Global Weak Solution of Planetary Geostrophic Equations with Inviscid Geostrophic Balance

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Abstract. A reformulation of the planetary geostrophic equations (PGEs) with inviscid balance equation is proposed and the existence of global weak solutions is established, provided that the mechanical forcing satisfies an integral constraint. There is only one prognostic equation for the temperature field and the velocity field is statically determined by the planetary geostrophic balance combined with the incompressibility condition. Furthermore, the velocity profile can be accurately represented as a functional of the temperature gradient. In particular, the vertical velocity depends only on the first order derivative of the temperature. As a result, the bound for the $L^\infty(0,t_1;L^2)\cap L^2(0,t_1;H^1)$ norm of the temperature field is sufficient to show the existence of the weak solution.

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1. Introduction

The planetary geostrophic equations (PGEs) have played an important role in large-scale ocean circulation since the pioneering work of A. Robinson and H. Stommel [6] and P. Welander [12]. This system arises as an asymptotic approximation to the primitive equations (PEs) for planetary-scale motions in the limit of small Rossby number. The PGEs are considerably simpler than the PEs, but retain the dynamics necessary to represent the large-scale, low-frequency dynamics of the mid-latitude oceans.

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The PGEs with viscous geostrophic balance has been analyzed at the PDE level in recent articles. See [1, 8, 9] for relevant discussions. One distinguishing feature of the PGEs is that there is only one prognostic equation in the system for the temperature field; the velocity field is diagnostically determined by the planetary geostrophic balance. The addition of a diffusion term in the geostrophic balance equation is for the sake of simplicity in mathematical analysis, due to the lack of regularity for the velocity field by a straightforward manipulation.

In this article, we consider the original formulation of the PGEs, with no viscous term in the geostrophic balance equations. In such a formulation, both horizontal and vertical velocity profiles are accurately represented as functionals of the temperature gradient. The representation formula for the horizontal velocity field is based on the planetary geostrophic balance. Since every variable can be uniquely determined by the combination of its mean (in vertical direction) and its vertical derivative, the horizontal velocity turns out to be the solution of a differential equation at each fixed horizontal point, depending only on the temperature gradient. The vertical velocity can be recovered by the continuity equation. Using the special form of the Coriolis parameter, we arrive at a two-point boundary value ordinary differential equations (O.D.E.) in the vertical direction at each fixed horizontal point for the vertical velocity, with the right hand side depending only on the first order derivative of the temperature field.

The new formulation is derived in Section 2 and the existence of the global (in time) weak solution of the reformulated PGEs is provided in Section 3. The approach of Galerkin approximation is used. Standard energy estimate for the temperature equation gives the bound of the $L^\infty(0,t_1;L^2)\cap L^2(0,t_1;H^1)$ norm of the temperature variable, which in turn shows the bound of the $L^2(0,t_1;L^2)$ norm of the horizontal velocity. Moreover, the $L^2(0,t_1;L^2)$ norm of the vertical velocity is also uniformly bounded since it satisfies the second order O.D.E., in which the force term only involves the temperature gradient. The compactness for the time derivative of the temperature field can be established in a similar manner. Thus the existence of the global weak solution is proven.

2. Reformulation of the Inviscid Planetary Geostrophic Equations

The non-dimensional PGEs can be written as

$$
\begin{cases}
T_t + (\boldsymbol{v}\cdot\nabla)T + w\dfrac{\partial T}{\partial z} = \Big(\dfrac{1}{Rt_1}\triangle + \dfrac{1}{Rt_2}\partial_z^2\Big)T\,, \\
fk\times\boldsymbol{v} + \nabla p = F\,, \\
\dfrac{\partial p}{\partial z} = T\,, \\
\nabla\cdot\boldsymbol{v} + \partial_z w = 0\,,
\end{cases}
\tag{2.1}
$$

where T represents the temperature, $\boldsymbol{v}=(u,v)$ the horizontal velocity, w the vertical velocity, and p the pressure. The term $fk\times\boldsymbol{v}$ corresponds to the Coriolis force with f depending only on the latitude y. As a typical example used in geophysical literatures, its $\beta-$plane approximation is given by $f=f_0+\beta y$. The parameters $1/Rt_1$, $1/Rt_2$ stand for the horizontal and vertical heat conductivity coefficients. The operators ∇, $\nabla^\perp$, $\nabla\cdot$, $\triangle$ stand for the gradient, perpendicular gradient, divergence and Laplacian in horizontal plane, respectively. For simplicity, we set $\kappa_1=1/Rt_1$, $\kappa_2=1/Rt_2$. The forcing term $F=(F^x,F^y)^\perp$ appearing in the geostrophic balance equation $(2.1)_2$ comes from the wind stress at the ocean surface, which is a boundary layer approximation. It may or may not depend on the vertical variable z. For simplicity, we assume in this article $F=F(x,y)=\big(F^x(x,y),F^y(x,y)\big)$. The discussion of a general case can be carried out in the same fashion and does not add any mathematical difficulty. See the relevant references on both the physical and mathematical descriptions of the PGEs in [2, 3, 4, 5, 6, 8, 9, 10, 12], etc.

