\left(\mu_h^m, \nu\right) + b\left(\phi_h^{m-1}, \mathcal{P}_h\left(\gamma \phi_h^{m-1} \nabla \mu_h^m\right), \nu\right) = 0, \quad \forall\, \nu \in S_h, \tag{2.18a}$$

$$\varepsilon^{-1}\left(\left(\phi_h^m\right)^3-\phi_h^{m-1},\psi\right)+\varepsilon\, a\left(\phi_h^m,\psi\right)-\left(\mu_h^m,\psi\right)=0,\quad \forall\,\psi\in S_h, \tag{2.18b}$$

or equivalently, with $\mathbf{u}_h^m := -\mathcal{P}_h\left(\gamma\phi_h^{m-1}\nabla\mu_h^m\right)\in\mathbf{L}^2$, as

$$\left(\delta_\tau\phi_h^m,\nu\right)+\varepsilon\, a\left(\mu_h^m,\nu\right)-b\left(\phi_h^{m-1},\mathbf{u}_h^m,\nu\right)=0,\quad \forall\,\nu\in S_h, \tag{2.19a}$$

$$\varepsilon^{-1}\left(\left(\phi_h^m\right)^3-\phi_h^{m-1},\psi\right)+\varepsilon\, a\left(\phi_h^m,\psi\right)-\left(\mu_h^m,\psi\right)=0,\quad \forall\,\psi\in S_h. \tag{2.19b}$$

We observe that, in general, $\mathbf{u}_h^m$ is a discontinuous function, its components are not in the finite element spaces so far described.

To remedy this we could formulate a scheme which keeps the velocity as a separate variable in some appropriate finite element space. To this end, we will also consider a scheme that uses a mixed method for the velocity and pressure: for any $1\le m\le M$, given $\phi_h^{m-1}\in S_h$, find $\phi_h^m,\mu_h^m\in S_h$ and $\mathbf{u}_h^m\in\mathbf{X}_h$, $p_h^m\in\mathring{Q}_h$ such that

$$\left(\delta_\tau\phi_h^m,\nu\right)+\varepsilon\, a\left(\mu_h^m,\nu\right)-b\left(\phi_h^{m-1},\mathbf{u}_h^m,\nabla\nu\right)=0,\quad \forall\,\nu\in S_h, \tag{2.20a}$$

$$\varepsilon^{-1}\left(\left(\phi_h^m\right)^3-\phi_h^{m-1},\psi\right)+\varepsilon\, a\left(\phi_h^m,\psi\right)-\left(\mu_h^m,\psi\right)=0,\quad \forall\,\psi\in S_h, \tag{2.20b}$$

$$\left(\mathbf{u}_h^m,\mathbf{v}\right)+c\left(\mathbf{v},p\right)+\gamma b\left(\phi_h^{m-1},\mathbf{v},\mu_h^m\right)=0,\quad \forall\,\mathbf{v}\in\mathbf{X}_h, \tag{2.20c}$$

$$c\left(\mathbf{u}_h^m,q\right)=0,\quad \forall\,q\in Q_h, \tag{2.20d}$$

where $\mathbf{X}_h\subset\mathbf{L}^2$ and $Q_h\subset H^1(\Omega)$ are compatible and inf-sup stable finite element spaces. Here we have used the so called primal mixed formulation. A finite element method based on the dual mixed formulation is also available. We will not pursue this further at this time.

2.2 Unconditional solvability and energy stability

In this subsection, we demonstrate some results from [13,19] that are important for the proof in the following section. These results show that our schemes are unconditionally uniquely solvable. We begin by defining some machinery for the solvability, as well as the stability and convergence analyses discussed later. First, consider the invertible linear operator $\mathsf{T}_h:\mathring{S}_h\to\mathring{S}_h$ defined via the following variational problem: given $\zeta\in\mathring{S}_h$, find $\mathsf{T}_h(\zeta)\in\mathring{S}_h$ such that

$$a\left(\mathsf{T}_h(\zeta),\chi\right)=\left(\zeta,\chi\right),\quad \forall\,\chi\in\mathring{S}_h. \tag{2.21}$$

This clearly has a unique solution because $a\left(\cdot,\cdot\right)$ is an inner product on $\mathring{S}_h$. We now wish to define a mesh-dependent "-1" norm, *i.e.*, a discrete analogue to the H^{-1} norm. The following result can be found in [13,19].

Lemma 2.4 *Let $\zeta,\ \xi \in \mathring{S}_h$ and set*

$$(\zeta, \xi)_{-1,h} := a\left(\mathsf{T}_h(\zeta), \mathsf{T}_h(\xi)\right) = \left(\zeta, \mathsf{T}_h(\xi)\right) = \left(\mathsf{T}_h(\zeta), \xi\right). \tag{2.22}$$

Therefore, $(\cdot\,,\,\cdot)_{-1,h}$ defines an inner product on $\mathring{S}_h$, and the induced negative norm satisfies

$$\|\zeta\|_{-1,h} := \sqrt{(\zeta,\zeta)_{-1,h}} = \sup_{0\neq\chi\in\mathring{S}_h} \frac{(\zeta,\chi)}{\|\nabla\chi\|}. \tag{2.23}$$

Consequently, for all $\chi \in S_h$ and all $\zeta \in \mathring{S}_h$,

$$|(\zeta,\chi)| \le \|\zeta\|_{-1,h}\,\|\nabla\chi\|\,. \tag{2.24}$$

The following Poincaré-type estimate holds:

$$\|\zeta\|_{-1,h} \le C\,\|\zeta\|\,,\quad \forall\ \zeta \in \mathring{S}_h, \tag{2.25}$$

for some $C > 0$ that is independent of h. Finally, if $\mathcal{T}_h$ is globally quasi-uniform, then the following inverse estimate holds:

$$\|\zeta\| \le Ch^{-1}\,\|\zeta\|_{-1,h}\,,\quad \forall\ \zeta \in \mathring{S}_h, \tag{2.26}$$

for some $C > 0$ that is independent of h.

The result for the uniquely solvability of the scheme can be found in [19]. The solutions to our scheme enjoy stability properties that are similar to those of the PDE solutions. Moreover, these properties hold regardless of the sizes of h and τ. The first property, the unconditional energy stability, is a direct result of the convex decomposition represented in the scheme [19].

Lemma 2.5 *Let $(\phi_h^m, \mu_h^m, p_h^m) \in S_h \times S_h \times \mathring{S}_h$ be the unique solution of* (2.1a), (2.1b). *Then the following energy law holds for any $h,\ \tau > 0$:*

$$\begin{aligned} &E\left(\phi_h^\ell\right) + \tau\varepsilon \sum_{m=1}^{\ell} \left\|\nabla\mu_h^m\right\|^2 + \tau\frac{1}{\gamma}\sum_{m=1}^{\ell}\left\|\mathbf{u}_h^m\right\|^2 \\ &\quad + \tau^2 \sum_{m=1}^{\ell}\left\{\frac{\varepsilon}{2}\left\|\nabla\left(\delta_\tau\phi_h^m\right)\right\|^2 + \frac{1}{4\varepsilon}\left\|\delta_\tau(\phi_h^m)^2\right\|^2\right. \\ &\quad \left. + \frac{1}{2\varepsilon}\left\|\phi_h^m\delta_\tau\phi_h^m\right\|^2 + \frac{1}{2\varepsilon}\left\|\delta_\tau\phi_h^m\right\|^2\right\} = E\left(\phi_h^0\right), \end{aligned} \tag{2.27}$$

for all $0 \le \ell \le M$.

The discrete energy law immediately implies the following uniform (in h and τ) *a priori* estimates for ϕ_h^m, μ_h^m, and $\mathbf{u}_h^m$. Note that, from this point, we will not track the dependence of the estimates on the interface parameter $\varepsilon > 0$, though this may be of importance, especially if ε is made smaller [19].

Lemma 2.6 *Suppose that Ω is convex polyhedral. Let $(\phi_h^m, \mu_h^m, p_h^m) \in S_h \times S_h \times \mathring{S}_h$ be the unique solution of (2.1a)–(2.1c). Assume that $E\left(\phi_h^0\right) < C_0$, independent of h. Then for any $0 \le m \le M$,*

$$\int_\Omega \phi_h^m \mathrm{d}\boldsymbol{x} = \int_\Omega \phi_h^0 \mathrm{d}\boldsymbol{x}, \tag{2.28}$$

and there is a constant $C > 0$ independent of h and τ such that the following estimates hold for any $h, \ \tau > 0$:

$$\max_{0\le m\le M} \left[\left\| \nabla\phi_h^m \right\|^2 + \left\| \left(\phi_h^m\right)^2 - 1 \right\|^2 \right] \le C, \tag{2.29}$$

$$\max_{0\le m\le M} \left[\left\| \phi_h^m \right\|_{L^4}^4 + \left\| \phi_h^m \right\|^2 + \left\| \phi_h^m \right\|_{H^1}^2 \right] \le C, \tag{2.30}$$

$$\tau \sum_{m=1}^{M} \left[\ \left\| \nabla\mu_h^m \right\|^2 + \left\| \mathbf{u}_h^m \right\|^2 \right] \le C, \tag{2.31}$$

$$\sum_{m=1}^{M} \left[\ \left\| \nabla \left(\phi_h^m - \phi_h^{m-1} \right) \right\|^2 + \left\| \phi_h^m - \phi_h^{m-1} \right\|^2 + \left\| \phi_h^m (\phi_h^m - \phi_h^{m-1}) \right\|^2 \right.$$
$$\left. + \left\| (\phi_h^m)^2 - (\phi_h^{m-1})^2 \right\|^2 \right] \le C, \tag{2.32}$$

$$\tau \sum_{m=1}^{M} \left[\ \left\| \Delta_h\phi_h^m \right\|^2 + \left\| \mu_h^m \right\|^2 + \left\| \phi_h^m \right\|_{L^\infty}^{\frac{4(6-d)}{d}} \right] \le C(T+1), \tag{2.33}$$

$$\tau \sum_{m=1}^{M} \left\| \delta_\tau \phi_h^m \right\|_{W_3^{1*}}^2 \le C, \tag{2.34}$$

for some constant $C > 0$ that is independent of h, τ, and T.

