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Stability and convergence of a second-order mixed finite element method for the Cahn–Hilliard equation

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In this paper, we devise and analyse an unconditionally stable, second-order-in-time numerical scheme for the Cahn–Hilliard equation in two and three space dimensions. We prove that our two-step scheme is unconditionally energy stable and unconditionally uniquely solvable. Furthermore, we show that the discrete phase variable is bounded in $L^\infty(0,T;L^\infty)$ and the discrete chemical potential is bounded in $L^\infty(0,T;L^2)$, for any time and space step sizes, in two and three dimensions, and for any finite final time T. We subsequently prove that these variables converge with optimal rates in the appropriate energy norms in both two and three dimensions. We include in this work a detailed analysis of the initialization of the two-step scheme.

Keywords: Cahn–Hilliard equation; spinodal decomposition; mixed finite element methods; energy stability; error estimates; second-order accuracy.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open polygonal or polyhedral domain. For all $\phi \in H^1(\Omega)$, consider the energy (Cahn & Hilliard, 1958)

$$E(\phi) = \int_\Omega \left\{ \frac{1}{4\varepsilon}\left(\phi^2 - 1\right)^2 + \frac{\varepsilon}{2}|\nabla\phi|^2 \right\} \mathrm{d}\mathbf{x}, \tag{1.1}$$

where ϕ is the concentration field and ε is a positive constant. The phase equilibria are represented by the values $\phi = \pm 1$. One version of the celebrated Cahn–Hilliard equation is given by Cahn (1961) and Cahn & Hilliard (1958):

$$\partial_t\phi = \varepsilon\Delta\mu, \quad \text{in } \Omega_T, \tag{1.2a}$$

$$\mu = \varepsilon^{-1}\left(\phi^3 - \phi\right) - \varepsilon\Delta\phi, \quad \text{in } \Omega_T, \tag{1.2b}$$

$$\partial_n\phi = \partial_n\mu = 0, \quad \text{on } \partial\Omega \times (0,T), \tag{1.2c}$$

where $\mu := \delta_\phi E$ is the chemical potential. The boundary conditions represent local thermodynamic equilibrium ($\partial_n \phi = 0$) and no-mass-flux ($\partial_n \mu = 0$). Clearly $E(\phi) \geqslant 0$ for all $\phi \in H^1(\Omega)$. Additionally, for all $\varepsilon > 0$ and $\phi \in H^1(\Omega)$, there exist positive constants $K_1 = K_1(\varepsilon)$ and $K_2 = K_2(\varepsilon)$ such that

$$0 < K_1 \|\phi\|_{H^1}^2 \leqslant E(\phi) + K_2. \tag{1.3}$$

A weak formulation of (1.2a–1.2c) may be written as follows: find (ϕ, μ) such that

$$\phi \in L^\infty\left(0, T; H^1(\Omega)\right) \cap L^4\left(0, T; L^\infty(\Omega)\right), \quad \partial_t \phi \in L^2\left(0, T; H^{-1}(\Omega)\right), \quad \mu \in L^2\left(0, T; H^1(\Omega)\right),$$

and there hold for almost all $t \in (0, T)$

$$\langle \partial_t \phi, \nu \rangle + \varepsilon a(\mu, \nu) = 0 \quad \forall \nu \in H^1(\Omega), \tag{1.4a}$$

$$(\mu, \psi) - \varepsilon a(\phi, \psi) - \varepsilon^{-1}\left(\phi^3 - \phi, \psi\right) = 0 \quad \forall \psi \in H^1(\Omega), \tag{1.4b}$$

where

$$a(u, v) := (\nabla u, \nabla v), \tag{1.5}$$

with the 'compatible' initial data

$$\phi(0) = \phi_0 \in H_N^2(\Omega) := \left\{ v \in H^2(\Omega) \mid \partial_n v = 0 \text{ on } \partial\Omega \right\}. \tag{1.6}$$

Here we use the notations $H^{-1}(\Omega) := (H^1(\Omega))^*$ and $\langle \cdot, \cdot \rangle$ as the duality paring between H^{-1} and H^1. Throughout the paper, we use the notation $\Phi(t) := \Phi(\cdot, t) \in X$, which views a spatiotemporal function as a map from the time interval $[0, T]$ into an appropriate Banach space, X. The system (1.4a) and (1.4b) is mass conservative: for almost every $t \in [0, T]$, $(\phi(t) - \phi_0, 1) = 0$. This observation rests on the fact that $a(\phi, 1) = 0$, for all $\phi \in L^2(\Omega)$. Observe that the homogeneous Neumann boundary conditions associated with the phase variables ϕ and μ are natural in this mixed weak formulation of the problem.

The existence of weak solutions is a straightforward exercise using the compactness/energy method, for example, Elliott & Zheng (1986). It is likewise straightforward to show that weak solutions of (1.4a) and (1.4b) dissipate the energy (1.1). In other words, (1.2a–1.2c) is a mass-conservative gradient flow with respect to the energy (1.1). Precisely, for any $t \in [0, T]$, we have the energy law

$$E(\phi(t)) + \int_0^t \varepsilon \|\nabla \mu(s)\|_{L^2}^2 \, \mathrm{d}s = E(\phi_0). \tag{1.7}$$

The Cahn–Hilliard equation is one of the most important models in mathematical physics. On its own, the equation is a model for spinodal decomposition (Cahn, 1961). However, the Cahn–Hilliard equation is more often paired with equations that describe important physical behaviour of a given physical system, typically through nonlinear coupling terms. Prominent examples include the Cahn–Hilliard–Navier–Stokes equation, describing two-phase flow (Liu & Shen, 2003; Feng, 2006; Kay & Welford, 2007; Shen & Yang, 2010b; Grün, 2013; Grün & Klingbeil, 2014), the Cahn–Hilliard–Hele–Shaw equation (Lee *et al.*, 2002a,b; Wise, 2010), which describes spinodal decomposition of a binary fluid in a Hele–Shaw cell, and the Cahn–Larché equation (Larché & Cahn, 1982; Fratzl *et al.*, 1999; Garcke & Weikard, 2005; Wise *et al.*, 2005) describing solid-state, binary phase transformations involving coherent, linear-elastic misfit.

The Cahn–Hilliard equation is a challenging fourth-order, nonlinear parabolic-type partial differential equation. Naive explicit methods suffer from severe time-step restrictions for stability. On the other hand, fully implicit numerical methods must contend with a potentially large, ill-conditioned nonlinear system of algebraic equations. There remains a great need for sophisticated stable and efficient numerical schemes for the Cahn–Hilliard equation. Indeed, extensive research has been conducted in this area, in particular for first-order-accurate-in-time schemes; see Aristotelous *et al.* (2013), Chen & Shen (1998), Elliott & Larsson (1992), Elliott & Stuart (1993, 1996), Feng (2006), Feng & Prohl (2004), Furihata (2001), Guan *et al.* (2014), He *et al.* (2006), Kay & Welford (2006, 2007), Kim *et al.* (2003), Wise (2010) and the references therein. Less commonly investigated are second-order-accurate-in-time numerical schemes. In general, the analysis of second-order schemes for nonlinear equations can be significantly more difficult than that for first-order methods. Nevertheless, such work has been reported in the following articles (Elliott, 1989; Du & Nicolaides, 1991; Chen & Shen, 1998; Furihata, 2001; Shen & Yang, 2010a; Shen *et al.*, 2012; Benešova *et al.*, 2014; Guillén-González & Tierra, 2014; Wu *et al.*, 2014; Aristotelous *et al.*, 2015). We mention, in particular, the secant-type algorithms described in Du & Nicolaides (1991) and Furihata (2001). With the notation $\Psi(\phi) := \frac{1}{4}(\phi^2-1)^2$, the secant scheme of Du & Nicolaides (1991) for the Cahn–Hilliard equation may be formulated as

$$\phi^{n+1} - \phi^n = s\varepsilon\Delta\mu^{n+1/2}, \quad \mu^{n+1/2} := \varepsilon^{-1}\frac{\Psi(\phi^{n+1}) - \Psi(\phi^n)}{\phi^{n+1}-\phi^n} - \frac{\varepsilon}{2}\left(\Delta\phi^{n+1} + \Delta\phi^n\right). \tag{1.8}$$

This scheme is energy stable. However, it may not be unconditionally uniquely solvable with respect to the time step size s. (See Elliott, 1989; Du & Nicolaides, 1991; Furihata, 2001 for details.) Lack of unconditional solvability may be problematic, as coarsening studies using the Cahn–Hilliard equation may involve very large time scales, requiring potentially very large time steps for efficiency.

There are few recent works examining second-order (in time) methods for the Cahn–Hilliard equation that we should highlight. Very recently, Benešova *et al.* (2014) introduced and analysed a temporally second-order numerical scheme for the approximate solution of a modified Cahn–Hilliard equation using an implicit midpoint rule and spatial discretization by the Fourier–Galerkin spectral method. (The model is the same as was considered in Aristotelous *et al.* (2013) and Diegel *et al.* (2015) using mixed finite element discretizations.) The stabilities proved in Benešova *et al.* (2014) may be viewed as *conditional*—in the sense that there is a restriction on the time step size for stability in terms of the model parameters—but the authors emphasize that the stabilities also may be viewed as *unconditional*, since the time step restrictions do not depend on any spatial discretization parameters. (This terminology is inconsistent in the literature.) The solvability issue aside, one very nice feature of the implicit midpoint rule is, of course, the small local truncation error, relative to the splitting-type methods examined herein and elsewhere. The authors show existence and uniqueness of their scheme along with several stability results, and they prove optimal convergence of their scheme. Specifically, they are able to demonstrate the $L^\infty(0,T;L^\infty)$ stability of their scheme, provided the time step is smaller than a constant depending only on the model parameters, which, as in our paper, is the key to proving convergence.