The computational domain is taken as $\mathcal{M}=\mathcal{M}_0\times[-H_0,0]$, where $\mathcal{M}_0$ is the surface of the ocean. The boundary condition at the top and bottom surfaces are given by

$$
\begin{aligned}
&w=0 \quad \text{and} \quad \kappa_2\frac{\partial T}{\partial z}=T^f\,, \quad \text{at } z=0\,,\\
&w=0 \quad \text{and} \quad \kappa_2\frac{\partial T}{\partial z}=0\,, \quad \text{at } z=-H_0\,,
\end{aligned}
\tag{2.2}
$$

where the term T^f represents the heat flux at the surface of the ocean. Usually T^f can be taken as either a fixed heat flux function or of the form $T^f=-\alpha(T-\theta^*)$, where θ^* is a reference temperature. Both boundary conditions can be dealt with in an efficient way. In this article for simplicity we choose T^f as a given flux. On the lateral boundary section $\partial\mathcal{M}_0\times[-H_0,0]$, the fixed boundary condition is prescribed for the temperature field

$$
T=T_{lb}\,, \quad \text{on } \partial\mathcal{M}_0\times[-H_0,0]\,,
\tag{2.3}
$$

where T_{lb} is a given distribution. As will be shown later, the purpose of the choice of a Dirichlet boundary condition is to facilitate the analysis of the system at the PDE level, although the no-flux boundary condition for the temperature field is physically more relevant. The boundary condition (2.3) can also be viewed as an approximation such that the disturbance of oceanic circulation motion is far away from the lateral boundary. For simplicity of the presentation, we set the homogeneous profile $T_{lb} = 0$ in the theoretical and numerical analysis. There is no real change for the non-homogeneous case. The normal component of the vertically averaged horizontal velocity turns out to have a vanishing flux

$$\overline{\boldsymbol{v}}\cdot\boldsymbol{n} = 0\,, \quad \text{on } \partial\mathcal{M}_0\,, \tag{2.4}$$

which is compatible with the continuity equation $(2.1)_4$. We recall that the average (in the vertical direction) of any 3-D field g is given by $\overline{g}(x,y) = \frac{1}{H_0}\int_{-H_0}^{0} g(x,y,z)\,dz$. See [4, 8] for a detailed explanation for the choice of this nonlocal boundary condition in the case where no viscosity is present in the geostrophic balance equation.

We now derive an equivalent formulation of the system of PGEs (2.1)-(2.4). The key point in this reformulation is that both the horizontal and vertical velocity variables can be determined by the first order derivative of the temperature field. This makes valid the analysis of the well-posedness for the system.

The horizontal velocity field is the solution of the following system

$$\begin{cases} \partial_z u = \dfrac{-T_y}{f}\,, \qquad \partial_z v = \dfrac{T_x}{f}\,, \\ \overline{u}(x,y) = \overline{u}_e\,, \quad \overline{v}(x,y) = \overline{v}_e = \dfrac{-\partial_y F^x + \partial_x F^y}{\partial_y f}\,, \end{cases} \tag{2.5}$$

where $\overline{u}_e$ is explicitly given below.

Equation (2.5) is derived from the geostrophic equation and hydrostatic equation. Taking the vertical derivative of the geostrophic balance equation $fk\times\boldsymbol{v}+\nabla p = F$ gives the thermal wind equation

$$\boldsymbol{v}_z = \frac{\nabla^{\perp} p_z + \partial_z F^{\perp}}{f} = \frac{\nabla^{\perp} T}{f}\,, \qquad \text{i.e.} \quad u_z = \frac{-T_y}{f}\,, \quad v_z = \frac{T_x}{f}\,, \tag{2.6}$$

where the hydrostatic balance $\partial p/\partial z = T$ and the independence on z of F and $f = f(y)$ were used. In other words, the profile $\boldsymbol{v}_z$ can be expressed by the temperature gradient.

Meanwhile, integrating the geostrophic balance equation $fk \times \boldsymbol{v} + \nabla p = F$ in the vertical direction and dividing by H_0, we find

$$fk \times \overline{\boldsymbol{v}} + \nabla \overline{p} = F\,. \tag{2.7}$$

Applying the curl operator $\nabla^\perp$ to (2.7) results in

$$(\partial_y f)\overline{v} + f(\overline{v}_y + \overline{u}_x) = \nabla \times F = -\partial_y F^x + \partial_x F^y\,. \tag{2.8}$$

Moreover, the continuity equation $\nabla \cdot \boldsymbol{v} + \partial_z w = 0$ and the boundary condition for the vertical velocity $w(\,\cdot, 0) = w(\,\cdot, -H_0) = 0$ show that the averaged horizontal velocity field is divergence-free, i.e.,

$$\nabla \cdot \overline{\boldsymbol{v}} = 0\,. \tag{2.9}$$

The combination of (2.8) and (2.9) yields

$$\overline{v}_e = \overline{v}(x, y) = \frac{-\partial_y F^x + \partial_x F^y}{\partial_y f}\,. \tag{2.10}$$

By taking the tangential part of (2.7) and applying (2.4), one obtains the vertically averaged tangential pressure gradient (and thus the pressure, aside from an arbitrary constant) on the boundary from the tangential component of the forcing F:

$$\frac{\partial \overline{p}}{\partial \tau} = F \cdot \tau\,, \tag{2.11}$$

where τ is the unit tangential vector on the boundary. Integrating this relation around the boundary, one obtains the constraint that the line integral of F around the boundary must be zero:

$$\int_{\partial \mathcal{M}'} F \cdot \tau \, dl = 0\,. \tag{2.12}$$

This means that the forcing must not give a net torque on the fluid.