We are able to prove the next set of *a priori* stability estimates without any restrictions of h and τ. Before we begin, we will need the discrete Laplacian, $\Delta_h : S_h \to \mathring{S}_h$, which is defined as follows: for any $v_h \in S_h$, $\Delta_h v_h \in \mathring{S}_h$ denotes the unique solution to the problem

$$(\Delta_h v_h, \chi) = -a\left(v_h, \chi\right), \quad \forall\ \chi \in S_h. \tag{2.35}$$

In particular, setting $\chi = \Delta_h v_h$ in (2.35), we obtain

$$\left\|\Delta_h v_h\right\|^2 = -a\left(v_h, \Delta_h v_h\right).$$

Lemma 2.7 *The discrete Laplacian has the following properties. For any $v_h \in S_h$,*

$$\left\|\Delta_h v_h\right\| \le \left\|\nabla v_h\right\|^{1/2} \left\|\nabla \Delta_h v_h\right\|^{1/2}, \tag{2.36}$$

and, there is some constant $C > 0$ such that

$$h\left\|\Delta_h v_h\right\| \le C\left\|\nabla v_h\right\|, \tag{2.37}$$

and

$$h^2 \|\Delta_h v_h\| \leq C \|v_h\| . \tag{2.38}$$

Proof The first inequality follows from (2.36) and the Cauchy–Schwarz inequality. For the second inequality, starting from the first and using a standard inverse inequality, we have

$$\|\Delta_h v_h\|^2 \leq \|\nabla v_h\| \cdot \|\nabla \Delta_h v_h\| \leq Ch^{-1} \|\nabla v_h\| \cdot \|\Delta_h v_h\| . \tag{2.39}$$

Applying the inverse inequality again, the third inequality follows as well. □

Next we need a kind of discrete Gagliardo–Nirenberg inequality in the finite element space. Noting that the functions in the conforming finite element space only have the regularity up to H^1, it is impossible to directly apply standard Gagliardo–Nirenberg inequalities involving higher order norms, such as H^2 or H^3. Now that we have the definition of Δ_h, we can prove the following discrete Gagliardo–Nirenberg inequality. Similar techniques can be found in the existing works [25,28] for related finite element estimates involved with higher order derivatives.

Theorem 2.8 *Suppose that Ω is convex and polyhedral. Then, for any $\psi_h \in S_h$*

$$\|\psi_h\|_{L^\infty} \leq C \|\Delta_h \psi_h\|^{\frac{d}{2(6-d)}} \|\psi_h\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} + C \|\psi_h\|_{L^6} , \quad (d = 2, 3), \tag{2.40}$$

$$\|\nabla \psi_h\|_{L^3} \leq C \|\Delta_h \psi_h\|^{\frac{d}{6}} \|\nabla \psi_h\|^{\frac{6-d}{6}} + C \|\nabla \psi_h\| , \quad (d = 2, 3), \tag{2.41}$$

and, consequently,

$$\left\|\psi_h - \overline{\psi_h}\right\|_{L^\infty} \leq C \|\nabla \Delta_h \psi_h\|^{\frac{d}{4(6-d)}} \|\nabla \psi_h\|^{\frac{24-5d}{4(6-d)}} + C \|\nabla \psi_h\| , \quad (d = 2, 3), \tag{2.42}$$

$$\|\nabla \psi_h\|_{L^3} \leq C \|\nabla \Delta_h \psi_h\|^{\frac{d}{12}} \|\nabla \psi_h\|^{\frac{12-d}{12}} + C \|\nabla \psi_h\| , \quad (d = 2, 3), \tag{2.43}$$

using the Poincaré inequality and estimate (2.36).

Proof Define $H_N^2 := \left\{\phi \in H^2(\Omega) \mid \partial_{\mathbf{n}} \phi = 0\right\}$. By elliptic regularity, for any $\psi_h \in S_h$, there is a unique function $\psi \in H_N^2$ such that

$$(\nabla \psi, \nabla w) = (-\Delta_h \psi_h, w) , \quad \forall w \in H^1, \quad (\psi - \psi_h, 1) = 0. \tag{2.44}$$

According to the definitions of R_h in (2.3) and the discrete Laplacian in (2.35), $\psi_h = R_h \psi$. Moreover, $\Delta \psi = \Delta_h \psi_h$ in $L^2(\Omega)$. Therefore, there is a constant $C > 0$ such that

$$|\psi|_{H^2} \leq C \|\Delta \psi\| = C \|\Delta_h \psi_h\| . \tag{2.45}$$

We summarize some standard inverse inequalities, which can be found in [6,11]:

$$\|\varphi_h\|_{W_q^m} \le C h^{\frac{d}{q}-\frac{d}{p}} h^{\ell-m} \|\varphi_h\|_{W_p^\ell}, \quad \forall \varphi_h \in S_h, \quad 1 \le p \le q \le \infty, \quad 0 \le \ell \le m \le 1, \tag{2.46}$$

for some constant $C > 0$. By $\mathcal{I}_h : H^2(\Omega) \to S_h$ we denote the $C^0(\Omega)$, piecewise-polynomial nodal interpolation operator, and we recall the following approximation estimate from [6,11]: for any $\phi \in H^2(\Omega)$, and any $2 \le q \le \infty$,

$$\|\phi - \mathcal{I}_h\phi\|_{W_q^m} \le C h^{\frac{d}{q}-\frac{d}{2}} h^{2-m} |\phi|_{H^2}, \tag{2.47}$$

for $m = 0, 1$, and some constant $C > 0$. Then, by approximation properties, an inverse inequality, and elliptic regularity, we have

$$\begin{aligned}
\|\psi - \psi_h\|_{L^6} &\le \|\psi_h - \mathcal{I}_h\psi\|_{L^6} + \|\mathcal{I}_h\psi - \psi\|_{L^6} \\
&\le C h^{-\frac{d}{3}} \|\psi_h - \mathcal{I}_h\psi\| + C h^{2-\frac{d}{3}} |\psi|_{H^2} \\
&\le C h^{-\frac{d}{3}} \|\psi_h - \psi\| + C h^{-\frac{d}{3}} \|\psi - \mathcal{I}_h\psi\| + C h^{2-\frac{d}{3}} |\psi|_{H^2} \\
&\le C h^{2-\frac{d}{3}} |\psi|_{H^2(\Omega)} \le C h^{2-\frac{d}{3}} \|\Delta_h\psi_h\| .
\end{aligned} \tag{2.48}$$

Therefore, by the triangle inequality,

$$\|\psi\|_{L^6} \le \|\psi_h\|_{L^6} + C h^{2-\frac{d}{3}} \|\Delta_h\psi_h\| . \tag{2.49}$$

On the other hand, using (2.37) and (2.46), we have

$$\|\Delta_h\psi_h\| \le C h^{-2+\frac{d}{3}} \|\psi_h\|_{L^6}, \tag{2.50}$$

and combining the last two inequalities, we have the (reciprocal stability) bound

$$\|\psi\|_{L^6} \le C \|\psi_h\|_{L^6}, \tag{2.51}$$

for some constant $C > 0$. Using the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
\|\psi\|_{L^\infty} &\le C \|\psi\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} |\psi|_{H^2}^{\frac{d}{2(6-d)}} + C \|\psi\|_{L^6} \\
&\le C \|\psi_h\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} \|\Delta_h\psi_h\|^{\frac{d}{2(6-d)}} + C \|\psi_h\|_{L^6} .
\end{aligned} \tag{2.52}$$

Using inverse inequalities, the approximation properties above, and the last inequality, we find

$$\begin{aligned}
\|\psi_h\|_{L^\infty} &\le \|\psi_h - \mathcal{I}_h\psi\|_{L^\infty} + \|\mathcal{I}_h\psi - \psi\|_{L^\infty} + \|\psi\|_{L^\infty} \\
&\le C h^{-\frac{d}{2}} \|\psi_h - \mathcal{I}_h\psi\| + C h^{2-\frac{d}{2}} |\psi|_{H^2} + \|\psi\|_{L^\infty} \\
&\le C h^{-\frac{d}{2}} \|\psi_h - \psi\| + C h^{-\frac{d}{2}} \|\psi - \mathcal{I}_h\psi\| + C h^{2-\frac{d}{2}} \|\Delta_h\psi_h\| + \|\psi\|_{L^\infty}
\end{aligned}$$

$$\begin{aligned}
&\leq Ch^{2-\frac{d}{2}} \|\Delta_h \psi_h\| + \|\psi\|_{L^\infty} \\
&\leq Ch^{2-\frac{d}{2}} \|\Delta_h \psi_h\|^{\frac{3(4-d)}{2(6-d)}} \|\Delta_h \psi_h\|^{\frac{d}{2(6-d)}} \\
&\quad + C \|\psi_h\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} \|\Delta_h \psi_h\|^{\frac{d}{2(6-d)}} + C \|\psi_h\|_{L^6} \\
&\leq C \|\psi_h\|_{L^6}^{\frac{3(4-d)}{2(6-d)}} \|\Delta_h \psi_h\|^{\frac{d}{2(6-d)}} + C \|\psi_h\|_{L^6}, \qquad (2.53)
\end{aligned}$$

where the inequality (2.50) is applied in the last step. The result (2.40) is proven.

Since ψ_h is the Ritz projection of ψ, the forward stability $\|\nabla \psi_h\| \leq \|\nabla \psi\|$ follows easily. To obtain the inequality in the other direction, by the definition of ψ, the triangle inequality, a standard approximation estimate for the Ritz projection, and the inverse inequality (2.37), it follows that

$$\begin{aligned}
\|\nabla \psi\| &\leq \|\nabla \psi - \nabla \phi_h\| + \|\nabla \psi_h\| \\
&\leq h \|\Delta_h \psi_h\| + \|\nabla \psi_h\| \\
&\leq C \|\nabla \psi_h\| + \|\nabla \psi_h\| = C \|\nabla \psi_h\|_, \qquad (2.54)
\end{aligned}$$

which is another type of reciprocal stability. Applying a different Gagliardo–Nirenberg inequality and using the reciprocal stability above, it follows that

$$\begin{aligned}
\|\nabla \psi\|_{L^3} &\leq C \|\nabla \psi\|^{\frac{6-d}{6}} |\psi|_{H^2}^{\frac{d}{6}} + C \|\nabla \psi\| \\
&\leq C \|\nabla \psi_h\|^{\frac{6-d}{6}} \|\Delta_h \psi_h\|^{\frac{d}{6}} + C \|\nabla \psi_h\|_. \qquad (2.55)
\end{aligned}$$