Wu *et al.* (2014) proposed a semi-discrete second-order convex-splitting scheme for a family of Cahn–Hilliard-type equations with applications to diffuse interface tumour growth models. Taking advantage of a (quadratic) cut-off of the double-well energy and artificial stabilization terms, they are able to show unconditional energy stability for their scheme. Moreover, their scheme has the advantage of being linear. However, they do not prove convergence of their scheme, and it is not clear if the analyses used here could be straightforwardly extended to gain the higher-order stabilities needed to prove convergence. In particular, Wu *et al.* (2014) use the standard Crank–Nicolson method to handle the highest-order linear diffusion term. Our analysis has suggested that, at least in the finite difference and finite element contexts, a more dissipative treatment of this term may be required in order to gain the higher-order stability estimates needed for convergence analysis.

Guillén-González & Tierra (2014) made a careful examination of several second-order in time numerical schemes for the Cahn–Hilliard problem (some of which have already been presented in the literature) and study the constraints on the physical and discrete parameters to assure energy stability, unique solvability, and in the case of nonlinear schemes, the convergence of Newton's method to the nonlinear schemes. To save computational cost, they develop a new adaptive time-stepping algorithm based on the numerical dissipation introduced in the discrete energy laws at each time step. The authors are able to show some useful energy stability estimates, but they do not establish higher-order stabilities or the convergence of their schemes.

In contrast to the papers referenced above, we propose and analyse a new second-order-accurate-in-time, fully discrete, mixed finite element scheme for the Cahn–Hilliard problem (1.2a–1.2c), which is closely related to the finite difference scheme proposed in Guo *et al.* (2015):

$$\phi_h^{n+1} - \phi_h^n = s\varepsilon \Delta_h \mu_h^{n+1/2}, \tag{1.9a}$$

$$\mu_h^{n+1/2} := \frac{1}{4\varepsilon}\left(\phi_h^{n+1}+\phi_h^n\right)\left(\left(\phi_h^{n+1}\right)^2+\left(\phi_h^n\right)^2\right) - \frac{1}{\varepsilon}\left(\frac{3}{2}\phi_h^n - \frac{1}{2}\phi_h^{n-1}\right) - \varepsilon\Delta_h\left(\frac{3}{4}\phi_h^{n+1}+\frac{1}{4}\phi_h^{n-1}\right), \tag{1.9b}$$

where Δ_h above is a finite difference stencil approximating the Laplacian, and ϕ_h and μ_h are grid variables. The formulation of the scheme (1.9a) and (1.9b) uses a convex splitting of the energy (Elliott & Stuart, 1993; Eyre, 1998; Wise *et al.*, 2009; Feng & Wise, 2012). Observe that the energy (1.1) may be represented as the difference between two purely convex energies:

$$E(\phi) = E_c(\phi) - E_e(\phi) = \frac{1}{4\varepsilon}\|\phi\|_{L^4}^4 + \frac{\varepsilon}{2}\|\nabla\phi\|_{L^2}^2 + \frac{|\Omega|}{4\varepsilon} - \frac{1}{2\varepsilon}\|\phi\|_{L^2}^2. \tag{1.10}$$

The idea is then to treat the variation of E_c implicitly and that of E_e, explicitly. The advantages of the scheme (1.9a) and (1.9b) are threefold. The scheme is unconditionally energy stable, unconditionally uniquely solvable and converges optimally in the energy norm. The scheme is nonlinear, but it results as the gradient of a strictly convex functional. In our finite element version of the scheme, the stability and solvability statements we prove are *completely unconditional with respect to the time and space step sizes*. In fact, *all* of our *a priori* stability estimates hold completely independently of the time and space step sizes. We use a bootstrapping technique to leverage the energy stabilities to achieve unconditional $L^\infty(0,T;L^\infty(\Omega))$ stability for the phase field variable ϕ_h and unconditional $L^\infty(0,T;L^2(\Omega))$ stability for the chemical potential μ_h. With these stabilities in hand, we are then able to prove optimal error estimates for ϕ_h and μ_h in the appropriate energy norms.

The remainder of the paper is organized as follows. In Section 2, we define our second-order mixed finite element version of the scheme and prove the unconditional solvability and stability. In Section 3, we prove error estimates for the scheme under suitable regularity assumptions for the PDE solution. In Section 4, we present the results of numerical tests that confirm the rates of convergence predicted by the error estimates.

2. A mixed finite element convex splitting scheme

2.1 *Definition of the scheme*

Let M be a positive integer and $0 = t_0 < t_1 < \cdots < t_M = T$ be a uniform partition of $[0,T]$, with $\tau = t_i - t_{i-1}$ and $i = 1,\ldots,M$. Suppose $\mathcal{T}_h = \{K\}$ is a conforming, shape-regular, quasi-uniform family of triangulations of Ω. For $q \in \mathbb{Z}^+$, define $S_h := \{v \in C^0(\Omega) \mid v|_K \in \mathcal{P}_q(K),\ \forall K \in \mathcal{T}_h\} \subset H^1(\Omega)$. Define $\mathring{S}_h := S_h \cap L_0^2(\Omega)$, with $L_0^2(\Omega)$ denoting those functions in $L^2(\Omega)$ with zero mean. Our mixed second-order splitting scheme is defined as follows: for any $1 \leqslant m \leqslant M-1$, given $\phi_h^m, \phi_h^{m-1} \in S_h$, find $\phi_h^{m+1}, \mu_h^{m+1/2} \in S_h$ such that

$$\left(\delta_\tau \phi_h^{m+1/2}, \nu\right) + \varepsilon a\left(\mu_h^{m+1/2}, \nu\right) = 0 \quad \forall \nu \in S_h, \tag{2.1a}$$

$$\varepsilon^{-1}\left(\chi\left(\phi_h^{m+1}, \phi_h^m\right), \psi\right) - \varepsilon^{-1}\left(\tilde{\phi}_h^{\,m+1/2}, \psi\right) + \varepsilon a\left(\check{\phi}_h^{\,m+1/2}, \psi\right) - \left(\mu_h^{m+1/2}, \psi\right) = 0 \quad \forall \psi \in S_h, \tag{2.1b}$$

where

$$\delta_\tau \phi_h^{m+1/2} := \frac{\phi_h^{m+1} - \phi_h^m}{\tau}, \quad \phi_h^{m+1/2} := \frac{1}{2}\phi_h^{m+1} + \frac{1}{2}\phi_h^m, \quad \tilde{\phi}_h^{\,m+1/2} := \frac{3}{2}\phi_h^m - \frac{1}{2}\phi_h^{m-1}, \tag{2.2}$$

$$\check{\phi}_h^{\,m+1/2} := \frac{3}{4}\phi_h^{m+1} + \frac{1}{4}\phi_h^{m-1}, \quad \chi\left(\phi_h^{m+1}, \phi_h^m\right) := \frac{1}{2}\left(\left(\phi_h^{m+1}\right)^2 + \left(\phi_h^m\right)^2\right)\phi^{m+1/2}. \tag{2.3}$$

Since this is a multi-step scheme, it requires a separate initialization process. For the first step, the scheme is as follows: given $\phi_h^0 \in S_h$, find $\phi_h^1, \mu_h^{1/2} \in S_h$ such that

$$\left(\delta_\tau \phi_h^{1/2}, \nu\right) + \varepsilon a\left(\mu_h^{1/2}, \nu\right) = 0 \quad \forall \nu \in S_h, \tag{2.4a}$$

$$\varepsilon^{-1}\left(\chi\left(\phi_h^1, \phi_h^0\right), \psi\right) - \varepsilon^{-1}\left(\phi_h^0, \psi\right) + \frac{\tau}{2} a\left(\mu_h^0, \psi\right) + \varepsilon a\left(\phi_h^{1/2}, \psi\right) - \left(\mu_h^{1/2}, \psi\right) = 0 \quad \forall \psi \in S_h, \tag{2.4b}$$

where $\phi_h^0 := R_h \phi_0$, and the operator $R_h : H^1(\Omega) \to S_h$ is a standard Ritz projection:

$$a\left(R_h \phi - \phi, \xi\right) = 0 \quad \forall \xi \in S_h, \quad \left(R_h \phi - \phi, 1\right) = 0. \tag{2.5}$$

Note that the scheme requires initial data for the chemical potential, $\mu_h^0 \in S_h$, which is defined as $\mu_h^0 := R_h \mu_0$, where

$$\mu_0 := \varepsilon^{-1}\left(\phi_0^3 - \phi_0\right) - \varepsilon \Delta \phi_0. \tag{2.6}$$

THEOREM 2.1 The scheme (2.1a) and (2.1b), coupled with the initial scheme (2.4a) and (2.4b), is uniquely solvable for any mesh parameters h and τ, and for any model parameters.

Proof. The proof is based on convexity arguments and follows in a similar manner as that of Theorem 5 from Hu *et al.* (2009). We omit the details for brevity. □

REMARK 2.2 Note that it is not necessary for solvability and some basic energy stabilities that the μ-space and the ϕ-space be equal. However, the proofs of the higher-order stability estimates, in particular, the proof in Lemma 2.9, do require the equivalence of these spaces. We do, however, use the equality of the spaces to our advantage in Lemma 2.7, although it should be noted that a proof for this lemma is possible without the equality of spaces.

REMARK 2.3 The elliptic projections are used in the initialization for simplicity in the forthcoming error analysis. However, other (simpler) projections may be used in the initialization step, as long as they have good approximation properties.

2.2 *Unconditional energy stability*

We now show that the solutions to our scheme enjoy stability properties that are similar to those of the PDE solutions, and moreover, these properties hold regardless of the sizes of h and τ. The first property, the unconditional energy stability, is a direct result of the convex decomposition. We begin the discussion with the definition of the discrete Laplacian, $\Delta_h : S_h \to S_h$, as follows: for any $v_h \in S_h$, $\Delta_h v_h \in S_h$ denotes the unique solution to the problem

$$(\Delta_h v_h, \xi) = -a\left(v_h, \xi\right) \quad \forall \xi \in S_h. \tag{2.7}$$

In particular, setting $\xi = \Delta_h v_h$ in (2.7), we obtain

$$\|\Delta_h v_h\|_{L^2}^2 = -a\left(v_h, \Delta_h v_h\right).$$

See, for example, Feng *et al.* (2007).