From (2.9), we can find a 2-D mean stream function $\overline{\psi}(x, y)$ for the vertically averaged velocity field, such that $(\overline{u}, \overline{v}) = (-\partial_y \overline{\psi}, \partial_x \overline{\psi})$. Moreover, the boundary condition (2.4) indicates that $\overline{\psi}$ is a constant on the lateral boundary. For simplicity of the discussion, we take $\overline{\psi} = 0$ on $\partial \mathcal{M}_0$. In addition, we denote $\gamma_1(y_0)$, $\gamma_2(y_0)$ by the x-coordinates of the intersection points between $\partial\mathcal{M}_0$ and $y = y_0$. The mean stream function $\overline{\psi}$ and the mean velocity $\overline{u}$ can be determined by the kinematic relationship and formula (2.10):

$$\overline{\psi}_e(x,y) = \int_{\gamma_1(y)}^{x} \frac{\nabla\times F}{\partial_y f}\,dx'\,, \qquad \overline{u}_e(x,y) = -\partial_y\overline{\psi}_e(x,y)\,, \tag{2.13}$$

with $\gamma_1(y)$ being a point on $\partial\mathcal{M}_0$. Evaluating $\overline{\psi}$ at another boundary point $(\gamma_2(y), y)$ with the same y-value, we obtain an additional constraint on the forcing:

$$\int_{\gamma_1(y)}^{\gamma_2(y)} \frac{\nabla\times F}{\partial_y f}\,dx' = 0\,, \tag{2.14}$$

since $\overline{\psi}$ is identically 0 on the lateral boundary. Constraint (2.14) amounts to saying that the average forcing across the domain at a fixed y must not give a torque on the fluid.

The combination of (2.6), (2.10) and (2.13) leads to the system (2.5). By the representation formula valid for any 3-D variable g:

$$g(x,y,z) = \int_{-H_0}^{z} g_z(x,y,z_1)\,dz_1 + \overline{g}(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} g_z(x,y,z_1)\,dz_1\,dz\,, \tag{2.15}$$

the solution of (2.5) can be expressed explicitly using an integration formula:

$$u(x,y,z) = -\int_{-H_0}^{z} \frac{T_y}{f}(x,y,z_1)\,dz_1 + \overline{u}_e(x,y) + \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{T_y}{f}(x,y,z_1)\,dz_1\,dz\,, \tag{2.16a}$$

$$v(x,y,z) = \int_{-H_0}^{z} \frac{T_x}{f}(x,y,z_1)\,dz_1 + \overline{v}_e(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{T_x}{f}(x,y,z_1)\,dz_1\,dz\,, \tag{2.16b}$$

with $\overline{u}_e$, $\overline{v}_e$ given by (2.10) and (2.13).

The vertical velocity can be calculated by integrating the horizontal divergence of the horizontal velocity field, due to the incompressibility condition

$$w(x,y,z) = -\int_{-H_0}^{z} \nabla\cdot\boldsymbol{v}(x,y,s)\,ds\,. \tag{2.17}$$

The substitution of (2.16) into (2.17) gives

$$\begin{aligned} w(x,y,z) = &\int_{-H_0}^{z}\int_{-H_0}^{z_2} \frac{(\partial_y f)T_x}{f^2}(x,y,z_1)\,dz_1\,dz_2 \\ &-\frac{1}{H_0}(z+H_0)\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{(\partial_y f)T_x}{f^2}(x,y,z_1)\,dz_1\,dz\,. \end{aligned} \tag{2.18}$$

Therefore, w can be expressed as a functional of the temperature gradient, like the horizontal velocity $\boldsymbol{v}$.

The vertical velocity can also be represented as the solution of a differential equation. By taking the vertical derivative of the continuity equation

$$\nabla \cdot \boldsymbol{v}_z + \partial_z^2 w = 0\,, \tag{2.19}$$

combined with (2.6), we arrive at

$$\partial_z^2 w = -\partial_x(u_z) - \partial_y(v_z) = \partial_x\Big(\frac{T_y}{f}\Big) - \partial_y\Big(\frac{T_x}{f}\Big) = \frac{(\partial_y f)T_x}{f^2}\,. \tag{2.20}$$

It can be observed that the second order derivatives for the temperature field cancel each other due to the special form of the Coriolis parameter $f = f(y)$. Therefore, the vertical velocity w can be reformulated as the solution of the following system of second order O.D.E.s

$$\begin{cases} \partial_z^2 w = \dfrac{(\partial_y f)T_x}{f^2}\,, \\ w = 0\,, \qquad \text{at} \quad z = 0\,, -H_0\,, \end{cases} \tag{2.21}$$

in which the right hand side includes only the first order derivative of the temperature. This key point is crucial to the analysis presented below.

We then have the following formulation, where the velocities are expressed as functionals of the temperature gradient.