To finish up, we argue as before

$$\begin{aligned}
\|\nabla \psi_h\|_{L^3} &\leq \|\nabla \psi_h - \nabla (\mathcal{I}_h \psi)\|_{L^3} + \|\nabla (\mathcal{I}_h \psi) - \nabla \psi\|_{L^3} + \|\nabla \psi\|_{L^3} \\
&\leq Ch^{-\frac{d}{6}} \|\nabla \psi_h - \nabla (\mathcal{I}_h \psi)\| + Ch^{1-\frac{d}{6}} \|\Delta_h \psi_h\| + \|\nabla \psi\|_{L^3} \\
&\leq Ch^{-\frac{d}{6}} \|\nabla \psi_h - \nabla \psi\| + Ch^{-\frac{d}{6}} \|\nabla \psi - \nabla (\mathcal{I}_h \psi)\| \\
&\quad + Ch^{1-\frac{d}{6}} \|\Delta_h \psi_h\| + \|\nabla \psi\|_{L^3} \\
&\leq Ch^{1-\frac{d}{6}} \|\Delta_h \psi_h\| + C \|\nabla \psi_h\|^{\frac{6-d}{6}} \|\Delta_h \psi_h\|^{\frac{d}{6}} + C \|\nabla \psi_h\| \\
&= Ch^{1-\frac{d}{6}} \|\Delta_h \psi_h\|^{\frac{6-d}{6}} \|\Delta_h \psi_h\|^{\frac{d}{6}} + C \|\nabla \psi_h\|^{\frac{6-d}{6}} \|\Delta_h \psi_h\|^{\frac{d}{6}} + C \|\nabla \psi_h\| \\
&= C \|\nabla \psi_h\|^{\frac{6-d}{6}} \|\Delta_h \psi_h\|^{\frac{d}{6}} + C \|\nabla \psi_h\|. \qquad (2.56)
\end{aligned}$$

□

Theorem 2.9 *Let* $(\phi_h^m, \mu_h^m, p_h^m) \in S_h \times S_h \times \mathring{S}_h$ *be the unique solution of* (2.1a)–(2.1c)*. Suppose that* $E\left(\phi_h^0\right) < C_0$*, independent of* h*, and that* Ω *is a convex polyhedral. The following estimate holds for any* $h,\ \tau > 0$*:*

$$\tau \sum_{m=1}^{M} \left[\left\|\nabla \Delta_h \phi_h^m\right\|^2 + \left\|\phi_h^m\right\|_{L^\infty}^{\frac{8(6-d)}{d}} \right] \leq C_4 (T+1), \qquad (2.57)$$

with some constant $C_4 > 0$ *independent of* h*,* τ*, and* T*.*

Proof We first observe that for any $v_h \in S_h$, $\Delta_h v_h, \Delta_h^2 v_h \in \mathring{S}_h$,

$$a\left(v_h, \Delta_h^2 v_h\right) = \|\nabla \Delta_h v_h\|^2 = \left\|\Delta_h^2 v_h\right\|_{-1,h}^2. \tag{2.58}$$

Taking $\psi = \Delta_h^2 \phi_h^m$ in (2.1b), we have

$$\begin{aligned}
\varepsilon \left\|\nabla \Delta_h \phi_h^m\right\|^2 &= \varepsilon^{-1}\left(\left(\phi_h^m\right)^3, \Delta_h^2 \phi_h^m\right) - \varepsilon^{-1}\left(\phi_h^{m-1}, \Delta_h^2 \phi_h^m\right) - \left(\mu_h^m, \Delta_h^2 \phi_h^m\right) \\
&= -\varepsilon^{-1}\left(\nabla\left(\phi_h^m\right)^3, \nabla \Delta_h \phi_h^m\right) + \varepsilon^{-1}\left(\nabla \phi_h^{m-1}, \nabla \Delta_h \phi_h^m\right) \\
&\quad + \left(\nabla \mu_h^m, \nabla \Delta_h \phi_h^m\right) \\
&= \varepsilon^{-1}\left\|\nabla\left(\phi_h^m\right)^3\right\| \left\|\nabla \Delta_h \phi_h^m\right\| + \varepsilon^{-1}\left\|\nabla \phi_h^{m-1}\right\| \left\|\nabla \Delta_h \phi_h^m\right\| \\
&\quad + \left\|\nabla \mu_h^m\right\| \left\|\nabla \Delta_h \phi_h^m\right\| \\
&\le C\varepsilon^{-3}\left\|\nabla\left(\phi_h^m\right)^3\right\|^2 + C\varepsilon^{-3}\left\|\nabla \phi_h^{m-1}\right\|^2 \\
&\quad + C\varepsilon^{-1}\left\|\nabla \mu_h^m\right\|^2 + \frac{\varepsilon}{2}\left\|\nabla \Delta_h \phi_h^m\right\|^2 \\
&\le C\varepsilon^{-3}\left\|\phi_h^m\right\|_{L^\infty}^4 \left\|\nabla \phi_h^m\right\|^2 + C + C\varepsilon^{-1}\left\|\nabla \mu_h^m\right\|^2 + \frac{\varepsilon}{2}\left\|\nabla \Delta_h \phi_h^m\right\|^2 \\
&\le C\varepsilon^{-3}\left\|\phi_h^m\right\|_{L^\infty}^4 + C + C\varepsilon^{-1}\left\|\nabla \mu_h^m\right\|^2 + \frac{\varepsilon}{2}\left\|\nabla \Delta_h \phi_h^m\right\|^2.
\end{aligned}$$

The first estimate follows upon summing and the result from (2.33).

To get the second estimate, we appeal to (2.42):

$$\begin{aligned}
\left\|\phi_h^m\right\|_{L^\infty} &\le \left\|\phi_h^m - \overline{\phi_h^m}\right\|_{L^\infty} + \left|\overline{\phi_h^m}\right| \le C\left\|\nabla \Delta_h \phi_h^m\right\|^{\frac{d}{4(6-d)}} \left\|\nabla \phi_h^m\right\|^{\frac{24-5d}{4(6-d)}} \\
&\quad + C\left\|\nabla \phi_h^m\right\| + \left|\overline{\phi_h^m}\right| \\
&\le C + \left|\overline{\phi_h^0}\right| + C\left\|\nabla \Delta_h \phi_h^m\right\|^{\frac{d}{4(6-d)}}.
\end{aligned} \tag{2.59}$$

Hence,

$$\left\|\phi_h^m\right\|_{L^\infty}^{\frac{8(6-d)}{d}} \le C + C\left\|\nabla \Delta_h \phi_h^m\right\|^2. \tag{2.60}$$

Summing gives the result. □

3 Error estimates for the fully discrete convex splitting scheme

3.1 Preliminary estimates

We utilize some notation to simplify the error analysis. To this end, define the time lag operator $L_\tau \phi(t) := \phi(t-\tau)$, and the backward difference operator $\delta_\tau \phi(t) :=$

$\frac{\phi(t)-L_\tau\phi(t)}{\tau}$. Define the approximation errors

$$\mathcal{E}_a^\phi := \phi - R_h\phi, \quad \mathcal{E}_a^\mu := \mu - R_h\mu, \tag{3.1}$$
$$\sigma^\phi := \delta_\tau R_h\phi - \partial_t\phi. \tag{3.2}$$

Define the piecewise constant (in time) functions, for $m = 1, \ldots M$ and for $t \in (t_{m-1}, t_m]$,

$$\hat{\phi}(t) := \phi_h^m, \quad \hat{\mu}(t) := \mu_h^m, \quad \hat{\mathbf{u}}(t) := \mathbf{u}_h^m, \quad \hat{p}(t) := p_h^m,$$

where ϕ_h^m, μ_h^m, $\mathbf{u}_h^m$, and p_h^m are the solutions of the fully discrete convex-splitting scheme (2.1a)–(2.1c). We take $\hat{\phi}(0) = \phi_h^0$, et cetera, as is natural. Finally, let us define

$$\mathcal{E}_h^\phi := R_h\phi - \hat{\phi}, \quad \mathcal{E}^\phi := \phi - \hat{\phi}, \quad \mathcal{E}_h^\mu := R_h\mu - \hat{\mu}, \quad \mathcal{E}^\mu := \mu - \hat{\mu}. \tag{3.3}$$

Proposition 3.1 *The following key error equation holds for all $t \in [\tau, T]$:*

$$\begin{aligned}
&\varepsilon \left\|\nabla\mathcal{E}_h^\mu\right\|^2 + \frac{\varepsilon}{2}\delta_\tau \left\|\nabla\mathcal{E}_h^\phi\right\|^2 + \frac{\varepsilon\tau}{2}\left\|\delta_\tau\nabla\mathcal{E}_h^\phi\right\|^2 \\
&\quad = \left(\sigma^\phi, \mathcal{E}_h^\mu\right) + b\left(\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) - b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \mathcal{E}_h^\mu\right) + \left(\mathcal{E}_a^\mu, \delta_\tau\mathcal{E}_h^\phi\right) + \frac{\tau}{\varepsilon}\left(\delta_\tau\phi, \delta_\tau\mathcal{E}_h^\phi\right) \\
&\qquad + \varepsilon^{-1}\left(L_\tau\mathcal{E}^\phi, \delta_\tau\mathcal{E}_h^\phi\right) - \varepsilon^{-1}\left(\phi^3 - \hat{\phi}^3, \delta_\tau\mathcal{E}_h^\phi\right).
\end{aligned} \tag{3.4}$$

Proof Weak solutions (ϕ, μ) with the higher regularities (3.9)–(3.12) solve the following variational problem:

$$\left(\partial_t\phi, \nu\right) + \varepsilon\, a\left(\mu, \nu\right) - b\left(\phi, \mathbf{u}, \nu\right) = 0, \quad \forall\, \nu \in H^1(\Omega), \tag{3.5a}$$
$$\left(\mu, \psi\right) - \varepsilon\, a\left(\phi, \psi\right) - \varepsilon^{-1}\left(\phi^3 - \phi, \psi\right) - \left(\xi, \psi\right) = 0, \quad \forall\, \psi \in H^1(\Omega), \tag{3.5b}$$

where $\mathbf{u} := -\mathcal{P}(\gamma\phi\nabla\mu)$. By definition of the Ritz projection, for all $\nu, \psi \in S_h$, we see that

$$\left(\delta_\tau R_h\phi, \nu\right) + \varepsilon\, a\left(R_h\mu, \nu\right) = \left(\sigma^\phi, \nu\right) + b\left(\phi, \mathbf{u}, \nu\right), \tag{3.6a}$$
$$\varepsilon\, a\left(R_h\phi, \psi\right) - \left(R_h\mu, \psi\right) = \left(\mathcal{E}_a^\mu, \psi\right) - \varepsilon^{-1}\left(\phi^3 - L_\tau\phi, \psi\right) + \frac{\tau}{\varepsilon}\left(\delta_\tau\phi, \psi\right). \tag{3.6b}$$