LEMMA 2.4 Let $\left(\phi_h^1, \mu_h^{1/2}\right) \in S_h \times S_h$ be the unique solution of the initialization scheme (2.4a) and (2.4b). Then the following first-step energy stability holds for any $h,\ \tau > 0$:

$$E\left(\phi_h^1\right) + \tau\varepsilon \left\|\nabla \mu_h^{1/2}\right\|_{L^2}^2 + \frac{1}{4\varepsilon}\left\|\phi_h^1 - \phi_h^0\right\|_{L^2}^2 \leqslant E\left(\phi_h^0\right) + \frac{\varepsilon\tau^2}{4}\left\|\Delta_h \mu_h^0\right\|_{L^2}^2, \tag{2.8}$$

where $E(\phi)$ is defined in (1.10).

Proof. Setting $\nu = \tau\mu_h^{1/2}$ in (2.4a) and $\psi = \tau\delta_\tau\phi_h^{1/2} = \phi_h^1 - \phi_h^0$ in (2.4b) yields the following:

$$\tau\left(\delta_\tau\phi_h^{1/2}, \mu_h^{1/2}\right) + \tau\varepsilon\left\|\nabla\mu_h^{1/2}\right\|_{L^2}^2 = 0, \tag{2.9}$$

$$\begin{aligned} &\varepsilon^{-1}\left(\chi\left(\phi_h^1, \phi_h^0\right), \phi_h^1 - \phi_h^0\right) - \varepsilon^{-1}\left(\phi_h^0, \phi_h^1 - \phi_h^0\right) + \varepsilon\, a\left(\phi_h^{1/2}, \phi_h^1 - \phi_h^0\right) \\ &\quad + \frac{\tau}{2} a\left(\mu_h^0, \phi_h^1 - \phi_h^0\right) - \tau\left(\mu_h^{1/2}, \delta_\tau\phi_h^{1/2}\right) = 0. \end{aligned} \tag{2.10}$$

Adding Eqs. (2.9) and (2.10), using Young's inequality, and the following identities:

$$\left(\chi\left(\phi_h^1, \phi_h^0\right), \phi_h^1 - \phi_h^0\right) = \tfrac{1}{4}\left(\left\|\phi_h^1\right\|_{L^4}^4 - \left\|\phi_h^0\right\|_{L^4}^4\right), \tag{2.11}$$

$$\left(\phi_h^0, \phi_h^1 - \phi_h^0\right) = \tfrac{1}{2}\left(\left\|\phi_h^1\right\|_{L^2}^2 - \left\|\phi_h^0\right\|_{L^2}^2 - \left\|\phi_h^1 - \phi_h^0\right\|_{L^2}^2\right), \tag{2.12}$$

the result is obtained. □

We now define a modified energy

$$F(\phi,\psi) := E(\phi) + \frac{1}{4\varepsilon}\left\|\phi-\psi\right\|_{L^2}^2 + \frac{\varepsilon}{8}\left\|\nabla\phi-\nabla\psi\right\|_{L^2}^2, \tag{2.13}$$

where $E(\phi)$ is defined as above.

LEMMA 2.5 Let $\left(\phi_h^{m+1},\mu_h^{m+1/2}\right)\in S_h\times S_h$ be the unique solution of (2.1a–2.1b), and $(\phi_h^1,\mu_h^{1/2})\in S_h\times S_h$, the unique solution of (2.4a) and (2.4b). Then the following energy law holds for any $h,\tau>0$:

$$\begin{aligned} F\left(\phi_h^{\ell+1},\phi_h^{\ell}\right) + \tau\varepsilon\sum_{m=1}^{\ell}\left\|\nabla\mu_h^{m+1/2}\right\|_{L^2}^2 + \sum_{m=1}^{\ell}\left[\frac{1}{4\varepsilon}\left\|\phi_h^{m+1}-2\phi_h^m+\phi_h^{m-1}\right\|_{L^2}^2\right. \\ \left. + \frac{\varepsilon}{8}\left\|\nabla\phi_h^{m+1}-2\nabla\phi_h^m+\nabla\phi_h^{m-1}\right\|_{L^2}^2\right] = F\left(\phi_h^1,\phi_h^0\right), \end{aligned} \tag{2.14}$$

for all $1\leqslant\ell\leqslant M-1$.

Proof. Setting $\nu=\mu_h^{m+1/2}$ in (2.1a) and $\psi=\delta_\tau\phi_h^{m+1/2}$ in (2.1b) gives

$$\left(\delta_\tau\phi_h^{m+1/2},\mu_h^{m+1/2}\right) + \varepsilon\left\|\nabla\mu_h^{m+1/2}\right\|_{L^2}^2 = 0, \tag{2.15}$$

$$\begin{aligned} \varepsilon^{-1}\left(\chi\left(\phi_h^{m+1},\phi_h^m\right),\delta_\tau\phi_h^{m+1/2}\right) - \varepsilon^{-1}\left(\tilde{\phi}_h^{m+1/2},\delta_\tau\phi_h^{m+1/2}\right) \\ + \varepsilon a\left(\check{\phi}_h^{m+1/2},\delta_\tau\phi_h^{m+1/2}\right) - \left(\mu_h^{m+1/2},\delta_\tau\phi_h^{m+1/2}\right) = 0. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), using the identities

$$\begin{aligned} &\left(\chi\left(\phi_h^{m+1},\phi_h^m\right),\delta_\tau\phi_h^{m+1/2}\right) - \left(\tilde{\phi}_h^{m+1/2},\delta_\tau\phi_h^{m+1/2}\right) \\ &= \frac{1}{4\tau}\left(\left\|\left(\phi_h^{m+1}\right)^2-1\right\|_{L^2}^2 - \left\|\left(\phi_h^m\right)^2-1\right\|_{L^2}^2\right) + \frac{1}{4\tau}\left(\left\|\phi_h^{m+1}-\phi_h^m\right\|_{L^2}^2 - \left\|\phi_h^m-\phi_h^{m-1}\right\|_{L^2}^2\right) \\ &\quad + \frac{1}{4\tau}\left\|\phi_h^{m+1}-2\phi_h^m+\phi_h^{m-1}\right\|_{L^2}^2 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} a\left(\check{\phi}_h^{m+1/2},\delta_\tau\phi_h^{m+1/2}\right) &= \frac{1}{2\tau}\left(\left\|\nabla\phi_h^{m+1}\right\|_{L^2}^2 - \left\|\nabla\phi_h^m\right\|_{L^2}^2\right) \\ &\quad + \frac{1}{8\tau}\left(\left\|\nabla\phi_h^{m+1}-\nabla\phi_h^m\right\|_{L^2}^2 - \left\|\nabla\phi_h^m-\nabla\phi_h^{m-1}\right\|_{L^2}^2\right) \\ &\quad + \frac{1}{8\tau}\left\|\nabla\phi_h^{m+1}-2\nabla\phi_h^m+\nabla\phi_h^{m-1}\right\|_{L^2}^2, \end{aligned} \tag{2.18}$$

and applying the operator $\tau\sum_{m=1}^{\ell}$ to the combined equation result in (2.14). □

In the sequel, we will make the following stability assumptions for the initial data:

$$E\left(\phi_h^0\right) + \tau^2 \left\|\Delta_h \mu_h^0\right\|_{L^2}^2 + \left\|\Delta_h \phi_h^0\right\|_{L^2}^2 \leqslant C, \tag{2.19}$$

for some constant $C > 0$ that is independent of h and τ. Here, we assume that $\varepsilon > 0$ is fixed. In fact, from this point in the stability and error analyses, we will not track the dependence of the estimates on the interface parameter ε, though this may be of importance, especially if ε tends to zero.

Lemma 2.6 Let $(\phi_h^{m+1}, \mu_h^{m+1/2}) \in S_h \times S_h$ be the unique solution of (2.1a) and (2.1b), and $(\phi_h^1, \mu_h^{1/2}) \in S_h \times S_h$, the unique solution of (2.4a) and (2.4b). Then the following estimates hold for any $h,\ \tau > 0$:

$$\max_{0 \leqslant m \leqslant M} \left[\left\|\nabla \phi_h^m\right\|_{L^2}^2 + \left\|\left(\phi_h^m\right)^2 - 1\right\|_{L^2}^2\right] \leqslant C, \tag{2.20}$$

$$\max_{0 \leqslant m \leqslant M} \left[\left\|\phi_h^m\right\|_{L^4}^4 + \left\|\phi_h^m\right\|_{L^2}^2 + \left\|\phi_h^m\right\|_{H^1}^2\right] \leqslant C, \tag{2.21}$$

$$\max_{1 \leqslant m \leqslant M} \left[\left\|\phi_h^m - \phi_h^{m-1}\right\|_{L^2}^2 + \left\|\nabla \phi_h^m - \nabla \phi_h^{m-1}\right\|_{L^2}^2\right] \leqslant C, \tag{2.22}$$

$$\tau \sum_{m=0}^{M-1} \left\|\nabla \mu_h^{m+1/2}\right\|_{L^2}^2 \leqslant C, \tag{2.23}$$

$$\sum_{m=1}^{M-1} \left[\left\|\phi_h^{m+1} - 2\phi_h^m + \phi_h^{m-1}\right\|_{L^2}^2 + \left\|\nabla \phi_h^{m+1} - 2\nabla \phi_h^m + \nabla \phi_h^{m-1}\right\|_{L^2}^2\right] \leqslant C, \tag{2.24}$$

for some constant $C > 0$ that is independent of h, τ and T.