Temperature Transport Equation

$$\begin{cases} T_t + (\boldsymbol{v}\cdot\nabla)T + w\dfrac{\partial T}{\partial z} = \Big(\kappa_1 \triangle + \kappa_2 \partial_z^2\Big)T\,, \\ \dfrac{\partial T}{\partial z} = T_f\,, \quad \text{at } z = 0\,, \qquad \dfrac{\partial T}{\partial z} = 0\,, \quad \text{at } z = -H_0\,, \\ T = 0\,, \quad \text{on } \partial\mathcal{M}_0 \times [-H_0, 0]\,; \end{cases} \tag{2.22a}$$

Recovery of the horizontal velocity

$$\begin{cases} \partial_z u = -\dfrac{T_y}{f}\,, \qquad \partial_z v = \dfrac{T_x}{f}\,, \\ \overline{u}(x,y) = \overline{u}_e\,, \quad \overline{v}(x,y) = \overline{v}_e\,; \end{cases} \tag{2.22b}$$

Recovery of the vertical velocity

$$\text{(2.22c)} \qquad \begin{cases} \partial_z^2 w = \dfrac{(\partial_y f) T_x}{f^2}\,, \\ w = 0\,, \qquad \text{at} \quad z = 0\,,\, -H_0\,. \end{cases}$$

Remark 2.1. *It is observed that the alternate formulation (2.22a-c) is equivalent to the original formulation (2.1)-(2.4) of the PGEs, from which they were derived. Indeed, to recover (2.1)-(2.4) from (2.22a-c), we need to show that $\phi = \nabla\cdot\boldsymbol{v} + \partial_z w \equiv 0$. A simple calculation utilizing (2.22b) and (2.22c) leads to*

$$\text{(2.23)} \quad \partial_z\phi = \partial_z\Big(\nabla\cdot\boldsymbol{v} + \partial_z w\Big) = \partial_x(\partial_z u) + \partial_y(\partial_z v) + \partial_z^2 w = -\partial_x\Big(\frac{T_y}{f}\Big) + \partial_y\Big(\frac{T_x}{f}\Big) + \frac{(\partial_y f) T_x}{f^2} = 0\,,$$

$$\text{(2.24)} \qquad \overline{\phi} = \overline{\Big(\nabla\cdot\boldsymbol{v} + \partial_z w\Big)} = \nabla\cdot\overline{\boldsymbol{v}} + \overline{\partial_z w} = 0\,,$$

since the average of $\boldsymbol{v}$ is divergence-free in the horizontal plane. The combination of (2.23) and (2.24) results in the incompressibility condition. In addition, a direct calculation

$$\text{(2.25)} \qquad \nabla\times(fk\times\overline{\boldsymbol{v}} - \overline{F}) = (\partial_y f)\overline{v} - \nabla\times F = 0\,,$$

indicates the existence of a mean pressure field $\overline{p}$ such that $fk\times\overline{\boldsymbol{v}} + \nabla\overline{p} = \overline{F}$. Accordingly, we define the total pressure field as

$$\text{(2.26)} \qquad p(x,y,z) = \int_{-H_0}^{z} T(x,y,s)\,ds + \overline{p}(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} T(x,y,s)\,ds\,dz\,.$$

Clearly, the hydrostatic balance is satisfied by taking the vertical derivative of (2.26). The geostrophic balance equation can also be verified by using the integration formulas for $\boldsymbol{v}$ in (2.16), which comes from the recovery equation (2.22b), combined with the horizontal gradient of (2.26):

$$\text{(2.27)} \qquad \nabla p = \int_{-H_0}^{z} \nabla T\,ds + \nabla\overline{p} - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \nabla T\,ds\,dz\,.$$

All the boundary conditions presented in (2.1)-(2.4) are included in the system (2.22). Hence, $(T, \boldsymbol{u}, p)$ is also a solution of (2.1)-(2.4). This completes the proof of the formal equivalence of smooth solutions between the two formulations.

3. Existence of a Global Weak Solution

Before starting the discussion on the weak solutions, we introduce the following functional setting:

$$
\begin{aligned}
&H = L^2(\mathcal{M})\,, \qquad V = \text{ the closure of } C^\infty_{lat,0}(\mathcal{M}) \text{ in } H^1(\mathcal{M})\;, \\
&C^m_{lat,0}(\mathcal{M}) = \{T \in C^m(\mathcal{M}) \mid \qquad T = 0 \quad \text{on } \partial\mathcal{M}_0 \times [-H_0, 0]\}\;.
\end{aligned} \tag{3.1}
$$

Note that the introduction of $C^m_{lat,0}(\mathcal{M})$ is motivated by the boundary condition for T on the lateral boundary sections. Let $(\,\cdot,\,\cdot)$ be the inner product in $L^2(\mathcal{M}) = L^2(\mathcal{M}_0 \times [-H_0, 0])$, and $\|\cdot\|$ the corresponding L^2 norm.