Thus, for $\tau \le t \le T$, and all $\nu, \psi \in S_h$,

$$\left(\delta_\tau\hat{\phi}, \nu\right) + \varepsilon\, a\left(\hat{\mu}, \nu\right) = b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \nu\right), \tag{3.7a}$$
$$\varepsilon\, a\left(\hat{\phi}, \psi\right) - \left(\hat{\mu}, \psi\right) = -\varepsilon^{-1}\left(\hat{\phi}^3 - L_\tau\hat{\phi}, \psi\right), \tag{3.7b}$$

where $\hat{\mathbf{u}} = -\mathcal{P}_h\left(\gamma L_\tau \hat{\phi} \nabla \hat{\mu}\right)$. Subtracting (3.7a), (3.7b) from (3.6a), (3.6b), we have, for all $\nu, \psi \in S_h$,

$$\left(\delta_\tau \mathcal{E}_h^\phi, \nu\right) + \varepsilon\, a\left(\mathcal{E}_h^\mu, \nu\right) = \left(\sigma^\phi, \nu\right) + b\left(\phi, \mathbf{u}, \nu\right) - b\left(L_\tau \hat{\phi}, \hat{\mathbf{u}}, \nu\right), \tag{3.8a}$$

$$\begin{aligned} \varepsilon\, a\left(\mathcal{E}_h^\phi, \psi\right) - \left(\mathcal{E}_h^\mu, \psi\right) = &\left(\mathcal{E}_a^\mu, \psi\right) + \frac{\tau}{\varepsilon}\left(\delta_\tau \phi, \psi\right) + \varepsilon^{-1}\left(L_\tau \mathcal{E}^\phi, \psi\right) \\ &- \varepsilon^{-1}\left(\phi^3 - \hat{\phi}^3, \psi\right). \end{aligned} \tag{3.8b}$$

Setting $\nu = \mathcal{E}_h^\mu$ in (3.8a), $\psi = \delta_\tau \mathcal{E}_h^\phi$ in (3.8b) and summing the two equations, we have the result. □

For the error estimates that we pursue in this section, we shall assume that weak solutions have the additional regularities:

$$\phi \in W_\infty^2\big(0, T; W_4^1(\Omega)\big) \cap L^\infty\left(0, T; W_\infty^1(\Omega)\right) \cap L^\infty\big(0, T; W_\infty^{q+1}(\Omega)\big), \tag{3.9}$$

$$\mu \in L^\infty\big(0, T; W_6^1(\Omega)\big) \cap L^\infty\big(0, T; W_\infty^{q+1}(\Omega)\big), \tag{3.10}$$

$$\mathbf{u} \in L^\infty\big(0, T; \mathbf{H}^q(\Omega)\big), \tag{3.11}$$

$$\phi \nabla \mu \in L^\infty\big(0, T; H^q(\Omega)\big), \tag{3.12}$$

where $q \geq 1$ is the spatial approximation order.

We need some preliminary estimates, the proofs of which can be found in [13].

Lemma 3.2 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any $h, \tau > 0$, there exists $C > 0$, independent of h and τ, such that*

$$\left\|\sigma^\phi(t)\right\|^2 \leq C h^{2q} + C\tau^2. \tag{3.13}$$

Lemma 3.3 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any $h, \tau > 0$,*

$$\left\|\nabla\left(\phi^3 - \hat{\phi}^3\right)\right\| \leq C\left(\left\|\hat{\phi}\right\|_{L^\infty}^2 + 1\right)\left\|\nabla \mathcal{E}^\phi\right\|. \tag{3.14}$$

Proof For $t \in [0, T]$, the following estimate is valid:

$$\begin{aligned} \left\|\nabla\left(\phi^3 - \hat{\phi}^3\right)\right\| &= 3\left\|\phi^2 \nabla \phi - \hat{\phi}^2 \nabla \hat{\phi}\right\| \leq 3\left\|\phi^2 \nabla \phi - \hat{\phi}^2 \nabla \phi\right\| + 3\left\|\hat{\phi}^2 \nabla \mathcal{E}^\phi\right\| \\ &\leq 3\left\|\nabla \phi\right\|_{L^6}\left\|\phi + \hat{\phi}\right\|_{L^6}\left\|\mathcal{E}^\phi\right\|_{L^6} + 3\left\|\hat{\phi}\right\|_{L^\infty}^2\left\|\nabla \mathcal{E}^\phi\right\| \\ &\leq C\left(\left\|\hat{\phi}\right\|_{L^\infty}^2 + 1\right)\left\|\nabla \mathcal{E}^\phi\right\|, \end{aligned} \tag{3.15}$$

where $C > 0$ is independent of $t \in [0, T]$. Then, using the unconditional *a priori* estimates in (2.33) and the assumption that $\phi \in L^\infty\left(0, T; H^1(\Omega)\right)$, the result follows.

In our error analysis we need to make use of some non-standard approximation results for the Ritz projection. The proof of the following can be gleaned from the material in [6, Ch. 8] and [23].

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^3$ be a convex polyhedral domain. Assume that the solution u of the Neumann–Poisson equation*

$$a(u, v) = (f, v), \quad \forall\, v \in H^1(\Omega),$$

has regularity $u \in W_p^1(\Omega)$, for some $p \in [2, \infty]$. Then there are constants $C > 0$ and h_0, such that the stability

$$\|R_h u\|_{W_p^1} \le C\, \|u\|_{W_p^1} \tag{3.16}$$

holds, provided $0 < h < h_0$. Furthermore, if $u \in W_p^{q+1}(\Omega)$,

$$\|u - R_h u\|_{W_p^1} \le C h^q\, |u|_{W_p^{q+1}}\,, \tag{3.17}$$

where q is the order of the polynomial approximation defining R_h.

Remark 3.5 If Ω is a convex polyhedral domain, it is proven in [23] that the following best approximation property holds for the homogeneous Dirichlet–Poisson problem:

$$\|\nabla(u - R_h u)\|_{L^\infty} \le C \inf_{\chi \in S_h} \|\nabla(u - \chi)\|_{L^\infty}\,, \tag{3.18}$$

where $u \in H_0^1 \cap W_\infty^1$. It is expected to be straightforward to prove such a result for homogeneous Neumann–Poisson problem as well. With such a result, the last theorem will follow.

Lemma 3.6 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any $h, \tau > 0$ and any arbitrary $\theta > 0$, there exists a constant $C > 0$, independent of h and τ, but dependent upon θ, such that*

$$\begin{aligned}
&\frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^\mu\right\|^2 + \frac{\varepsilon}{2}\delta_\tau \left\|\nabla \mathcal{E}_h^\phi\right\|^2 + \frac{\varepsilon\tau}{2}\left\|\delta_\tau \nabla \mathcal{E}_h^\phi\right\|^2 \\
&\quad \le C\tau^2 + Ch^{2q} + b\left(\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) - b\left(L_\tau \hat{\phi}, \hat{\mathbf{u}}, \mathcal{E}_h^\mu\right) \\
&\qquad + C\left\|\nabla L_\tau \mathcal{E}_h^\phi\right\|_{L^3}^2 + \theta \left\|\delta_\tau \mathcal{E}_h^\phi\right\|_{W_3^{1*}}^2 - \varepsilon^{-1}\left((R_h\phi)^3 - \hat{\phi}^3, \delta_\tau \mathcal{E}_h^\phi\right).
\end{aligned} \tag{3.19}$$

Proof Using Lemma 3.2, the Cauchy–Schwarz inequality, the Poincaré inequality, and the fact that $\left(\sigma^\phi, 1\right) = 0$, we get the following estimates: if $\overline{\mathcal{E}_h^\mu}(t)$ is the spatial average of $\mathcal{E}_h^\mu(t)$, for $0 < t \le T$, then

$$\begin{aligned}
\left|\left(\sigma^\phi, \mathcal{E}_h^\mu\right)\right| = \left|\left(\sigma^\phi, \mathcal{E}_h^\mu - \overline{\mathcal{E}_h^\mu}\right)\right| &\le C\left\|\sigma^\phi\right\| \cdot \left\|\nabla \mathcal{E}_h^\mu\right\| \le C\left\|\sigma^\phi\right\|^2 + \frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^\mu\right\|^2 \\
&\le Ch^{2q} + C\tau^2 + \frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^\mu\right\|^2.
\end{aligned} \tag{3.20}$$

An application of Theorem 3.4 implies that

$$\left\| \mathcal{E}_a^\mu \right\|_{W_3^1} = \| R_h \mu - \mu \|_{W_3^1} \le C h^q \, |\mu|_{W_3^{q+1}} .$$

As a consequence, we arrive at

$$\begin{aligned} \left| \left(\mathcal{E}_a^\mu, \delta_\tau \mathcal{E}_h^\phi \right) \right| &\le \left\| \mathcal{E}_a^\mu \right\|_{W_3^1} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}} \le C h^q \, |\mu|_{W_3^{q+1}} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}} \\ &\le C h^{2q} + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 . \end{aligned} \tag{3.21}$$

Now, it follows that

$$\| \tau \nabla \delta_\tau \phi(t) \|_{L^3}^2 \le \tau^{\frac{4}{3}} \left(\int_{t-\tau}^{t} \| \nabla \partial_s \phi(s) \|_{L^3}^3 \, ds \right)^{\frac{2}{3}} \le C \tau^2, \tag{3.22}$$

and, therefore, using a Poincaré-type inequality, for any $\theta > 0$,

$$\frac{\tau}{\varepsilon} \left| \left(\delta_\tau \phi, \delta_\tau \mathcal{E}_h^\phi \right) \right| \le C \, \| \tau \nabla \delta_\tau \phi \|_{L^3} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}} \le C \tau^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 . \tag{3.23}$$

With similar steps, the next-to-last term in (3.4) is controlled by,

$$\begin{aligned} \varepsilon^{-1} \left| \left(L_\tau \mathcal{E}^\phi, \delta_\tau \mathcal{E}_h^\phi \right) \right| &\le C \left\| \nabla L_\tau \mathcal{E}^\phi \right\|_{L^3} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}} \\ &\le C h^{2q} \, |L_\tau \phi|_{W_3^{q+1}}^2 + C \left\| \nabla L_\tau \mathcal{E}_h^\phi \right\|_{L^3}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 \\ &\le C h^{2q} + C \left\| \nabla L_\tau \mathcal{E}_h^\phi \right\|_{L^3}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 , \end{aligned} \tag{3.24}$$

using Theorem 3.4 in the second step.