Proof. Starting with the stability of the initial step, inequality (2.8), and considering the stability of the initial data, inequality (2.19), we immediately have

$$\left\|\nabla \phi_h^1\right\|_{L^2}^2 + \left\|\left(\phi_h^1\right)^2 - 1\right\|_{L^2}^2 + \left\|\phi_h^1\right\|_{L^4}^4 + \left\|\phi_h^1\right\|_{L^2}^2 + \left\|\phi_h^1\right\|_{H^1}^2 + \tau \left\|\nabla \mu_h^{1/2}\right\|_{L^2}^2 \leqslant C. \tag{2.25}$$

The triangle inequality immediately implies

$$F\left(\phi_h^1, \phi_h^0\right) = E(\phi_h^1) + \frac{1}{4\varepsilon}\left\|\phi_h^1 - \phi_h^0\right\|_{L^2}^2 + \frac{\varepsilon}{8}\left\|\nabla \phi_h^1 - \nabla \phi_h^0\right\|_{L^2}^2 \leqslant C.$$

This, together with (2.14) and the fact that $F(\phi_h^{m+1}, \phi_h^m) \geqslant E(\phi_h^{m+1})$, for all $0 \leqslant m \leqslant M-1$, establishes all of the inequalities. □

We are able to prove the next set of *a priori* stability estimates without any restrictions on h and τ. See Diegel *et al.* (2015) for a definition of discrete negative norm $\|\cdot\|_{-1,h}$.

LEMMA 2.7 Let $(\phi_h^{m+1}, \mu_h^{m+1/2}) \in S_h \times S_h$ be the unique solution of (2.1a) and (2.1b), and $\left(\phi_h^1, \mu_h^{1/2}\right) \in S_h \times S_h$, the unique solution of (2.4a) and (2.4b). Then the following estimates hold for any $h,\ \tau > 0$:

$$\tau \sum_{m=0}^{M-1} \left[\left\| \delta_\tau \phi_h^{m+1/2} \right\|_{H^{-1}}^2 + \left\| \delta_\tau \phi_h^{m+1/2} \right\|_{-1,h}^2 \right] \leqslant C, \tag{2.26}$$

$$\tau \sum_{m=0}^{M-1} \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 \leqslant C(T+1), \tag{2.27}$$

$$\tau \sum_{m=1}^{M-1} \left[\left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 + \left\| \check{\phi}_h^{\,m+1/2} \right\|_{L^\infty}^{4(6-d)/d} \right] \leqslant C(T+1), \tag{2.28}$$

for some constant $C > 0$ that is independent of h, τ and T.

Proof. Let $\mathcal{Q}_h : L^2(\Omega) \to S_h$ be the L^2 projection, i.e., $(\mathcal{Q}_h \nu - \nu, \xi) = 0$ for all $\xi \in S_h$. Suppose $\nu \in H^1(\Omega)$. Then, by (2.1a) and (2.4a), for all $0 < m < M-1$

$$\begin{aligned}\left(\delta_\tau \phi_h^{m+1/2}, \nu\right) &= \left(\delta_\tau \phi_h^{m+1/2}, \mathcal{Q}_h \nu\right) = -\varepsilon \left(\nabla \mu_h^{m+1/2}, \nabla \mathcal{Q}_h \nu\right) \leqslant \varepsilon \left\| \nabla \mu_h^{m+1/2} \right\|_{L^2} \|\nabla \mathcal{Q}_h \nu\|_{L^2} \\ &\leqslant C\varepsilon \left\| \nabla \mu_h^{m+1/2} \right\|_{L^2} \|\nabla \nu\|_{L^2}, \end{aligned} \tag{2.29}$$

where we used the H^1 stability of the L^2 projection in the last step. Applying $\tau \sum_{m=0}^{M-1}$ and using (2.23) we obtain the first estimate of (2.26). The second estimate of (2.26) follows from the inequality $\|\nu\|_{-1,h} \leqslant \|\nu\|_{H^{-1}}$, which holds for all $\nu \in S_h$.

To prove (2.27), for $1 \leqslant m \leqslant M-1$ we set $\psi = \mu_h^{m+1/2}$ in (2.1b) to obtain

$$\begin{aligned}\left\| \mu_h^{m+1/2} \right\|_{L^2}^2 &= \varepsilon^{-1} \left(\chi\left(\phi_h^{m+1}, \phi_h^m\right), \mu_h^{m+1/2} \right) - \varepsilon^{-1} \left(\tilde{\phi}_h^{\,m+1/2}, \mu_h^{m+1/2} \right) + \varepsilon a \left(\check{\phi}_h^{\,m+1/2}, \mu_h^{m+1/2} \right) \\ &\leqslant C \left\| \chi\left(\phi_h^{m+1}, \phi_h^m\right) \right\|_{L^2}^2 + \frac{1}{4} \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 + C \left\| \tilde{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 + \frac{1}{4} \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 \\ &\quad + C \left\| \nabla \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla \mu_h^{m+1/2} \right\|_{L^2}^2 .\end{aligned}$$

And, similarly, setting $\psi = \mu_h^{1/2}$ in (2.4b), we have

$$\begin{aligned}\left\| \mu_h^{1/2} \right\|_{L^2}^2 &\leqslant C \left\| \chi\left(\phi_h^1, \phi_h^0\right) \right\|_{L^2}^2 + \frac{1}{6} \left\| \mu_h^{1/2} \right\|_{L^2}^2 + C \left\| \phi_h^0 \right\|_{L^2}^2 + \frac{1}{6} \left\| \mu_h^{1/2} \right\|_{L^2}^2 + C \left\| \nabla \phi_h^{1/2} \right\|_{L^2}^2 \\ &\quad + \frac{1}{2} \left\| \nabla \mu_h^{1/2} \right\|_{L^2}^2 + \frac{1}{6} \left\| \mu_h^{1/2} \right\|_{L^2}^2 + C\tau^2 \left\| \Delta_h \mu_h^0 \right\|_{L^2}^2 .\end{aligned}$$

Hence, using the triangle inequality, (2.21), and the initial stability (2.19), we have for all $0 \leqslant m \leqslant M-1$,

$$\frac{1}{2} \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 \leqslant C \left\| \chi\left(\phi_h^{m+1}, \phi_h^m\right) \right\|_{L^2}^2 + \frac{1}{2} \left\| \nabla \mu_h^{m+1/2} \right\|_{L^2}^2 + C.$$

Now, using Lemma 2.6, we have the following bound for all $0 \leqslant m \leqslant M-1$

$$\begin{aligned}\left\|\chi\left(\phi_h^{m+1},\phi_h^m\right)\right\|_{L^2}^2 &= \frac{1}{16}\left\|\left(\phi_h^{m+1}\right)^3+\left(\phi_h^{m+1}\right)^2\phi_h^m+\phi_h^{m+1}\left(\phi_h^m\right)^2+\left(\phi_h^m\right)^3\right\|_{L^2}^2\\ &\leqslant C\left\|\left(\phi_h^{m+1}\right)^3\right\|_{L^2}^2+C\left\|\left(\phi_h^{m+1}\right)^2\phi_h^m\right\|_{L^2}^2+C\left\|\phi_h^{m+1}\left(\phi_h^m\right)^2\right\|_{L^2}^2+C\left\|\left(\phi_h^m\right)^3\right\|_{L^2}^2\\ &\leqslant C\left\|\phi_h^{m+1}\right\|_{L^6}^6+C\left\|\phi_h^m\right\|_{L^6}^6\leqslant C\left\|\phi_h^{m+1}\right\|_{H^1}^6+C\left\|\phi_h^m\right\|_{H^1}^6\leqslant C,\end{aligned}\tag{2.30}$$

where we used Young's inequality and the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$, for $d=2,3$. Hence,

$$\left\|\mu_h^{m+1/2}\right\|_{L^2}^2\leqslant\left\|\nabla\mu_h^{m+1/2}\right\|_{L^2}^2+C.\tag{2.31}$$

Applying $\tau\sum_{m=0}^{M-1}$, estimate (2.27) now follows from (2.23).

Setting $\psi_h=\Delta_h\check{\phi}_h^{m+1/2}$ in (2.1b) and using the definition of the discrete Laplacian (2.7), it follows that for all $1\leqslant m\leqslant M-1$

$$\begin{aligned}\varepsilon\left\|\Delta_h\check{\phi}_h^{m+1/2}\right\|_{L^2}^2 &= -\varepsilon a\left(\check{\phi}_h^{m+1/2},\Delta_h\check{\phi}_h^{m+1/2}\right)\\ &= -\left(\mu_h^{m+1/2},\Delta_h\check{\phi}_h^{m+1/2}\right)-\varepsilon^{-1}\left(\tilde{\phi}_h^{m+1/2},\Delta_h\check{\phi}_h^{m+1/2}\right)\\ &\quad+\varepsilon^{-1}\left(\chi\left(\phi_h^{m+1},\phi_h^m\right),\Delta_h\check{\phi}_h^{m+1/2}\right)\\ &= a\left(\mu_h^{m+1/2},\check{\phi}_h^{m+1/2}\right)-\varepsilon^{-1}\left(\tilde{\phi}_h^{m+1/2},\Delta_h\check{\phi}_h^{m+1/2}\right)\\ &\quad+\varepsilon^{-1}\left(\chi\left(\phi_h^{m+1},\phi_h^m\right),\Delta_h\check{\phi}_h^{m+1/2}\right)\\ &\leqslant\frac{1}{2}\left\|\nabla\mu_h^{m+1/2}\right\|_{L^2}^2+\frac{1}{2}\left\|\nabla\check{\phi}_h^{m+1/2}\right\|_{L^2}^2+C\left\|\tilde{\phi}_h^{m+1/2}\right\|_{L^2}^2+\frac{\varepsilon}{4}\left\|\Delta_h\check{\phi}_h^{m+1/2}\right\|_{L^2}^2\\ &\quad+C\left\|\chi\left(\phi_h^{m+1},\phi_h^m\right)\right\|_{L^2}^2+\frac{\varepsilon}{4}\left\|\Delta_h\check{\phi}_h^{m+1/2}\right\|_{L^2}^2.\end{aligned}$$

Using the triangle inequality, (2.21) and (2.30), we have

$$\varepsilon\left\|\Delta_h\check{\phi}_h^{m+1/2}\right\|_{L^2}^2\leqslant\left\|\nabla\mu_h^{m+1/2}\right\|_{L^2}^2+C.\tag{2.32}$$

Applying $\tau\sum_{m=1}^{M-1}$, the first estimate of (2.28) now follows from (2.23).