For any positive final time $t_1 > 0$, the functions (T, u, v, w)

$$
T \in L^\infty\Big(0, t_1\,; H\Big) \cap L^2\Big(0, t_1\,; V\Big)\,, \quad u\,, v\,, w \in L^2\Big(0, t_1\,; L^2(\mathcal{M})\Big)\,,
$$

are called a weak solution of the original PGEs formulated in (2.22) if

(3.2a)

$$
\begin{aligned}
&\int_{\mathcal{M}_0}\int_{-H_0}^{0} \Big(\partial_t(T\phi) + uT\phi_x + vT\phi_y + wT\phi_z + \kappa_1(\nabla T)\cdot(\nabla\phi) + \kappa_2(\partial_z T)\cdot(\partial_z\phi)\Big)\, d\boldsymbol{x}\, dz \\
&\qquad + \kappa_2 \int_{\mathcal{M}_0} \frac{\alpha}{\kappa_2}(T(\boldsymbol{x}, 0) - \theta^*)\phi(\boldsymbol{x}, 0)\, d\boldsymbol{x} = 0\,, \qquad \forall \phi \in C^1_{lat,0}(\mathcal{M}) \cap H^3(\mathcal{M})\,,
\end{aligned}
$$

where

$$
\begin{aligned}
u(x,y,z) &= -\int_{-H_0}^{z} \frac{T_y}{f}(x,y,z_1)\, dz_1 + \overline{u}_e(x,y) + \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{T_y}{f}(x,y,z_1)\, dz_1\, dz\,, \\
v(x,y,z) &= \int_{-H_0}^{z} \frac{T_x}{f}(x,y,z_1)\, dz_1 + \overline{v}_e(x,y) - \frac{1}{H_0}\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{T_x}{f}(x,y,z_1)\, dz_1\, dz\,,
\end{aligned} \tag{3.2b}
$$

$$
\begin{aligned}
w(x,y,z) = &\int_{-H_0}^{z}\int_{-H_0}^{z_2} \frac{(\partial_y f)T_x}{f^2}(x,y,z_1)\, dz_1\, dz_2 \\
&- \frac{1}{H_0}(z + H_0)\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{(\partial_y f)T_x}{f^2}(x,y,z_1)\, dz_1\, dz\,.
\end{aligned} \tag{3.2c}
$$

Theorem 3.1 *Suppose $F \in H^2(\mathcal{M}_0)$ is given and the constraint (2.14) is satisfied so that $\overline{u}_e$, $\overline{v}_e$ can be consistently determined. Let $T_0 = T(\cdot, 0) \in L^2(\mathcal{M})$. Then there exists at least one global weak solution for the PGEs (2.22), such that for any $t_1 > 0$*

$$
\begin{aligned}
&T \in L^\infty\Big(0, t_1\,; H\Big) \cap L^2\Big(0, t_1\,; V\Big)\,, \quad \partial_t T \in L^{\frac{4}{3}}\Big(0, t_1\,; H^{-2}(\mathcal{M})\Big)\,, \\
&u\,, v\,, w \in L^2\Big(0, t_1\,; L^2(\mathcal{M})\Big)\,, \quad \partial_z u\,, \partial_z v\,, \partial_z^2 w \in L^2\Big(0, t_1\,; L^2(\mathcal{M})\Big)\,,
\end{aligned} \tag{3.3}
$$

(3.4)
$$\|T(\cdot,t)\|^2 + 2\int_0^t \Big(\kappa_1\|\nabla T(\cdot,s)\|^2 + \kappa_2\|\partial_z T(\cdot,s)\|^2\Big)\,ds \le \|T_0\|^2 + C^* t\,, \quad \text{for } 0<t<t_1\,,$$
$$\text{with } C^* = \alpha \int_{\mathcal{M}_0} (\theta^*)^2\, d\boldsymbol{x}\,,$$

$$\nabla\cdot\int_{-H_0}^0 \boldsymbol{v}\,dz = 0\,, \qquad \text{in the sense of distribution}\,,$$
(3.5)
$$\int_{-H_0}^0 \boldsymbol{v}\,dz\cdot\boldsymbol{n} = 0\,, \qquad \text{on} \quad \partial\mathcal{M}_0\,,$$
$$w = 0\,, \qquad \text{at} \quad z = 0\,, -H_0\,.$$

Proof. The proof can be accomplished by the Galerkin procedure. A standard energy estimate is used to obtain the uniform bound for the $L^\infty(0,t_1\,;L^2(\mathcal{M}))$ and $L^2(0,t_1\,;H^1(\mathcal{M}))$ norms of the temperature field.

Let $\{\Phi_j\}_{j\ge 1}\subset H^2(\mathcal{M})$ be the eigenvectors of the diffusion operator corresponding to the eigenvalues λ_j, $j=1,2,...$, such that

(3.6)
$$\begin{cases} \Big(\kappa_1\triangle + \kappa_2\partial_z^2\Big)\Phi_j = \lambda_j\Phi_j\,, \qquad \lambda_j\to\infty\,,\\ \dfrac{\partial\Phi_j}{\partial z} = -\alpha\dfrac{\Phi_j}{\kappa_2}\,, \quad \text{at } z=0\,, \qquad \dfrac{\partial\Phi_j}{\partial z} = 0\,, \quad \text{at } z=-H_0\,,\\ \Phi_j = 0\,, \quad \text{on } \partial\mathcal{M}_0\times[-H_0,0]\,. \end{cases}$$

The diffusion operator $A=\kappa_1\triangle+\kappa_2\partial_z^2$ with the given boundary condition is a self-adjoint linear operator and admits a compact inverse. Then $\{\Phi_n\}_{n\ge 1}$ defines a complete orthogonal basis in $L^2(\mathcal{M})$. To seek a weak solution of the reformulated PGEs defined in (3.2), we find an approximate solution $\{T_m\}$ such that