The last term in (3.4) can be divided into

$$-\left(\phi^3 - \hat{\phi}^3, \delta_\tau \mathcal{E}_h^\phi \right) = -\left(\phi^3 - (R_h \phi)^3, \delta_\tau \mathcal{E}_h^\phi \right) - \left((R_h \phi)^3 - \hat{\phi}^3, \delta_\tau \mathcal{E}_h^\phi \right). \tag{3.25}$$

Using the stability $\| R_h \phi \|_{W_1^\infty} \le C \, \| \phi \|_{W_1^\infty}$ and the non-standard approximation results from Theorem 3.4, and the assumed regularities of the PDE solution, the first term above can be bounded as follows: for any $\theta > 0$,

$$\begin{aligned} & -\left(\phi^3 - (R_h \phi)^3, \delta_\tau \mathcal{E}_h^\phi \right) \\ & \le \; C \left\| \phi^3 - (R_h \phi)^3 \right\|_{W_3^1}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 \\ & \le \; C \left\| \left(\phi^2 + \phi R_h \phi + (R_h \phi)^2 \right) \mathcal{E}_a^\phi \right\|_{L^3}^2 \end{aligned}$$

$$
\begin{aligned}
&+ C \left\| \phi^2 \nabla \phi - (R_h \phi)^2 \nabla R_h \phi \right\|_{L^3}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 \\
\leq\ & C \|\phi\|_{L^\infty}^4 \left\| \mathcal{E}_a^\phi \right\|_{L^3}^2 + C \left\| (\phi + R_h \phi) \mathcal{E}_a^\phi \nabla \phi \right\|_{L^3}^2 \\
&+ C \left\| (R_h \phi)^2 \nabla \mathcal{E}_a^\phi \right\|_{L^3}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 \\
\leq\ & C \|\phi\|_{W_\infty^1}^4 \left\| \mathcal{E}_a^\phi \right\|_{L^3}^2 + C \|\phi\|_{W_\infty^1}^2 \|\nabla \phi\|_{L^6}^2 \left\| \mathcal{E}_a^\phi \right\|_{L^6}^2 \\
&+ C \|\phi\|_{W_\infty^1}^2 \left\| \nabla \mathcal{E}_a^\phi \right\|_{L^3}^2 + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 \\
\leq\ & C h^{2q} + \frac{\theta}{4} \left\| \delta_\tau \mathcal{E}_h^\phi \right\|_{W_3^{1*}}^2 .
\end{aligned} \tag{3.26}
$$

Combining (3.20)–(3.26) leads to the result. □

Now, let us consider the error of the triple form in (3.4). Define

$$
I_4 := b\left(\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) - b\left(L_\tau \hat{\phi}, \hat{\mathbf{u}}, \mathcal{E}_h^\mu\right). \tag{3.27}
$$

Lemma 3.7 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any h, $\tau > 0$, there exists a constant $C > 0$, independent of h and τ, such that*

$$
I_4 \leq -\gamma \left\| \mathcal{P}_h \left(L_\tau \hat{\phi} \nabla \mathcal{E}_h^\mu \right) \right\|^2 + C \hat{D}_0 (\tau^2 + h^{2q}) + C \hat{D}_0 \left\| \nabla L_\tau \mathcal{E}_h^\phi \right\|^2 + \frac{\varepsilon}{4} \left\| \nabla \mathcal{E}_h^\mu \right\|^2 , \tag{3.28}
$$

where

$$
\hat{D}_0 := \left\| L_\tau \hat{\phi} \right\|_{L^\infty}^4 + 1. \tag{3.29}
$$

Proof By adding and subtracting appropriate terms, we have

$$
I_4 = b\left(\mathcal{E}_a^\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(L_\tau \mathcal{E}_h^\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(\tau \delta_\tau R_h \phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(L_\tau \hat{\phi}, \mathbf{u} - \hat{\mathbf{u}}, \mathcal{E}_h^\mu\right). \tag{3.30}
$$

The last term is the only one that will give us any concern.

Recall that the discrete and continuous velocities can be described as

$$
\mathbf{u} = -\mathcal{P}(\gamma \phi \nabla \mu), \quad \hat{\mathbf{u}} = -\mathcal{P}_h(\gamma L_\tau \hat{\phi} \nabla \hat{\mu}). \tag{3.31}
$$

We obtain the following useful decomposition:

$$
\begin{aligned}
&- \gamma^{-1} \left(\mathbf{u} - \hat{\mathbf{u}} \right) \\
&\quad = \mathcal{P}(\phi \nabla \mu) - \mathcal{P}_h (L_\tau \hat{\phi} \nabla \hat{\mu})
\end{aligned}
$$

$$
\begin{aligned}
&= \mathcal{P}(\phi\nabla\mu) - \mathcal{P}_h(\phi\nabla\mu) + \mathcal{P}_h(\phi\nabla\mu) - \mathcal{P}_h(L_\tau\hat{\phi}\nabla\hat{\mu}) \\
&= \mathcal{P}(\phi\nabla\mu) - \mathcal{P}_h(\phi\nabla\mu) + \mathcal{P}_h(\tau\delta_\tau\phi\nabla\mu) + \mathcal{P}_h(L_\tau\phi\nabla\mu) - \mathcal{P}_h(L_\tau\hat{\phi}\nabla\hat{\mu}) \\
&= \mathcal{P}(\phi\nabla\mu) - \mathcal{P}_h(\phi\nabla\mu) + \mathcal{P}_h(\tau\delta_\tau\phi\nabla\mu) + \mathcal{P}_h(L_\tau\mathcal{E}^\phi\nabla\mu) + \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}^\mu).
\end{aligned} \tag{3.32}
$$

Let's deal with all the above terms except for the last one. Define

$$
I_5 := \mathcal{P}(\phi\nabla\mu) - \mathcal{P}_h(\phi\nabla\mu) + \mathcal{P}_h(\tau\delta_\tau\phi\nabla\mu) + \mathcal{P}_h(L_\tau\mathcal{E}^\phi\nabla\mu). \tag{3.33}
$$

Then

$$
\begin{aligned}
\|I_5\|^2 &\le 3\,\|\mathcal{P}(\phi\nabla\mu) - \mathcal{P}_h(\phi\nabla\mu)\|^2 + 3\,\|\mathcal{P}_h(\tau\delta_\tau\phi\nabla\mu)\|^2 + 3\,\left\|\mathcal{P}_h(L_\tau\mathcal{E}^\phi\nabla\mu)\right\|^2 \\
&\le Ch^{2q}\,\|\phi\nabla\mu\|_{H^q}^2 + C\tau^2\,\|\nabla\mu\|_{L^6}^2\,\|\partial_t\phi\|_{L^3}^2 + 6\,\left\|L_\tau\mathcal{E}_a^\phi\nabla\mu\right\|^2 + 6\,\left\|L_\tau\mathcal{E}_h^\phi\nabla\mu\right\|^2 \\
&\le C\left(h^{2q}+\tau^2\right) + Ch^{2q}\,\|\nabla\mu\|_{L^6}^2\,\|\phi\|_{H^{q+1}}^2 + \|\nabla\mu\|_{L^6}^2\,\left\|L_\tau\mathcal{E}_h^\phi\right\|_{L^3}^2 \\
&\le C\left(h^{2q}+\tau^2\right) + C\,\left\|L_\tau\nabla\mathcal{E}_h^\phi\right\|^2.
\end{aligned} \tag{3.34}
$$

From (3.30) we have

$$
\begin{aligned}
I_4 &= b\left(\mathcal{E}_a^\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(L_\tau\mathcal{E}_h^\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(\tau\delta_\tau R_h\phi, \mathbf{u}, \mathcal{E}_h^\mu\right) + b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, \mathcal{E}_h^\mu\right) \\
&\le \left\|\mathcal{E}_a^\phi\right\|_{L^6}\|\mathbf{u}\|_{L^3}\left\|\nabla\mathcal{E}_h^\mu\right\| + \left\|L_\tau\mathcal{E}_h^\phi\right\|_{L^6}\|\mathbf{u}\|_{L^3}\left\|\nabla\mathcal{E}_h^\mu\right\| \\
&\quad + \left(\left\|\mathcal{E}_a^\phi\right\| + \left\|L_\tau\mathcal{E}_a^\phi\right\| + \|\tau\delta_\tau\phi\|\right)\|\mathbf{u}\|\,\left\|\nabla\mathcal{E}_h^\mu\right\| - b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, \mathcal{E}_h^\mu\right) \\
&\le Ch^{2q} + \frac{\varepsilon}{24}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 + C\left\|\nabla L_\tau\mathcal{E}_h^\phi\right\|^2 + \frac{\varepsilon}{24}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 \\
&\quad + C\tau^2 + \frac{\varepsilon}{24}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 + b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, \mathcal{E}_h^\mu\right).
\end{aligned} \tag{3.35}
$$

Now, using (3.34) we have

$$
\begin{aligned}
b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, \mathcal{E}_h^\mu\right) &= -\gamma b\left(L_\tau\hat{\phi}, I_5, \mathcal{E}_h^\mu\right) - \gamma b\left(L_\tau\hat{\phi}, \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_a^\mu), \mathcal{E}_h^\mu\right) \\
&\quad -\gamma b\left(L_\tau\hat{\phi}, \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu), \mathcal{E}_h^\mu\right) \\
&\le C\left\|L_\tau\hat{\phi}\right\|_{L^\infty}^2\|I_5\|^2 + \frac{\varepsilon}{16}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 + Ch^{2q}\left\|L_\tau\hat{\phi}\right\|_{L^\infty}^4\|\mu\|_{H^{q+1}}^2 \\
&\quad + \frac{\varepsilon}{16}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 - \gamma b\left(L_\tau\hat{\phi}, \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu), \mathcal{E}_h^\mu\right) \\
&\le C\hat{D}_0(\tau^2 + h^{2q}) + C\hat{D}_0\left\|L_\tau\mathcal{E}_h^\phi\right\|^2 + \frac{\varepsilon}{8}\left\|\nabla\mathcal{E}_h^\mu\right\|^2 \\
&\quad -\gamma\left\|\mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu)\right\|^2.
\end{aligned} \tag{3.36}
$$

To finish up, adding (3.35) and (3.36) leads to the result. □

Combining Lemmas 3.6 and 3.7, we get immediately the following result:

Lemma 3.8 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any $h, \tau > 0$, and any arbitrary $\theta > 0$, there exists a constant $C > 0$, independent of h and τ, but dependent on θ, such that*

$$\begin{aligned}
&\frac{\varepsilon}{4}\left\|\nabla\mathcal{E}_h^{\mu}\right\|^2 + \frac{\varepsilon}{2}\delta_\tau\left\|\nabla\mathcal{E}_h^{\phi}\right\|^2 + \frac{\varepsilon\tau}{2}\left\|\delta_\tau\nabla\mathcal{E}_h^{\phi}\right\|^2 + \gamma\left\|\mathcal{P}_h\left(L_\tau\hat{\phi}\nabla\mathcal{E}_h^{\mu}\right)\right\|^2 \\
&\quad\le C\hat{D}_0(\tau^2 + h^{2q}) + C\hat{D}_0\left\|\nabla L_\tau\mathcal{E}_h^{\phi}\right\|^2 + C\left\|\nabla L_\tau\mathcal{E}_h^{\phi}\right\|_{L^3}^2 \\
&\qquad + \theta\left\|\delta_\tau\mathcal{E}_h^{\phi}\right\|_{W_3^{1*}}^2 - \varepsilon^{-1}\left((R_h\phi)^3 - \hat{\phi}^3, \delta_\tau\mathcal{E}_h^{\phi}\right).
\end{aligned} \tag{3.37}$$

The next step is to prove that the dual norm $\left\|\delta_\tau\mathcal{E}_h^{\phi}\right\|_{W_3^{1*}}$ can be bounded in a convenient way.