To prove the second estimate of (2.28), we use the discrete Gagliardo–Nirenberg inequality Diegel (2015):

$$\left\|\psi_h^m\right\|_{L^\infty}\leqslant C\left\|\Delta_h\psi_h^m\right\|_{L^2}^{d/2(6-d)}\left\|\psi_h^m\right\|_{L^6}^{3(4-d)/2(6-d)}+C\left\|\psi_h^m\right\|_{L^6}\quad\forall\psi\in S_h,\quad(d=2,3).\tag{2.33}$$

Applying $\tau\sum_{m=1}^{M-1}$ and using $H^1(\Omega)\hookrightarrow L^6(\Omega)$, (2.21) and the first estimate of (2.28), the second estimate of (2.28) follows. □

REMARK 2.8 We point out that the T-dependence in the estimate (2.26) may not be optimal in the finite element context. While one might expect that there should be no dependence upon T at all in the estimates, other Galerkin finite element analyses, for example, Aristotelous *et al.* (2013), Diegel *et al.* (2015) and Kay *et al.* (2009), have observed similar linear-in-T stability estimates for first-order-in-time schemes. We point out that when using finite differences in space on a cubic domain, one can eliminate the T-dependence using techniques that may (or may not) extend to the Galerkin finite element setting; see Guo *et al.* (2015). In any case, this linear dependence upon the final time in the estimate (2.26) will propagate through the following stability estimates.

LEMMA 2.9 Let $(\phi_h^{m+1}, \mu_h^{m+1/2}) \in S_h \times S_h$ be the unique solution of (2.1a–2.1b), and $(\phi_h^1, \mu_h^{1/2}) \in S_h \times S_h$, the unique solution of (2.4a–2.4b). Assume that $\left\|\mu_h^0\right\|_{L^2}^2 \leqslant C$, independent of h. Then the following estimates hold for any $h,\ \tau > 0$:

$$\tau \sum_{m=0}^{M-1} \left\|\delta_\tau \phi_h^{m+1/2}\right\|_{L^2}^2 \leqslant C(T+1), \tag{2.34}$$

$$\max_{0 \leqslant m \leqslant M-1} \left\|\mu_h^{m+1/2}\right\|_{L^2}^2 \leqslant C(T+1), \tag{2.35}$$

for some constant $C > 0$ that is independent of h, τ and T.

Proof. The proof is divided into three parts.

Part 1. We first establish

$$\left\|\mu_h^{1/2}\right\|_{L^2}^2 + \tau \left\|\delta_\tau \phi_h^{1/2}\right\|_{L^2}^2 \leqslant C. \tag{2.36}$$

To this end, setting $\nu = \tau \delta_\tau \phi_h^{1/2}$ in (2.4a) and $\psi = 2\mu_h^{1/2}$ in (2.4b) and, adding the resulting equations, we have

$$\begin{aligned} 2 &\left\|\mu_h^{1/2}\right\|_{L^2}^2 + \tau \left\|\delta_\tau \phi_h^{1/2}\right\|_{L^2}^2 \\ &= \frac{2}{\varepsilon} \left(\chi\left(\phi_h^1, \phi_h^0\right), \mu_h^{1/2}\right) - \frac{2}{\varepsilon}\left(\phi_h^0, \mu_h^{1/2}\right) - \tau\left(\Delta_h \mu_h^0, \mu_h^{1/2}\right) - 2\varepsilon\left(\Delta_h \phi_h^0, \mu_h^{1/2}\right) \\ &\leqslant \left\|\mu_h^{1/2}\right\|_{L^2}^2 + C\left\|\chi\left(\phi_h^1, \phi_h^0\right)\right\|_{L^2}^2 + C\left\|\phi_h^0\right\|_{L^2}^2 + C\tau^2\left\|\Delta_h \mu_h^0\right\|_{L^2}^2 + C\left\|\Delta_h \phi_h^0\right\|_{L^2}^2. \end{aligned}$$

Thus,

$$\left\|\mu_h^{1/2}\right\|_{L^2}^2 + \tau \left\|\delta_\tau \phi_h^{1/2}\right\|_{L^2}^2 \leqslant C, \tag{2.37}$$

considering the initial stability (2.19), (2.21) and (2.30).

Part 2. Next we prove that

$$\left\|\mu_h^{3/2}\right\|_{L^2}^2 + \tau \left\|\delta_\tau \phi_h^{3/2}\right\|_{L^2}^2 \leqslant C. \tag{2.38}$$

Setting $m=1$ in (2.1b) and subtracting (2.4b), we obtain

$$\begin{aligned}\left(\mu_h^{3/2}-\mu_h^{1/2},\psi\right)&=\varepsilon a\left(\check{\phi}_h^{\,3/2}-\phi_h^{1/2},\psi\right)-\frac{3}{2\varepsilon}\left(\phi_h^1-\phi_h^0,\psi\right)-\frac{\tau}{2}a\left(\mu_h^0,\psi\right)\\&\quad+\varepsilon^{-1}\left(\chi\left(\phi_h^2,\phi_h^1\right)-\chi\left(\phi_h^1,\phi_h^0\right),\psi\right)\end{aligned}\tag{2.39}$$

$$\begin{aligned}&=\varepsilon a\left(\frac{3}{4}\tau\delta_\tau\phi_h^{3/2}+\frac{1}{4}\tau\delta_\tau\phi_h^{1/2},\psi\right)-\frac{3}{2\varepsilon}\left(\phi_h^1-\phi_h^0,\psi\right)-\frac{\tau}{2}a\left(\mu_h^0,\psi\right)\\&\quad+\varepsilon^{-1}\left(\chi\left(\phi_h^2,\phi_h^1\right)-\chi\left(\phi_h^1,\phi_h^0\right),\psi\right).\end{aligned}\tag{2.40}$$

Additionally, we take a weighted average of (2.1a) with $m=1$ and (2.4a) with the weights $\frac{3}{4}$ and $\frac{1}{4}$, respectively, to obtain,

$$\left(\tfrac{3}{4}\delta_\tau\phi_h^{3/2}+\tfrac{1}{4}\delta_\tau\phi_h^{1/2},\nu\right)=-\varepsilon a\left(\tfrac{3}{4}\mu_h^{3/2}+\tfrac{1}{4}\mu_h^{1/2},\nu\right)\quad\forall\nu\in S_h.\tag{2.41}$$

Taking $\psi=\frac{3}{4}\mu_h^{3/2}+\frac{1}{4}\mu_h^{1/2}$ in (2.40), $\nu=(3\tau/4)\delta_\tau\phi_h^{3/2}+(\tau/4)\delta_\tau\phi_h^{1/2}$ in (2.41), and adding the results yields

$$\begin{aligned}&\left(\mu_h^{3/2}-\mu_h^{1/2},\frac{3}{4}\mu_h^{3/2}+\frac{1}{4}\mu_h^{1/2}\right)+\tau\left\|\frac{3}{4}\delta_\tau\phi_h^{3/2}+\frac{1}{4}\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2\\&=-\frac{3}{8\varepsilon}\left(\phi_h^1-\phi_h^0,3\mu_h^{3/2}+\mu_h^{1/2}\right)-\frac{\tau}{8\varepsilon}a\left(\mu_h^0,3\mu_h^{3/2}+\mu_h^{1/2}\right)\\&\quad+\frac{1}{4\varepsilon}\left(\chi\left(\phi_h^2,\phi_h^1\right)-\chi\left(\phi_h^1,\phi_h^0\right),3\mu_h^{3/2}+\mu_h^{1/2}\right)\\&=-\frac{3}{8\varepsilon}\left(\phi_h^1-\phi_h^0,3\mu_h^{3/2}+\mu_h^{1/2}\right)+\frac{\tau}{8\varepsilon}\left(\Delta_h\mu_h^0,3\mu_h^{3/2}+\mu_h^{1/2}\right)\\&\quad+\frac{1}{4\varepsilon}\left(\chi\left(\phi_h^2,\phi_h^1\right)-\chi\left(\phi_h^1,\phi_h^0\right),3\mu_h^{3/2}+\mu_h^{1/2}\right)\\&\leqslant\frac{1}{4}\left\|\mu_h^{3/2}\right\|_{L^2}^2+C\left\|\mu_h^{1/2}\right\|_{L^2}^2+C\left\|\phi_h^1\right\|_{L^2}^2+C\left\|\phi_h^0\right\|_{L^2}^2+C\tau^2\left\|\Delta_h\mu_h^0\right\|_{L^2}^2\\&\quad+C\left\|\chi\left(\phi_h^2,\phi_h^1\right)\right\|_{L^2}^2+C\left\|\chi\left(\phi_h^1,\phi_h^0\right)\right\|_{L^2}^2\\&\leqslant\frac{1}{4}\left\|\mu_h^{3/2}\right\|_{L^2}^2+C\left\|\mu_h^{1/2}\right\|_{L^2}^2+C,\end{aligned}$$

where we have used Young's inequality, (2.19), (2.21) and (2.30). Considering Part 1 and the inequalities

$$\begin{aligned}\left\|\frac{3}{4}\delta_\tau\phi_h^{3/2}+\frac{1}{4}\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2&=\frac{9}{16}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}^2+\frac{3}{8}\left(\delta_\tau\phi_h^{3/2},\delta_\tau\phi_h^{1/2}\right)+\frac{1}{16}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2\\&\geqslant\frac{9}{16}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}^2-\frac{3}{8}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}+\frac{1}{16}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2\\&\geqslant\frac{3}{8}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}^2-\frac{1}{8}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2,\end{aligned}$$

and

$$\left(\mu_h^{3/2} - \mu_h^{1/2}, \frac{3}{4}\mu_h^{3/2} + \frac{1}{4}\mu_h^{1/2}\right) = \frac{3}{4}\left\|\mu_h^{3/2}\right\|_{L^2}^2 - \frac{1}{2}\left(\mu_h^{3/2}, \mu_h^{1/2}\right) - \frac{1}{4}\left\|\mu_h^{1/2}\right\|_{L^2}^2$$
$$\geqslant \frac{1}{2}\left\|\mu_h^{3/2}\right\|_{L^2}^2 - \frac{1}{2}\left\|\mu_h^{1/2}\right\|_{L^2}^2,$$

we have

$$\frac{1}{4}\left\|\mu_h^{3/2}\right\|_{L^2}^2 + \frac{3\tau}{8}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}^2 \leqslant C\left\|\mu_h^{1/2}\right\|_{L^2}^2 + \frac{\tau}{8}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2 + C \leqslant C. \tag{2.42}$$