(3.7)
$$T_m(\boldsymbol{x},z;t) = \sum_{j=1}^m \beta_j^m(t)\Phi_j(\boldsymbol{x},z)\,,$$

(3.8)
$$\frac{d}{dt}\Big(T_m,\Phi_j\Big) + \Big(\boldsymbol{v}_m T_m,\nabla\Phi_j\Big) + \Big(w_m T_m,\partial_z\Phi_j\Big) + \kappa_1\Big(\nabla T_m,\nabla\Phi_j\Big) + \kappa_2\Big(\partial_z T_m,\partial_z\Phi_j\Big)$$
$$+\kappa_2\int_{\mathcal{M}_0}\frac{\alpha}{\kappa_2}(T_m(\boldsymbol{x},0)-\theta^*)\Phi_j(\boldsymbol{x},0)\,d\boldsymbol{x} = 0\,, \qquad j=1,2,...,m\,,$$

(3.9)
$$u_m(x,y,z) = -\int_{-H_0}^z \frac{\partial_y T_m}{f}(x,y,z_1)\,dz_1 + \overline{u}_e(x,y) + \frac{1}{H_0}\int_{-H_0}^0\int_{-H_0}^z \frac{\partial_y T_m}{f}(x,y,z_1)\,dz_1\,dz\,,$$
$$v_m(x,y,z) = \int_{-H_0}^z \frac{\partial_x T_m}{f}(x,y,z_1)\,dz_1 + \overline{v}_e(x,y) - \frac{1}{H_0}\int_{-H_0}^0\int_{-H_0}^z \frac{\partial_x T_m}{f}(x,y,z_1)\,dz_1\,dz\,,$$

$$
\begin{aligned}
w_m(x,y,z) = & \int_{-H_0}^{z}\int_{-H_0}^{z_2} \frac{(\partial_y f)\partial_x T_m}{f^2}(x,y,z_1)\,dz_1\,dz_2 \\
& -\frac{1}{H_0}(z+H_0)\int_{-H_0}^{0}\int_{-H_0}^{z} \frac{(\partial_y f)\partial_x T_m}{f^2}(x,y,z_1)\,dz_1\,dz\,,
\end{aligned} \tag{3.10}
$$

$$
T_m\mid_{t=0} = P_m T_0\,, \tag{3.11}
$$

where P_m is the orthogonal projection operator in $L^2(\mathcal{M})$:
P_m: $L^2(\mathcal{M}) \to Span\,\{\Phi_1, ..., \Phi_m\}$.

The scheme (3.8) and (3.11) proposes an initial value problem for a system of m O.D.E.s, with the velocities determined by (3.9) and (3.10). Therefore, it is straightforward to conclude the local (in time) existence of the approximate solution. To get the global (in time) solution, the energy estimates are necessary.

We observe that the approximated velocity field $\boldsymbol{u}_m = (\boldsymbol{v}_m, w_m)$ satisfies

$$
\begin{aligned}
& \nabla\cdot\boldsymbol{v}_m + \partial_z w_m = 0\,, \\
& \boldsymbol{v}_m\cdot\boldsymbol{n}\mid_{\partial\mathcal{M}_0\times[-H_0,0]} = \overline{\boldsymbol{v}}_e\cdot\boldsymbol{n}\mid_{\partial\mathcal{M}_0\times[-H_0,0]} = 0\,, \qquad w_m\mid_{z=0,-H_0} = 0\,,
\end{aligned} \tag{3.12}
$$

which comes from the construction (3.9), (3.10) and the homogeneous Dirichlet boundary condition for T_m on the lateral boundary. As a result, we find by integration by parts that

$$
\Big(\boldsymbol{v}_m T_m, \nabla T_m\Big) + \Big(w_m T_m, \partial_z T_m\Big) = 0\,. \tag{3.13}
$$

Multiplying (3.8) by $\beta_j^m(t)$ and adding up the resulting equations leads to

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|T_m\|^2 + \kappa_1\|\nabla T_m\|^2 + \kappa_2\|\partial_z T_m\|^2 = & -\kappa_2\int_{\mathcal{M}_0}\frac{\alpha}{\kappa_2}\Big(T_m(\cdot,0)-\theta^*\Big)T_m(\cdot,0)\,d\boldsymbol{x} \\
= & -\alpha\int_{\mathcal{M}_0}\Big(T_m(\cdot,0)-\theta^*\Big)T_m(\cdot,0)\,d\boldsymbol{x} \\
\leq & -\frac{\alpha}{2}\int_{\mathcal{M}_0} T_m^2(\cdot,0)\,d\boldsymbol{x} + \frac{\alpha}{2}\int_{\mathcal{M}_0}(\theta^*)^2\,d\boldsymbol{x} \\
\leq & \ \frac{\alpha}{2}\int_{\mathcal{M}_0}(\theta^*)^2\,d\boldsymbol{x}\,.
\end{aligned} \tag{3.14}
$$

Applying the Gronwall inequality to (3.14) results in

$$
\|T_m(\cdot,t)\|^2 + 2\int_0^t (\kappa_1\|\nabla T_m\|^2 + \kappa_2\|\partial_z T_m\|^2)\,ds \leq \|T_0\|^2 + C^* t\,, \quad \text{with } C^* = \alpha\int_{\mathcal{M}_0}(\theta^*)^2\,d\boldsymbol{x}\,, \tag{3.15}
$$

which in turn indicates that

$$T_m \in \text{a bounded set of } L^\infty\Big(0,t_1\,;L^2(\mathcal{M})\Big)\cap L^2\Big(0,t_1\,;H^1(\mathcal{M})\Big)\,. \tag{3.16}$$