Lemma 3.9 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any $h, \tau > 0$,*

$$\left\|\delta_\tau\mathcal{E}_h^{\phi}\right\|_{W_3^{1*}} \le C\varepsilon\left\|\nabla\mathcal{E}_h^{\mu}\right\| + C\gamma\left\|\mathcal{P}_h\left(L_\tau\hat{\phi}\nabla\mathcal{E}_h^{\mu}\right)\right\| + C\left\|L_\tau\nabla\mathcal{E}_h^{\phi}\right\| + C\hat{D}_0^{\frac{1}{2}}(h^q + \tau), \tag{3.38}$$

where $C > 0$ is independent of h and τ.

Proof Here we follow the ideas in [19]. Let $\mathcal{Q}_h$ be the standard L^2 projection into S_h. For any $\nu \in W^{1,3}(\Omega)$, denote $\nu_h = \mathcal{Q}_h\nu$ in (3.8a). Recall the estimate for σ^ϕ from Lemma 3.2,

$$\begin{aligned}
\left(\delta_\tau\mathcal{E}_h^{\phi}, \nu\right) &= \left(\delta_\tau\mathcal{E}_h^{\phi}, \nu_h\right) \\
&= -\varepsilon\, a\left(\mathcal{E}_h^{\mu}, \nu_h\right) + \left(\sigma^\phi, \nu_h\right) + b\left(\phi, \mathbf{u}, \nu_h\right) - b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \nu_h\right) \\
&\le \varepsilon\left\|\nabla\mathcal{E}_h^{\mu}\right\|\left\|\nabla\nu_h\right\| + \left\|\sigma^\phi\right\|\left\|\nu_h\right\| + b\left(\phi, \mathbf{u}, \nu_h\right) - b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \nu_h\right) \\
&\le C\left(\varepsilon\left\|\nabla\mathcal{E}_h^{\mu}\right\| + h^q + \tau\right)\left\|\nu_h\right\|_{W_3^1} + b\left(\phi, \mathbf{u}, \nu_h\right) - b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \nu_h\right).
\end{aligned} \tag{3.39}$$

For the last two terms above, we repeat the techniques used to analyze I_4 in (3.30). Define

$$I_6 := b\left(\phi, \mathbf{u}, \nu_h\right) - b\left(L_\tau\hat{\phi}, \hat{\mathbf{u}}, \nu_h\right). \tag{3.40}$$

Recalling the estimates in (3.34)–(3.36), we can estimate I_6 as follows:

$$
\begin{aligned}
I_6 &= b\left(\mathcal{E}_a^\phi, \mathbf{u}, v_h\right) + b\left(L_\tau \mathcal{E}_h^\phi, \mathbf{u}, v_h\right) + b\left(\tau\delta_\tau R_h\phi, \mathbf{u}, v_h\right) + b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, v_h\right) \\
&\le \left\|\mathcal{E}_a^\phi\right\|_{L^6} \|\mathbf{u}\|_{L^3} \|\nabla v_h\| + \left\|L_\tau \mathcal{E}_h^\phi\right\|_{L^6} \|\mathbf{u}\|_{L^3} \|\nabla v_h\| + \|\tau\delta_\tau R_h\phi\| \, \|\mathbf{u}\| \, \|\nabla v_h\| \\
&\quad - b\left(L_\tau\hat{\phi}, \mathbf{u}-\hat{\mathbf{u}}, v_h\right) \\
&\le C\left(h^q + \tau + \left\|L_\tau \nabla \mathcal{E}_h^\phi\right\|\right) \|\nabla v_h\| + \gamma b\left(L_\tau\hat{\phi}, I_5, v_h\right) \\
&\quad + \gamma b\left(L_\tau\hat{\phi}, \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_a^\mu), v_h\right) + \gamma b\left(L_\tau\hat{\phi}, \mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu), v_h\right) \\
&\le C\left(h^q + \tau + \left\|L_\tau \nabla \mathcal{E}_h^\phi\right\|\right) \|\nabla v_h\| + \gamma \left\|L_\tau\hat{\phi}\right\|_{L^6} \|I_5\| \, \|\nabla v_h\|_{L^3} \\
&\quad + C h^q \left\|L_\tau\hat{\phi}\right\|_{L^\infty} \|\mu\|_{H^{q+1}} \|\nabla v_h\|_{L^3} + \gamma \left\|L_\tau\hat{\phi}\right\|_{L^6} \left\|\mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu)\right\| \|\nabla v_h\|_{L^3} \\
&\le C\left(\hat{D}_0^{\frac{1}{2}} h^q + \tau + \gamma \left\|\mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu)\right\| + \left\|L_\tau \nabla \mathcal{E}_h^\phi\right\|\right) \|\nabla v_h\|_{L^3} .
\end{aligned}
\tag{3.41}
$$

Combining (3.39) and (3.41), we get

$$
\begin{aligned}
\left(\delta_\tau \mathcal{E}_h^\phi, v\right) &\le C\left(\hat{D}_0^{\frac{1}{2}} h^q + \tau + \varepsilon \left\|\nabla \mathcal{E}_h^\mu\right\| + \gamma \left\|\mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu)\right\| + \left\|L_\tau \nabla \mathcal{E}_h^\phi\right\|\right) \|v_h\|_{W_3^1} \\
&= C\left(\hat{D}_0^{\frac{1}{2}} h^q + \tau + \varepsilon \left\|\nabla \mathcal{E}_h^\mu\right\| + \gamma \left\|\mathcal{P}_h(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu)\right\| + \left\|L_\tau \nabla \mathcal{E}_h^\phi\right\|\right) \|v\|_{W_3^1} .
\end{aligned}
\tag{3.42}
$$

The last estimate is due to the W_3^1 stability of the L^2 projection into the finite element space. See, for example [7]. □

Now, if we choose θ in (3.37) sufficiently small, and apply Lemma 3.9, the following result could be easily obtained:

Lemma 3.10 *Suppose that* (ϕ, μ) *is a weak solution to* (3.5a), (3.5b)*, with the additional regularities* (3.9)–(3.12)*. Then, for any* $h, \tau > 0$*, there exists a constant* $C > 0$*, independent of* h *and* τ*, such that*

$$
\begin{aligned}
&\frac{\varepsilon}{8} \left\|\nabla \mathcal{E}_h^\mu\right\|^2 + \frac{\varepsilon}{2}\delta_\tau \left\|\nabla \mathcal{E}_h^\phi\right\|^2 + \frac{\gamma}{2} \left\|\mathcal{P}_h\left(L_\tau\hat{\phi}\nabla\mathcal{E}_h^\mu\right)\right\|^2 \\
&\quad \le C\hat{D}_0(\tau^2 + h^{2q}) + C\hat{D}_0 \left\|\nabla L_\tau \mathcal{E}_h^\phi\right\|^2 + C\left\|\nabla L_\tau \mathcal{E}_h^\phi\right\|_{L^3}^2 \\
&\qquad - \varepsilon^{-1}\left((R_h\phi)^3 - \hat{\phi}^3, \delta_\tau \mathcal{E}_h^\phi\right).
\end{aligned}
\tag{3.43}
$$

3.2 Estimates for the cubic nonlinear error term

Now that all the preliminary estimates have been done, we will then elaborate how to deduce the stability for the error function (3.4). The result (3.37) is not enough to get what we want, since the last term of the right side has not been estimated yet. If it is estimated in the normal way, such as using the Cauchy–Schwarz inequality directly and summing every step, what we get is at most a stable inequality coupled with an implicit term like $\tau \hat{C} \left\| \nabla \mathcal{E}_h^\phi \right\|^2$ on the right side with $\hat{C}$ is dependent on some norm of the numerical solution $\hat{\phi}$. In this case, τ needs to be small enough in order to be absorbed by the left side. In addition, the high nonlinearity of the last term in (3.37) is another difficulty to be overcome. If we do not use dual norm estimates, what we get from (3.37) is a discrete nonlinear Gronwall inequality which leads us to the sub-optimal convergence rate. The main result is demonstrated below.