Part 3. Finally, we will establish

$$\left\|\mu_h^{\ell+1/2}\right\|_{L^2}^2 + \frac{\tau}{8}\sum_{m=2}^{\ell}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2 \leqslant C(T+1). \tag{2.43}$$

For $2 \leqslant m \leqslant M-1$, we subtract (2.1b) from itself at consecutive time steps to obtain

$$\begin{aligned}\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \psi\right) &= \varepsilon a\left(\check{\phi}_h^{\,m+1/2} - \check{\phi}_h^{\,m-1/2}, \psi\right) - \varepsilon^{-1}\left(\tilde{\phi}_h^{\,m+1/2} - \tilde{\phi}_h^{\,m-1/2}, \psi\right)\\ &\quad + \varepsilon^{-1}\left(\chi\left(\phi_h^{m+1}, \phi_h^m\right) - \chi\left(\phi_h^m, \phi_h^{m-1}\right), \psi\right)\\ &= \varepsilon a\left(\tfrac{3}{4}\tau\delta_\tau\phi_h^{m+1/2} + \tfrac{1}{4}\tau\delta_\tau\phi_h^{m-3/2}, \psi\right) - \varepsilon^{-1}\left(\tfrac{3}{2}\tau\delta_\tau\phi_h^{m-1/2} - \tfrac{1}{2}\tau\delta_\tau\phi_h^{m-3/2}, \psi\right)\\ &\quad + \tfrac{1}{4\varepsilon}\left(\omega_h^m\left(\phi_h^{m+1} - \phi_h^{m-1}\right), \psi\right),\end{aligned} \tag{2.44}$$

for all $\psi \in S_h$, where $\omega_h^m := \omega\left(\phi_h^{m+1}, \phi_h^m, \phi_h^{m-1}\right)$ and

$$\omega(a, b, c) := a^2 + b^2 + c^2 + ab + bc + ac.$$

Additionally, we take a weighted average of the $m + \frac{1}{2}$ and $m - \frac{3}{2}$ time steps with the weights $\frac{3}{4}$ and $\frac{1}{4}$, respectively, of (2.1a) to obtain

$$\left(\tfrac{3}{4}\delta_\tau\phi_h^{m+1/2} + \tfrac{1}{4}\delta_\tau\phi_h^{m-3/2}, \nu\right) = -\varepsilon\, a\left(\tfrac{3}{4}\mu_h^{m+1/2} + \tfrac{1}{4}\mu_h^{m-3/2}, \nu\right), \tag{2.45}$$

for all $\nu \in S_h$, which is well defined for all $2 \leqslant m \leqslant M-1$. Taking $\psi = \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}$ in (2.44), $\nu = \tau(\frac{3}{4}\delta_\tau\phi_h^{m+1/2} + \frac{1}{4}\delta_\tau\phi_h^{m-3/2})$ in (2.45), and adding the results yields

$$\begin{aligned}&\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) + \tau\left\|\frac{3}{4}\delta_\tau\phi_h^{m+1/2} + \frac{1}{4}\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2\\ &= -\frac{\tau}{\varepsilon}\left(\frac{3}{2}\delta_\tau\phi_h^{m-1/2} - \frac{1}{2}\delta_\tau\phi_h^{m-3/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right)\\ &\quad + \frac{1}{4\varepsilon}\left(\omega_h^m\left(\phi_h^{m+1} - \phi_h^{m-1}\right), \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{\tau}{\varepsilon}\left(\frac{3}{2}\delta_\tau\phi_h^{m-1/2} - \frac{1}{2}\delta_\tau\phi_h^{m-3/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) \\
&\quad + \frac{\tau}{4\varepsilon}\left(\omega_h^m\delta_\tau\phi_h^{m+1/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) \\
&\quad + \frac{\tau}{4\varepsilon}\left(\omega_h^m\delta_\tau\phi_h^{m-1/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) \\
&\leqslant \frac{3\tau}{8\varepsilon}\left\|\delta_\tau\phi_h^{m-1/2}\right\|_{L^2}\left\|3\mu_h^{m+1/2} + \mu_h^{m-3/2}\right\|_{L^2} \\
&\quad + \frac{\tau}{8\varepsilon}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}\left\|3\mu_h^{m+1/2} + \mu_h^{m-3/2}\right\|_{L^2} \\
&\quad + \frac{\tau}{16\varepsilon}\left\|\omega_h^m\right\|_{L^3}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}\left\|3\mu_h^{m+1/2} + \mu_h^{m-3/2}\right\|_{L^6} \\
&\quad + \frac{\tau}{16\varepsilon}\left\|\omega_h^m\right\|_{L^3}\left\|\delta_\tau\phi_h^{m-1/2}\right\|_{L^2}\left\|3\mu_h^{m+1/2} + \mu_h^{m-3/2}\right\|_{L^6}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) + \tau\left\|\frac{3}{4}\delta_\tau\phi_h^{m+1/2} + \frac{1}{4}\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2 \\
&\quad\leqslant \frac{\tau}{8}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2 + \frac{\tau}{32}\left\|\delta_\tau\phi_h^{m-1/2}\right\|_{L^2}^2 + \frac{\tau}{32}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2 \\
&\qquad + C\tau\left\|\mu_h^{m+1/2}\right\|_{H^1}^2 + C\tau\left\|\mu_h^{m-3/2}\right\|_{H^1}^2,
\end{aligned}
$$

where we use the $H^1(\Omega) \hookrightarrow L^6(\Omega)$ embedding to achieve the following bound:

$$
\begin{aligned}
\left\|\omega_h^m\right\|_{L^3} &= \left\|\left(\phi_h^{m+1}\right)^2 + \left(\phi_h^m\right)^2 + \left(\phi_h^{m-1}\right)^2 + \phi_h^{m+1}\phi_h^m + \phi_h^{m+1}\phi_h^{m-1} + \phi_h^m\phi_h^{m-1}\right\|_{L^3} \\
&\leqslant C\left\|\phi_h^{m+1}\right\|_{L^6}^2 + C\left\|\phi_h^m\right\|_{L^6}^2 + C\left\|\phi_h^{m-1}\right\|_{L^6}^2 \leqslant C.
\end{aligned}
$$

Applying $\sum_{m=2}^{\ell}$ and using the following properties:

$$
\begin{aligned}
&\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \frac{3}{4}\mu_h^{m+1/2} + \frac{1}{4}\mu_h^{m-3/2}\right) \\
&\quad = \frac{1}{2}\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \mu_h^{m+1/2} + \mu_h^{m-1/2}\right) \\
&\qquad + \frac{1}{4}\left(\mu_h^{m+1/2} - \mu_h^{m-1/2}, \mu_h^{m+1/2} - 2\mu_h^{m-1/2} + \mu_h^{m-3/2}\right) \\
&\quad = \frac{1}{2}\left\|\mu_h^{m+1/2}\right\|_{L^2}^2 - \frac{1}{2}\left\|\mu_h^{m-1/2}\right\|_{L^2}^2 + \frac{1}{8}\left\|\mu_h^{m+1/2} - \mu_h^{m-1/2}\right\|_{L^2}^2 \\
&\qquad - \frac{1}{8}\left\|\mu_h^{m-1/2} - \mu_h^{m-3/2}\right\|_{L^2}^2 + \frac{1}{8}\left\|\mu_h^{m+1/2} - 2\mu_h^{m-1/2} + \mu_h^{m-3/2}\right\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\frac{3}{4}\delta_\tau\phi_h^{m+1/2}+\frac{1}{4}\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2\\
&\quad=\frac{9}{16}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2+\frac{3}{8}\left(\delta_\tau\phi_h^{m+1/2},\delta_\tau\phi_h^{m-3/2}\right)+\frac{1}{16}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2\\
&\quad\geqslant\frac{9}{16}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2-\frac{3}{8}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}+\frac{1}{16}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2\\
&\quad\geqslant\frac{3}{8}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2-\frac{1}{8}\left\|\delta_\tau\phi_h^{m-3/2}\right\|_{L^2}^2,
\end{aligned}
$$

we conclude

$$
\begin{aligned}
\frac{1}{2}\left\|\mu_h^{\ell+1/2}\right\|_{L^2}^2+\frac{\tau}{16}\sum_{m=2}^{\ell}\left\|\delta_\tau\phi_h^{m+1/2}\right\|_{L^2}^2&\leqslant\frac{1}{8}\left\|\mu_h^{3/2}-\mu_h^{1/2}\right\|_{L^2}^2+\frac{3\tau}{16}\left\|\delta_\tau\phi_h^{3/2}\right\|_{L^2}^2+\frac{1}{2}\left\|\mu_h^{3/2}\right\|_{L^2}^2\\
&\quad+\frac{5\tau}{32}\left\|\delta_\tau\phi_h^{1/2}\right\|_{L^2}^2+C\tau\sum_{m=0}^{\ell}\left\|\mu_h^{m+1/2}\right\|_{H^1}^2\leqslant C(T+1),
\end{aligned}
$$

for any $2\leqslant\ell\leqslant M-1$, where we have used Parts 1 and 2 and estimates (2.23) and (2.27). The proof is completed by combining all three parts. □

LEMMA 2.10 Let $(\phi_h^{m+1},\mu_h^{m+1/2})\in S_h\times S_h$ be the unique solution of (2.1a) and (2.1b), and $(\phi_h^1,\mu_h^{1/2})\in S_h\times S_h$, the unique solution of (2.4a) and (2.4b). Then the following estimates hold for any $h,\ \tau>0$:

$$
\left\|\Delta_h\phi_h^{1/2}\right\|_{L^2}^2+\left\|\phi_h^{1/2}\right\|_{L^\infty}^2\leqslant C, \tag{2.46}
$$

$$
\max_{1\leqslant m\leqslant M-1}\left[\left\|\Delta_h\check{\phi}_h^{\,m+1/2}\right\|_{L^2}^2+\left\|\check{\phi}_h^{\,m+1/2}\right\|_{L^\infty}^{4(6-d)/d}\right]\leqslant C(T+1), \tag{2.47}
$$

for some constant $C>0$ that is independent of h, τ and T.