Moreover, by the recovery formulation (3.9), (3.10), we have

$$\begin{aligned} &u_m\,,v_m\,,w_m \in \text{a bounded set of } L^2\Big(0,t_1\,;L^2(\mathcal{M})\Big)\,,\\ &\partial_z u_m\,,\partial_z v_m\,,\partial_z^2 w_m \in \text{a bounded set of } L^2\Big(0,t_1\,;L^2(\mathcal{M})\Big)\,. \end{aligned} \tag{3.17}$$

Furthermore, we need an estimate for $\partial_t T_m$ so that we can obtain compactness and a strong convergence result. Consider $\tilde{T}\in H^2(\mathcal{M})$ given by $\tilde{T}=\sum_{j=1}^\infty \tilde{\beta}_j\Phi_j$. Equation (3.8) shows that

$$\begin{aligned} \Big(\partial_t T_m\,,\tilde{T}\Big) &= \Big(\partial_t T_m\,,P_m\tilde{T}\Big)\\ &= \Big(\boldsymbol{v}_m T_m\,,\nabla P_m\tilde{T}\Big)+\Big(w_m T_m\,,\partial_z P_m\tilde{T}\Big)+\kappa_1\Big(\nabla T_m,\nabla P_m\tilde{T}\Big)+\kappa_2\Big(\partial_z T_m,\partial_z P_m\tilde{T}\Big)\\ &\quad+\kappa_2\int_{\mathcal{M}_0}\frac{\alpha}{\kappa_2}(T_m(\boldsymbol{x},0)-\theta^*)P_m\tilde{T}(\,\cdot,0)\,d\boldsymbol{x}\,. \end{aligned} \tag{3.18}$$

Regarding the nonlinear term, we have

$$\left|\int_{\mathcal{M}}\boldsymbol{v}_m T_m\,\nabla P_m\tilde{T}\,d\boldsymbol{x}\right| \le \|\boldsymbol{v}_m\|_{L^2}\|T_m\|_{L^3}\|\nabla P_m\tilde{T}\|_{L^6}\le C\|\boldsymbol{v}_m\|\|T_m\|^{1/2}\|T_m\|_{H^1}^{1/2}\|P_m\tilde{T}\|_{H^2}\,. \tag{3.19}$$

It is observed that

$$\begin{aligned} &\frac{\partial P_m\tilde{T}}{\partial z}=-\alpha\frac{P_m\tilde{T}}{\kappa_2}\,,\quad \text{at } z=0\,,\qquad \frac{\partial P_m\tilde{T}}{\partial z}=0\,,\quad \text{at } z=-H_0\,,\\ &P_m\tilde{T}=0\,,\quad \text{on } \partial\mathcal{M}_0\times[-H_0,0]\,, \end{aligned} \tag{3.20}$$

since $P_m\tilde{T}=\sum_{j=1}^m\tilde{\beta}_j\Phi_j$ and each Φ_j satisfies the boundary condition in (3.6). Consequently, an application of the elliptic regularity for $P_m\tilde{T}$ gives

$$\|P_m\tilde{T}\|_{H^2}\le C\|A(P_m\tilde{T})\|\,. \tag{3.21}$$

In more detail, we have

$$\begin{aligned} &A(P_m\tilde{T})=A\Big(\sum_{j=1}^m\tilde{\beta}_j\Phi_j\Big)=\sum_{j=1}^m\tilde{\beta}_j A\Phi_j=\sum_{j=1}^m\lambda_j\tilde{\beta}_j\Phi_j\,,\qquad \text{by}\quad(3.6)\,,\\ &A\tilde{T}=A\Big(\sum_{j=1}^\infty\tilde{\beta}_j\Phi_j\Big)=\sum_{j=1}^\infty\tilde{\beta}_j A\Phi_j=\sum_{j=1}^\infty\lambda_j\tilde{\beta}_j\Phi_j\,,\qquad \text{by}\quad(3.6)\,, \end{aligned} \tag{3.22}$$

which along with the orthogonality of $\{\Phi_j\}_{j\geq 1}$ in $L^2(\mathcal{M})$ leads to

$$\|A(P_m\tilde{T})\|^2 = \sum_{j=1}^{m} \lambda_j^2\tilde{\beta}_j^2\|\Phi_j\|^2 \leq \sum_{j=1}^{\infty} \lambda_j^2\tilde{\beta}_j^2\|\Phi_j\|^2 = \|A\tilde{T}\|^2\,. \tag{3.23}$$

The combination of (3.21) and (3.22) results in

$$\|P_m\tilde{T}\|_{H^2} \leq C\|A\tilde{T}\| \leq C\|\tilde{T}\|_{H^2}\,. \tag{3.24}$$

The substitution of (3.24) into (3.19) leads to

$$\left|\int_{\mathcal{M}} \boldsymbol{v}_m T_m\,\nabla P_m\tilde{T}\,d\boldsymbol{x}\right| \leq C\|\boldsymbol{v}_m\|\|T_m\|_{H^1}^{1/2}\|T_m\|^{1/2}\|\tilde{T}\|_{H^2} \leq C\|T_m\|_{H^1}^{3/2}\|T_m\|^{1/2}\|\tilde{T}\|_{H^2}\,. \tag{3.25}$$