Lemma 3.11 *Suppose that* (ϕ, μ) *is a weak solution to* (3.5a), (3.5b)*, with the additional regularities* (3.9)–(3.12)*. Then, for any* $h, \tau > 0$*, there exists a constant* $C > 0$*, independent of* h *and* τ*, such that*

$$
\begin{aligned}
&\varepsilon \left\| \nabla \mathcal{E}_h^\phi(t_m) \right\|^2 + \frac{\varepsilon \tau}{4} \sum_{j=1}^{m} \left\| \nabla \mathcal{E}_h^\mu(t_j) \right\|^2 + \gamma \tau \sum_{j=1}^{m} \left\| \mathcal{P}_h \left(\phi_h^{j-1} \nabla \mathcal{E}_h^\mu(t_j) \right) \right\|^2 \\
&\quad \le C\tau \sum_{j=1}^{m} \hat{D}_0^j (\tau^2 + h^{2q}) + C \sum_{j=0}^{m-1} \mathcal{A}^j \left\| \nabla \mathcal{E}_h^\phi(t_j) \right\|^2 + \frac{\varepsilon^3 \tau}{8} \sum_{j=0}^{m-1} \left\| \nabla \Delta_h \mathcal{E}_h^\phi(t_j) \right\|^2,
\end{aligned}
\tag{3.44}
$$

where $\hat{D}_0^j := \hat{D}_0(t_j)$ *and*

$$
\begin{aligned}
\mathcal{A}^j := \tau \hat{D}_0^{j+1} &+ \left\| \tau \delta_\tau \nabla \phi_h^{j+1} \right\|^2 + \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}} \\
&+ \tau^{-\frac{1}{7}} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{\frac{8}{7}} + \tau^{-\frac{1}{3}} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{\frac{4}{3}}.
\end{aligned}
\tag{3.45}
$$

Proof Our starting point is estimate (3.43). The last term of (3.43) can be rewritten as

$$
\left((R_h \phi(t_m))^3 - (\phi_h^m)^3, \delta_\tau \mathcal{E}_h^\phi(t_m) \right) = \left(\zeta^m \mathcal{E}_h^\phi(t_m), \delta_\tau \mathcal{E}_h^\phi(t_m) \right), \tag{3.46}
$$

where

$$
\zeta^m := (R_h \phi(t_m))^2 + \phi_h^m R_h \phi(t_m) + \left(\phi_h^m \right)^2 \ge 0. \tag{3.47}
$$

By Lemma 4.2,

$$
\begin{aligned}
&- \frac{\tau}{\varepsilon} \sum_{j=1}^{m} \left(\zeta^m \mathcal{E}_h^\phi(t_m), \delta_\tau \mathcal{E}_h^\phi(t_m) \right) \\
&\quad = \frac{\tau}{2\varepsilon} \sum_{j=1}^{m} \left(\delta_\tau \zeta^m, \left(\mathcal{E}_h^\phi(t_{m-1}) \right)^2 \right) - \frac{\tau^2}{2\varepsilon} \sum_{j=1}^{m} \left(\zeta^m, \left(\delta_\tau \mathcal{E}_h^\phi(t_m) \right)^2 \right) \\
&\qquad - \frac{1}{2\varepsilon} \left(\zeta^m, \left(\mathcal{E}_h^\phi(t_m) \right)^2 \right).
\end{aligned}
\tag{3.48}
$$

Observe that the last two terms on the right-hand-side of the last identity are non-positive and can be dropped in the analysis. For any $1 \le m \le M$, summation of (3.43) implies that

$$
\begin{aligned}
&\varepsilon \left\| \nabla \mathcal{E}_h^{\phi}(t_m) \right\|^2 + \frac{\varepsilon\tau}{4} \sum_{j=1}^{m} \left\| \nabla \mathcal{E}_h^{\mu}(t_j) \right\|^2 + \gamma\tau \sum_{j=1}^{m} \left\| \mathcal{P}_h \left(\phi_h^{j-1} \nabla \mathcal{E}_h^{\mu}(t_j) \right) \right\|^2 \\
&\quad \le C(\tau^2 + h^{2q})\tau \sum_{j=1}^{m} \hat{D}_0^j + C\tau \sum_{j=0}^{m-1} \left(\hat{D}_0^{j+1} \left\| \nabla \mathcal{E}_h^{\phi}(t_j) \right\|^2 + C \left\| \nabla \mathcal{E}_h^{\phi}(t_j) \right\|_{L^3}^2 \right) \\
&\qquad + \frac{\tau}{\varepsilon} \sum_{j=1}^{m} \left(\delta_\tau \zeta^m, \left(\mathcal{E}_h^{\phi}(t_{m-1}) \right)^2 \right),
\end{aligned}
\tag{3.49}
$$

where we have dropped the indicated non-positive terms from the right-hand-side. Due to the definition of ζ^j,

$$
\begin{aligned}
\zeta^{j+1} - \zeta^j &= \tau\delta_\tau R_h\phi(t_{j+1}) \left(R_h\phi(t_{j+1}) + R_h\phi(t_j) \right) + \tau\delta_\tau R_h\phi(t_{j+1})\phi_h^j + \left(\tau\delta_\tau \phi_h^{j+1} \right)^2 \\
&\quad + \tau\delta_\tau \phi_h^{j+1} \left(R_h\phi(t_{j+1}) + 2R_h\phi(t_j) - 2\mathcal{E}_h^{\phi}(t_j) \right).
\end{aligned}
\tag{3.50}
$$

Then for every step t_j, the following estimate is available:

$$
\begin{aligned}
&\left(\zeta^{j+1} - \zeta^j, (\mathcal{E}_h^{\phi}(t_j))^2 \right) \\
&\quad \le C \left\| \phi(t_{j+1}) + \phi(t_j) \right\|_{W_\infty^1} \left\| \tau\delta_\tau \phi(t_{j+1}) \right\|_{L^3} \left\| \mathcal{E}_h^{\phi}(t_j) \right\|_{L^3}^2 \\
&\qquad + C \left\| \phi_h^j \right\|_{L^4} \left\| \tau\delta_\tau \phi(t_{j+1}) \right\|_{L^4} \left\| \mathcal{E}_h^{\phi}(t_j) \right\|_{L^4}^2 \\
&\qquad + C \left\| \tau\delta_\tau \phi_h^{j+1} \right\|_{L^4}^2 \left\| \mathcal{E}_h^{\phi}(t_j) \right\|_{L^4}^2 + C \left| \left(\tau\delta_\tau \phi_h^{j+1}, (\mathcal{E}_h^{\phi}(t_j))^2 \right) \right| \\
&\qquad + C \left| \left(\tau\delta_\tau \phi_h^{j+1}, (\mathcal{E}_h^{\phi}(t_j))^3 \right) \right| \\
&\quad \le C\tau \left\| \nabla \mathcal{E}_h^{\phi}(t_j) \right\|^2 + C \left\| \tau\delta_\tau \nabla \phi_h^{j+1} \right\|^2 \left\| \nabla \mathcal{E}_h^{\phi}(t_j) \right\|^2 \\
&\qquad + C \left\| \tau\delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}} \left(\left\| (\mathcal{E}_h^{\phi}(t_j))^2 \right\|_{W_3^1} + \left\| (\mathcal{E}_h^{\phi}(t_j))^3 \right\|_{W_3^1} \right).
\end{aligned}
\tag{3.51}
$$

Now define

$$
I_7 := \left\| (\mathcal{E}_h^{\phi}(t_j))^2 \right\|_{W_3^1} + \left\| (\mathcal{E}_h^{\phi}(t_j))^3 \right\|_{W_3^1}.
$$

We observe that I_7 can be analyzed as

$$
\begin{aligned}
I_7 \le\ & C\left(\left\|(\mathcal{E}_h^{\phi}(t_j))^2\right\|_{L^3} + \left\|\mathcal{E}_h^{\phi}(t_j)\nabla\mathcal{E}_h^{\phi}(t_j)\right\|_{L^3} + \left\|(\mathcal{E}_h^{\phi}(t_j))^3\right\|_{L^3}\right.\\
&\left. + \left\|(\mathcal{E}_h^{\phi}(t_j))^2\nabla\mathcal{E}_h^{\phi}(t_j)\right\|_{L^3}\right)\\
\le\ & C\left(\left\|\mathcal{E}_h^{\phi}(t_j)\right\|_{L^\infty} + 1\right)\left\|\mathcal{E}_h^{\phi}(t_j)\right\|_{L^6}^2 + C\left(\left\|\mathcal{E}_h^{\phi}(t_j)\right\|_{L^\infty} + \left\|\mathcal{E}_h^{\phi}(t_j)\right\|_{L^\infty}^2\right)\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|_{L^3}\\
\le\ & C\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + C\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{4}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + C\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{4}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{7}{4}}\\
& + C\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{2}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{5}{2}}\\
\le\ & C\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + C\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{4}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{7}{4}} + C\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{2}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{3}{2}}.
\end{aligned}
\tag{3.52}
$$

Here we reduce the power of $\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|$ in some terms above according to the $L^\infty(H^1)$ bound of $\mathcal{E}_h^{\phi}$. We also appeal to the discrete Gagliardo–Nirenberg inequality (2.42) and (2.43). This is then fed into (3.51) to obtain

$$
\begin{aligned}
\left(\zeta^{j+1} - \zeta^j, \left(\mathcal{E}_h^{\phi}(t_j)\right)^2\right) \le\ & C\tau\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2\\
& + C\left(\left\|\tau\delta_\tau\nabla\phi_h^{j+1}\right\|^2 + \left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}\right)\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2\\
& + C\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{4}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{7}{4}}\\
& + C\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{1}{2}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^{\frac{3}{2}}\\
\le\ & C\tau\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2\\
& + C\left(\left\|\tau\delta_\tau\nabla\phi_h^{j+1}\right\|^2 + \left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}\right)\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2\\
& + C\tau^{-\frac{1}{7}}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^{\frac{8}{7}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + \frac{\varepsilon^3\tau}{32}\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^2\\
& + C\tau^{-\frac{1}{3}}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^{\frac{4}{3}}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + \frac{\varepsilon^3\tau}{32}\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^2.
\end{aligned}
\tag{3.53}
$$

Due to the definition of $\mathcal{A}^j$ from (3.45), we arrive at

$$
\left(\zeta^{j+1} - \zeta^j, \left(\mathcal{E}_h^{\phi}(t_j)\right)^2\right) \le\ C\mathcal{A}^j\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + \frac{\varepsilon^3\tau}{16}\left\|\nabla\Delta_h\mathcal{E}_h^{\phi}(t_j)\right\|^2. \tag{3.54}
$$

For the term $\tau \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|_{L^3}^2$ in (3.49), we apply the discrete Gagliardo–Nirenberg inequality and Young's inequality again:

$$
\begin{aligned}
.\tau \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|_{L^3}^2 &\le C\tau \left\|\nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^{\frac{1}{4}} \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|^{\frac{7}{4}} + C\tau \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|^2 \\
&\le C\tau \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|^2 + \frac{\varepsilon^3 \tau}{16} \left\|\nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2
\end{aligned}
\tag{3.55}
$$

Combining (3.49)–(3.55), we finish the proof. □

The following lemma demonstrates an approach to deal with the term $\tau \left\|\nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2$ on the right-hand-side in (3.44).