Proof. To prove the first estimate of (2.46), set $\psi=\Delta_h\phi_h^{1/2}$ in (2.4b) and use the definition of the discrete Laplacian (2.7) to obtain

$$
\begin{aligned}
\varepsilon\left\|\Delta_h\phi_h^{1/2}\right\|_{L^2}^2&=-\varepsilon a\left(\phi_h^{1/2},\Delta_h\phi_h^{1/2}\right)\\
&=\varepsilon^{-1}\left(\chi\left(\phi_h^1,\phi_h^0\right),\Delta_h\phi_h^{1/2}\right)-\varepsilon^{-1}\left(\phi_h^0,\Delta_h\phi_h^{1/2}\right)-\left(\mu_h^{1/2},\Delta_h\phi_h^{1/2}\right)+\frac{\tau}{2}a\left(\mu_h^0,\Delta_h\phi_h^{1/2}\right)\\
&\leqslant\frac{\varepsilon}{2}\left\|\Delta_h\phi_h^{1/2}\right\|_{L^2}^2+C\left\|\chi\left(\phi_h^1,\phi_h^0\right)\right\|_{L^2}^2+C\left\|\phi_h^0\right\|_{L^2}^2+C\left\|\mu_h^{1/2}\right\|_{L^2}^2+C\tau^2\left\|\Delta_h\mu_h^0\right\|_{L^2}^2\\
&\leqslant\frac{\varepsilon}{2}\left\|\Delta_h\phi_h^{1/2}\right\|_{L^2}^2+C.
\end{aligned}
$$

The result now follows. The second estimate of (2.46) follows from (2.33), the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$, (2.21) and the first estimate of (2.46).

Setting $\psi = \Delta_h \check{\phi}_h^{\,m+1/2}$ in (2.1b) and using the definition of the discrete Laplacian (2.7), we get

$$\begin{aligned}
\varepsilon \left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 &= -\varepsilon a \left(\check{\phi}_h^{\,m+1/2}, \Delta_h \check{\phi}_h^{\,m+1/2} \right) \\
&= -\left(\mu_h^{m+1/2}, \Delta_h \check{\phi}_h^{\,m+1/2} \right) - \varepsilon^{-1} \left(\tilde{\phi}_h^{\,m+1/2}, \Delta_h \check{\phi}_h^{\,m+1/2} \right) \\
&\quad + \varepsilon^{-1} \left(\chi\left(\phi_h^{m+1}, \phi_h^m\right), \Delta_h \check{\phi}_h^{\,m+1/2} \right) \\
&\leqslant C \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 + \frac{\varepsilon}{2} \left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 + C \left\| \tilde{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 \\
&\quad + C \left\| \chi\left(\phi_h^{m+1}, \phi_h^m\right) \right\|_{L^2}^2 \\
&\leqslant C + C \left\| \mu_h^{m+1/2} \right\|_{L^2}^2 + \frac{\varepsilon}{2} \left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2,
\end{aligned}$$

where we have used the triangle inequality and (2.30). Hence, $\left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 \leqslant C + C \left\| \mu_h^{m+1/2} \right\|_{L^2}^2$, for $1 \leqslant m \leqslant M-1$, and the first estimate of (2.47) follows from (2.35). The second estimate of (2.47) follows from (2.33), the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, (2.21) and the first estimate of (2.47). □

LEMMA 2.11 Let $(\phi_h^{m+1}, \mu_h^{m+1/2}) \in S_h \times S_h$ be the unique solution of (2.1a) and (2.1b), and $(\phi_h^1, \mu_h^{1/2}) \in S_h \times S_h$, the unique solution of (2.4a) and (2.4b). The following estimates hold for any $h,\ \tau > 0$:

$$\max_{0 \leqslant m \leqslant M} \left[\left\| \Delta_h \phi_h^m \right\|_{L^2}^2 + \left\| \phi_h^m \right\|_{L^\infty}^{4(6-d)/d} \right] \leqslant C(T+1), \tag{2.48}$$

for some constant $C > 0$ that is independent of h, τ and T.

Proof. We begin by proving the stability for the first time step. A simple application of the triangle inequality gives the first estimate of (2.48) for $m = 1$ as follows:

$$\begin{aligned}
\left\| \Delta_h \phi_h^1 \right\|_{L^2} = \left\| \Delta_h \phi_h^1 + \Delta_h \phi_h^0 - \Delta_h \phi_h^0 \right\|_{L^2} &\leqslant \left\| \Delta_h \phi_h^1 + \Delta_h \phi_h^0 \right\|_{L^2} + \left\| \Delta_h \phi_h^0 \right\|_{L^2} \\
&\leqslant 2 \left\| \Delta_h \phi_h^{1/2} \right\|_{L^2} + \left\| \Delta_h \phi_h^0 \right\|_{L^2} \leqslant C,
\end{aligned}$$

where we have used the stability of the initial data, inequality (2.19) and the first estimate of (2.46). Next, using (2.33), $H^1(\Omega) \hookrightarrow L^6(\Omega)$, (2.21) and the first estimate of (2.48), we get the second estimate of (2.48) for $m = 1$. For $2 \leqslant m \leqslant M-1$, by definition:

$$\begin{aligned}
\left\| \Delta_h \check{\phi}_h^{\,m+1/2} \right\|_{L^2}^2 &= \left\| \Delta_h \left(\tfrac{3}{4} \phi_h^{m+1} + \tfrac{1}{4} \phi_h^{m-1} \right) \right\|_{L^2}^2 \\
&= \tfrac{9}{16} \left\| \Delta_h \phi_h^{m+1} \right\|_{L^2}^2 + \tfrac{3}{8} \left(\Delta_h \phi_h^{m+1}, \Delta_h \phi_h^{m-1} \right) + \tfrac{1}{16} \left\| \Delta_h \phi_h^{m-1} \right\|_{L^2}^2 \\
&\geqslant \tfrac{9}{16} \left\| \Delta_h \phi_h^{m+1} \right\|_{L^2}^2 - \tfrac{3}{16} \left\| \Delta_h \phi_h^{m+1} \right\|_{L^2}^2 - \tfrac{3}{16} \left\| \Delta_h \phi_h^{m-1} \right\|_{L^2}^2 + \tfrac{1}{16} \left\| \Delta_h \phi_h^{m-1} \right\|_{L^2}^2 \\
&= \tfrac{3}{8} \left\| \Delta_h \phi_h^{m+1} \right\|_{L^2}^2 - \tfrac{1}{8} \left\| \Delta_h \phi_h^{m-1} \right\|_{L^2}^2 .
\end{aligned}$$

With repeated application of the last estimate and using the first estimate of (2.47), we find

$$\begin{aligned}\left\|\Delta_h\phi_h^{2m}\right\|_{L^2}^2 &\leqslant \tfrac{8}{3}\left(1+\tfrac{1}{3}+\left(\tfrac{1}{3}\right)^2+\cdots+\left(\tfrac{1}{3}\right)^{m-1}\right)C(T+1)+\left(\tfrac{1}{3}\right)^m\left\|\Delta_h\phi_h^0\right\|_{L^2}^2\\ &\leqslant \tfrac{8}{3}\cdot\tfrac{3}{2}C(T+1)+\left(\tfrac{1}{3}\right)^m\cdot C\leqslant C(T+1),\end{aligned}$$

and

$$\begin{aligned}\left\|\Delta_h\phi_h^{2m+1}\right\|_{L^2}^2 &\leqslant \tfrac{8}{3}\left(1+\tfrac{1}{3}+\left(\tfrac{1}{3}\right)^2+\cdots+\left(\tfrac{1}{3}\right)^{m-1}\right)C(T+1)+\left(\tfrac{1}{3}\right)^m\left\|\Delta_h\phi_h^1\right\|_{L^2}^2\\ &\leqslant \tfrac{8}{3}\cdot\tfrac{3}{2}C(T+1)+\left(\tfrac{1}{3}\right)^m\cdot C\leqslant C(T+1),\end{aligned}$$

and the first estimate of (2.48) follows. The second estimate of (2.48) follows from (2.33), the first estimate of (2.48) and the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$. □

3. Error estimates for the fully discrete convex splitting scheme

In this section, we provide a rigorous convergence analysis for our scheme in the appropriate energy norms. We shall assume that weak solutions have the additional regularities

$$\begin{aligned}&\phi\in L^\infty\left(0,T;W^{1,6}(\Omega)\right)\cap H^1\left(0,T;H^{q+1}(\Omega)\right)\cap H^2(0,T;H^3(\Omega))\cap H^3(0,T;L^2(\Omega)),\\ &\phi^2\in H^2\left(0,T;H^1(\Omega)\right),\\ &\mu\in L^2\left(0,T;H^{q+1}(\Omega)\right),\end{aligned}\tag{3.1}$$

where $q\geqslant 1$. The norm bounds associated with the assumed regularities above are not necessarily global-in-time, and therefore can involve constants that depend upon the final time T. We also assume that the initial data are sufficiently regular so that the stability (2.19) holds. Weak solutions (ϕ,μ) to (1.4a) and (1.4b) with the higher regularities (3.1) solve the following variational problem: for all $t\in[0,T]$,

$$(\partial_t\phi,\nu)+\varepsilon a(\mu,\nu)=0\quad\forall\nu\in H^1(\Omega),\tag{3.2a}$$

$$(\mu,\psi)-\varepsilon a(\phi,\psi)-\varepsilon^{-1}\left(\phi^3-\phi,\psi\right)=0\quad\forall\psi\in H^1(\Omega).\tag{3.2b}$$