Similarly, we have the following estimates

$$\begin{aligned}\left|\int_{\mathcal{M}} w_m T_m\,\partial_z P_m\tilde{T}\,d\boldsymbol{x}\right| \leq\ & \|w_m\|_{L^2}\|T_m\|_{L^3}\|\partial_z P_m\tilde{T}\|_{L^6}\\ \leq\ & C\|T_m\|_{H^1}^{3/2}\|T_m\|^{1/2}\|\tilde{T}\|_{H^2}\,,\end{aligned} \tag{3.26}$$

$$\left|\Big(\nabla T_m, \nabla P_m\tilde{T}\Big) + \Big(\partial_z T_m, \partial_z P_m\tilde{T}\Big)\right| \leq\ C\|T_m\|_{H^1}\|\tilde{T}\|_{H^2}\,, \tag{3.27}$$

$$\left|\int_{\mathcal{M}_0} \frac{\alpha}{\kappa_2}(T_m(\boldsymbol{x},0)-\theta^*)P_m\tilde{T}(\cdot,0)\,d\boldsymbol{x}\right| \leq\ C\|T_m\|_{H^1}\|\tilde{T}\|_{H^2}\,, \tag{3.28}$$

By the combination of (3.16), (3.17), (3.25)-(3.28), we arrive at

$$\partial_t T_m \in \text{a bounded set of } L^{4/3}\Big(0,t_1\,;H^{-2}(\mathcal{M})\Big)\,, \tag{3.29}$$

for any $t_1 > 0$ and independent of m.

The estimates (3.16), (3.17), (3.29) imply the existence of $T\in L^\infty(0,t_1\,;L^2)\cap L^2(0,t_1\,;H^1)$ and $u\,,v\,,w\in L^2(0,t_1\,;L^2)$ and a subsequence $\{T_{m'},\boldsymbol{v}_{m'},w_{m'}\}$ such that

$$\begin{aligned}&T_{m'} \rightharpoonup T \text{ weakly in } L^2(0,t_1\,;H^1)\,,\\ &T_{m'} \overset{*}{\rightharpoonup} T \text{ weak-star in } L^\infty(0,t_1\,;L^2)\,,\\ &\boldsymbol{v}_{m'}\,,\,w_{m'} \rightharpoonup \boldsymbol{v}\,,\,w \text{ weakly in } L^2(0,t_1\,;L^2)\,,\end{aligned} \tag{3.30}$$

With the use of (3.29), (3.30) and Aubin's compactness theorem, we also have

$$T_{m'} \longrightarrow T \text{ strongly in } L^2(0,t_1\,;L^2)\,. \tag{3.31}$$

Then it is standard to pass to the limit in (3.9)-(3.11) and prove that the limit function $(T,\boldsymbol{v},w)$ is indeed a weak solution as defined in (3.2). The details are omitted for brevity. The proof for the first part of Theorem 3.1 is completed.

Furthermore, we multiply (3.15) by $\phi(t)$, where $\phi\in\mathcal{D}\big((0,t_1)\big)$, $\phi(t)\geq 0$, and integrate in time:

$$\begin{aligned}\int_0^{t_1}\Big(\|T_m(\cdot,t)\|^2+2\int_0^t(\kappa_1\|\nabla T_m(\cdot,s)\|^2+\kappa_2\|\partial_z T_m(\cdot,s)\|^2)\,ds\Big)\,\phi(t)\,dt\\ \leq\int_0^{t_1}\Big(\|T_0\|^2+C^*t\Big)\phi(t)\,dt\,.\end{aligned} \tag{3.32}$$

Using the weak convergence (3.30) we pass the lower limit in this inequality and obtain

$$\begin{aligned}\int_0^{t_1}\Big(\|T(\cdot,t)\|^2+2\int_0^t(\kappa_1\|\nabla T(\cdot,s)\|^2+\kappa_2\|\partial_z T(\cdot,s)\|^2)\,ds\Big)\,\phi(t)\,dt\\ \leq\int_0^{t_1}\Big(\|T_0\|^2+C^*t\Big)\phi(t)\,dt\,,\end{aligned} \tag{3.33}$$

for all $\phi\in\mathcal{D}\big((0,t_1)\big)$, $\phi\geq 0$. This amounts to saying that the energy inequality (3.4) is satisfied for almost every $t\in[0,t_1]$.

For the second part, we note that, by a direct calculation using the representation formulas for the horizontal velocity field in (3.2b),

$$\int_{-H_0}^0 \boldsymbol{v}(x,y,z)\,dz=\overline{\boldsymbol{v}}_e(x,y)\,,\qquad \forall(x,y)\in\mathcal{M}_0\,. \tag{3.34}$$

This leads to the first identity of (3.5), due to the free divergence of $\overline{\boldsymbol{v}}_e$ given by formulas (2.10) and (2.13). Moreover, the second identity of (3.5) is valid since $\overline{\boldsymbol{v}}_e$ satisfies the specified boundary condition on the lateral boundary. The third identity of (3.5) is also found by direct verification using formula (3.2c). This completes the proof of Theorem 3.1. ∎

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