Lemma 3.12 *Suppose that (ϕ, μ) is a weak solution to (3.5a), (3.5b), with the additional regularities (3.9)–(3.12). Then, for any h, $\tau > 0$,*

$$
\begin{aligned}
\varepsilon^2 \left\|\nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2 &\le \left\|\nabla \mathcal{E}_h^\mu(t_j)\right\|^2 \\
&+ C\hat{D}_0^{j+1} \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|^2 + C \left\|\nabla \mathcal{E}_h^\phi(t_{j-1})\right\|^2 + C\hat{D}_0^{j+1}(\tau^2 + h^{2q}).
\end{aligned}
\tag{3.56}
$$

Proof Since

$$
\varepsilon^2 \left\|\nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2 \le 2 \left\|\nabla \mathcal{E}_h^\mu(t_j)\right\|^2 + 2 \left\|\nabla \mathcal{E}_h^\mu(t_j) + \varepsilon \nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2, \tag{3.57}
$$

what we need to estimate is the last term above. To bound $\left\|\nabla \mathcal{E}_h^\mu + \varepsilon \nabla \Delta_h \mathcal{E}_h^\phi\right\|$, set $\psi = \Delta_h \mathcal{E}_h^\mu + \varepsilon \Delta_h^2 \mathcal{E}_h^\phi$ in (3.8b), which in turn implies that

$$
\begin{aligned}
&\left\|\nabla \mathcal{E}_h^\mu + \varepsilon \nabla \Delta_h \mathcal{E}_h^\phi\right\|^2 \\
&= \left(\mathcal{E}_a^\mu, \Delta_h \mathcal{E}_h^\mu + \varepsilon \Delta_h^2 \mathcal{E}_h^\phi\right) + \frac{\tau}{\varepsilon} \left(\delta_\tau \phi, \Delta_h \mathcal{E}_h^\mu + \varepsilon \Delta_h^2 \mathcal{E}_h^\phi\right) \\
&\quad + \varepsilon^{-1} \left(L_\tau \mathcal{E}^\phi, \Delta_h \mathcal{E}_h^\mu + \varepsilon \Delta_h^2 \mathcal{E}_h^\phi\right) - \varepsilon^{-1} \left(\phi^3 - \hat{\phi}^3, \Delta_h \mathcal{E}_h^\mu + \varepsilon \Delta_h^2 \mathcal{E}_h^\phi\right) \\
&\le C \left(\left\|\nabla \mathcal{E}_a^\mu\right\|^2 + \tau \left\|\delta_\tau \nabla \phi\right\|^2 + \left\|\nabla \left(\phi^3 - \hat{\phi}^3\right)\right\|^2 + \left\|\nabla L_\tau \mathcal{E}^\phi\right\|^2 \right) \\
&\quad + \frac{1}{2} \left\|\nabla \mathcal{E}_h^\mu + \varepsilon \nabla \Delta_h \mathcal{E}_h^\phi\right\|^2.
\end{aligned}
\tag{3.58}
$$

Using techniques from Lemmas 3.3 and 3.6, the above norm can be controlled as

$$
\begin{aligned}
\left\|\nabla \mathcal{E}_h^\mu(t_j) + \varepsilon \nabla \Delta_h \mathcal{E}_h^\phi(t_j)\right\|^2 &\le C\tau^2 + C \left(\left\|\hat{\phi}(t_j)\right\|_{L^\infty}^4 + 1 \right) h^{2q} + C \left\|\nabla \mathcal{E}_h^\phi(t_{j-1})\right\|^2 \\
&+ C \left(\left\|\hat{\phi}(t_j)\right\|_{L^\infty}^4 + 1 \right) \left\|\nabla \mathcal{E}_h^\phi(t_j)\right\|^2
\end{aligned}
$$

$$\le C\hat{D}_0^{j+1}(\tau^2+h^{2q}) + C\hat{D}_0^{j+1}\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2 + C\left\|\nabla\mathcal{E}_h^{\phi}(t_{j-1})\right\|^2. \tag{3.59}$$

□

A combination of Lemmas 3.12 and 3.11 yields the following theorem.

Theorem 3.13 *Suppose that* (ϕ,μ) *is a weak solution to* (3.5a), (3.5b)*, with the additional regularities* (3.9)–(3.12)*. Then, for any* h, $\tau>0$*, there exists a constant* $C>0$*, independent of* h *and* τ*, such that*

$$\varepsilon\left\|\nabla\mathcal{E}_h^{\phi}(t_m)\right\|^2 + \frac{\varepsilon\tau}{8}\sum_{j=1}^{m}\left\|\nabla\mathcal{E}_h^{\mu}(t_j)\right\|^2 + \gamma\tau\sum_{j=1}^{m}\left\|\mathcal{P}_h\left(\phi_h^{j-1}\nabla\mathcal{E}_h^{\mu}(t_j)\right)\right\|^2$$

$$\le C\tau\sum_{j=1}^{m}\left(\hat{D}_0^{j+1}+\hat{D}_0^{j}\right)(\tau^2+h^{2q}) + C\sum_{j=0}^{m-1}\mathcal{A}^j\left\|\nabla\mathcal{E}_h^{\phi}(t_j)\right\|^2. \tag{3.60}$$

The summability of the sequence $\mathcal{A}^j$ is then essential to apply the discrete Gronwall inequality. We have the following lemma:

Lemma 3.14 *Suppose that* (ϕ,μ) *is a weak solution to* (3.5a), (3.5b)*, with the additional regularities* (3.9)–(3.12)*. Then, for any* $1\le m\le M$ *and any* h, $\tau>0$*, there exists a constant* $C>0$*, independent of* h *and* τ*, such that*

$$\sum_{j=0}^{m}\mathcal{A}^j \le C. \tag{3.61}$$

Proof Recalling (3.45) for the definition of $\mathcal{A}^j$, $\tau\hat{D}_0^{j+1}$ is summable due to Theorem 2.9. $\left\|\tau\delta_\tau\nabla\phi_h^{j+1}\right\|^2$ and $\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}$ are summable due to (2.32) and (2.34) respectively. For the last two terms in (3.45), it can be estimated due to the Cauchy–Schwarz inequality

$$\begin{aligned}\sum_{j=0}^{m}\tau^{-\frac{1}{7}}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^{\frac{8}{7}} &\le \left(\sum_{j=0}^{m}1\right)^{\frac{3}{7}}\left(\sum_{j=0}^{m}\tau^{-\frac{1}{4}}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^2\right)^{\frac{4}{7}}\\ &\le C\tau^{-\frac{3}{7}}\left(\sum_{j=0}^{m}\tau^{-\frac{1}{4}}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^2\right)^{\frac{4}{7}}\\ &\le C\left(\sum_{j=0}^{m}\tau^{-1}\left\|\tau\delta_\tau\phi_h^{j+1}\right\|_{W_3^{1*}}^2\right)^{\frac{4}{7}} \le C,\end{aligned} \tag{3.62}$$

and

$$
\begin{aligned}
\sum_{j=0}^{m} \tau^{-\frac{1}{3}} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{\frac{4}{3}} &\le \left(\sum_{j=0}^{m} 1 \right)^{\frac{1}{3}} \left(\sum_{j=0}^{m} \tau^{-\frac{1}{2}} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{2} \right)^{\frac{2}{3}} \\
&\le C \tau^{-\frac{1}{3}} \left(\sum_{j=0}^{m} \tau^{-\frac{1}{2}} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{2} \right)^{\frac{2}{3}} \\
&\le C \left(\sum_{j=0}^{m} \tau^{-1} \left\| \tau \delta_\tau \phi_h^{j+1} \right\|_{W_3^{1*}}^{2} \right)^{\frac{4}{7}} \le C. \qquad (3.63)
\end{aligned}
$$

□

3.3 Main convergence result

Applying the discrete Gronwall inequality to (3.60), we get the optimal convergence rate for the numerical scheme.

Theorem 3.15 *Suppose that* (ϕ, μ) *is a weak solution to* (3.5a), (3.5b)*, with the additional regularities* (3.9)–(3.12)*. Then, for any* h*,* $\tau > 0$*, there exists a constant* $C > 0$*, independent of* h *and* τ*, such that*

$$
\begin{aligned}
&\varepsilon \left\| \nabla \mathcal{E}_h^{\phi}(t_m) \right\|^2 + \frac{\varepsilon \tau}{8} \sum_{j=1}^{m} \left\| \nabla \mathcal{E}_h^{\mu}(t_j) \right\|^2 \\
&\quad + \gamma \tau \sum_{j=1}^{m} \left\| \mathcal{P}_h \left(\phi_h^{j-1} \nabla \mathcal{E}_h^{\mu}(t_j) \right) \right\|^2 \le \; C(\tau^2 + h^{2q}). \qquad (3.64)
\end{aligned}
$$

Remark 3.16 A combination of (3.59) and (3.64) yields that

$$
\tau \sum_{j=1}^{m} \left\| \nabla \Delta_h \mathcal{E}_h^{\phi}(t_j) \right\|^2 \le C(\tau^2 + h^{2q}). \qquad (3.65)
$$

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Appendix 1: Discrete Gronwall inequality

We need the following discrete Gronwall inequality, cited in [26,30]:

Lemma 4.1 *Fix $T > 0$, and suppose $\{a_m\}_{m=1}^{M}$, $\{b_m\}_{m=1}^{M}$ and $\{c_m\}_{m=1}^{M-1}$ are non-negative sequences such that $\tau \sum_{m=1}^{M-1} c_m \le C_1$, where C_1 is independent of τ and M, and $M \cdot \tau = T$. Suppose that, for all $\tau > 0$,*

$$a_M + \tau \sum_{m=1}^{M} b_m \le C_2 + \tau \sum_{m=1}^{M-1} a_m c_m, \tag{4.1}$$

where $C_2 > 0$ is a constant independent of τ and M. Then, for all $\tau > 0$,

$$a_M + \tau \sum_{m=1}^{M} b_m \le C_2 \exp\left(\tau \sum_{m=1}^{M-1} c_m\right) \le C_2 \exp(C_1). \tag{4.2}$$

Note that the sum on the right-hand-side of (4.1) must be explicit.

Lemma 4.2 *Suppose $\{a_m\}_{m=1}^{M}$ and $\{b_m\}_{m=0}^{M}$ are sequences such that $b_0 = 0$. Define, for any integer m, $1 \le m \le M$,*

$$I_m := \sum_{j=1}^{m} a_j b_j (b_j - b_{j-1}). \tag{4.3}$$

Then the following identity is valid:

$$I_m = -\frac{1}{2} \sum_{j=1}^{m} (a_j - a_{j-1}) b_{j-1}^2 + \frac{1}{2} \sum_{j=1}^{m} a_j (b_j - b_{j-1})^2 + \frac{1}{2} a_m b_m^2. \tag{4.4}$$

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