We define the following: for any *real* number $m\in[0,M]$,

$$t_m:=m\tau,\quad \phi^m:=\phi(t_m),\quad \mathcal{E}_a^{\phi,m}:=\phi^m-R_h\phi^m,\quad \mathcal{E}_a^{\mu,m}:=\mu^m-R_h\mu^m;$$

and for any integer $0\leqslant m\leqslant M-1$,

$$\begin{aligned}&\delta_\tau\phi^{m+1/2}:=\frac{\phi^{m+1}-\phi^m}{\tau},\quad \sigma_1^{m+1/2}:=\delta_\tau R_h\phi^{m+1/2}-\delta_\tau\phi^{m+1/2},\\ &\sigma_2^{m+1/2}:=\delta_\tau\phi^{m+1/2}-\partial_t\phi^{m+1/2},\quad \sigma_3^{m+1/2}:=\tfrac{1}{2}\phi^{m+1}+\tfrac{1}{2}\phi^m-\phi^{m+1/2}\\ &\sigma_4^{m+1/2}:=\chi\left(\phi^{m+1},\phi^m\right)-\left(\phi^{m+1/2}\right)^3.\end{aligned}$$

Then the PDE solution, evaluated at the half-integer time steps $t_{m+1/2}$, satisfies

$$\left(\delta_\tau R_h\phi^{m+1/2}, \nu\right) + \varepsilon a\left(R_h\mu^{m+1/2}, \nu\right) = \left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, \nu\right), \tag{3.3a}$$

$$\begin{aligned}\varepsilon a\left(\frac{1}{2}R_h\phi^{m+1} + \frac{1}{2}R_h\phi^m, \psi\right) - \left(R_h\mu^{m+1/2}, \psi\right)& \\ = \left(\mathcal{E}_a^{\mu,m+1/2}, \psi\right) - \frac{1}{\varepsilon}\left(\chi\left(\phi^{m+1}, \phi^m\right), \psi\right) + \frac{1}{\varepsilon}\left(\phi^{m+1/2}, \psi\right)& \\ + \varepsilon a\left(\sigma_3^{m+1/2}, \psi\right) + \frac{1}{\varepsilon}\left(\sigma_4^{m+1/2}, \psi\right),&\end{aligned} \tag{3.3b}$$

for all $\nu, \psi \in S_h$. Restating the fully discrete splitting scheme, Eqs. (2.1a), (2.1b), (2.4a) and (2.4b), we have, for all $\nu, \psi \in S_h$,

$$\left(\delta_\tau\phi_h^{1/2}, \nu\right) + \varepsilon a\left(\mu_h^{1/2}, \nu\right) = 0, \tag{3.4a}$$

$$\varepsilon a\left(\phi_h^{1/2}, \psi\right) - \left(\mu_h^{1/2}, \psi\right) = -\frac{1}{\varepsilon}\left(\chi\left(\phi_h^1, \phi_h^0\right), \psi\right) + \frac{1}{\varepsilon}\left(\phi_h^0 + \frac{\tau}{2}\partial_t\phi^0, \psi\right); \tag{3.4b}$$

and, for $1 \leqslant m \leqslant M - 1$, and all $\nu, \psi \in S_h$,

$$\left(\delta_\tau\phi_h^{m+1/2}, \nu\right) + \varepsilon a\left(\mu_h^{m+1/2}, \nu\right) = 0, \tag{3.5a}$$

$$\begin{aligned}\varepsilon a\left(\phi_h^{m+1/2}, \psi\right) + \frac{\varepsilon}{4}a\left(\phi_h^{m+1} - 2\phi_h^m + \phi_h^{m-1}, \psi\right) - \left(\mu_h^{m+1/2}, \psi\right)& \\ = -\frac{1}{\varepsilon}\left(\chi\left(\phi_h^{m+1}, \phi_h^m\right), \psi\right) + \frac{1}{\varepsilon}\left(\tilde{\phi}_h^{m+1/2}, \psi\right).&\end{aligned} \tag{3.5b}$$

Now let us define the following additional error terms: for any integers $0 \leqslant m \leqslant M$,

$$\mathcal{E}_h^{\phi,m} := R_h\phi^m - \phi_h^m, \quad \mathcal{E}^{\phi,m} := \phi^m - \phi_h^m, \tag{3.6}$$

and, for any integers $0 \leqslant m \leqslant M - 1$

$$\mathcal{E}_h^{\mu,m+1/2} := R_h\mu^{m+1/2} - \mu_h^{m+1/2}, \quad \mathcal{E}^{\mu,m+1/2} := \mu^{m+1/2} - \mu_h^{m+1/2}. \tag{3.7}$$

Setting $m = 0$ in (3.3a) and (3.3b) and subtracting (3.4a) and (3.4b), we have

$$\left(\delta_\tau\mathcal{E}_h^{\phi,1/2}, \nu\right) + \varepsilon a\left(\mathcal{E}_h^{\mu,1/2}, \nu\right) = \left(\sigma_1^{1/2} + \sigma_2^{1/2}, \nu\right), \tag{3.8a}$$

$$\begin{aligned}\frac{\varepsilon}{2}a\left(\mathcal{E}_h^{\phi,1} + \mathcal{E}_h^{\phi,0}, \psi\right) - \left(\mathcal{E}_h^{\mu,1/2}, \psi\right) = \left(\mathcal{E}_a^{\mu,1/2}, \psi\right) - \frac{1}{\varepsilon}\left(\chi\left(\phi^1, \phi^0\right) - \chi\left(\phi_h^1, \phi_h^0\right), \psi\right)& \\ + \frac{1}{\varepsilon}\left(\phi^{1/2} - \phi_h^0 - \frac{\tau}{2}\partial_t\phi^0, \psi\right) + \varepsilon a\left(\sigma_3^{1/2}, \psi\right)& \\ + \frac{1}{\varepsilon}\left(\sigma_4^{1/2}, \psi\right).&\end{aligned} \tag{3.8b}$$

Similarly, subtracting (3.5a) and (3.5b) from (3.3a) and (3.3b), yields, for $1 \leqslant m \leqslant M-1$,

$$\left(\delta_\tau \mathcal{E}_h^{\phi,m+1/2}, \nu\right) + \varepsilon a\left(\mathcal{E}_h^{\mu,m+1/2}, \nu\right) = \left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, \nu\right), \tag{3.9a}$$

$$\begin{aligned}
&\frac{\varepsilon}{2} a\left(\mathcal{E}_h^{\phi,m+1} + \mathcal{E}_h^{\phi,m}, \psi\right) + \frac{\varepsilon\tau^2}{4} a\left(\delta_\tau^2 \mathcal{E}_h^{\phi,m}, \psi\right) - \left(\mathcal{E}_h^{\mu,m+1/2}, \psi\right) \\
&\quad = \left(\mathcal{E}_a^{\mu,m+1/2}, \psi\right) - \frac{1}{\varepsilon}\left(\chi\left(\phi^{m+1}, \phi^m\right) - \chi\left(\phi_h^{m+1}, \phi_h^m\right), \psi\right) \\
&\qquad + \frac{1}{\varepsilon}\left(\phi^{m+1/2} - \tilde{\phi}_h^{m+1/2}, \psi\right) + \varepsilon\, a\left(\sigma_3^{m+1/2}, \psi\right) \\
&\qquad + \frac{1}{\varepsilon}\left(\sigma_4^{m+1/2}, \psi\right) + \frac{\varepsilon\tau^2}{4} a\left(\delta_\tau^2 \phi^m, \psi\right),
\end{aligned} \tag{3.9b}$$

where $\tau^2 \delta_\tau^2 \psi^m := \psi^{m+1} - 2\psi^m + \psi^{m-1}$.

Now, define the additional error terms

$$\sigma_5^{m+1/2} := \chi\left(\phi_h^{m+1}, \phi_h^m\right) - \chi\left(\phi^{m+1}, \phi^m\right), \tag{3.10}$$

$$\sigma_6^{m+1/2} := \phi^{m+1/2} - \begin{cases} \phi_h^0 + \dfrac{\tau}{2}\partial_t \phi^0, & \text{for } m = 0 \\ \tilde{\phi}_h^{m+1/2}, & \text{for } 1 \leqslant m \leqslant M-1. \end{cases} \tag{3.11}$$

Then, setting $\nu = \mathcal{E}_h^{\mu,1/2}$ in (3.8a) and $\psi = \delta_\tau \mathcal{E}_h^{\phi,1/2}$ in (3.8b), setting $\nu = \mathcal{E}_h^{\mu,m+1/2}$ in (3.9a) and $\psi = \delta_\tau \mathcal{E}_h^{\phi,m+1/2}$ in (3.9b), and adding the resulting equations, we have

$$\begin{aligned}
&\frac{\varepsilon}{2} a\left(\mathcal{E}_h^{\phi,m+1} + \mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\gamma_m \varepsilon\tau^2}{4} a\left(\delta_\tau^2 \mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \varepsilon\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 \\
&\quad = \left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, \mathcal{E}_h^{\mu,m+1/2}\right) + \left(\mathcal{E}_a^{\mu,m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \varepsilon a\left(\sigma_3^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) \\
&\qquad + \frac{1}{\varepsilon}\left(\sigma_4^{m+1/2} + \sigma_5^{m+1/2} + \sigma_6^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\gamma_m \varepsilon\tau^2}{4} a\left(\delta_\tau^2 \phi^m, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right),
\end{aligned} \tag{3.12}$$

for all $0 \leqslant m \leqslant M-1$, where $\gamma_m := 1 - \delta_{0,m}$ and $\delta_{k,\ell}$ is the Kronecker delta function. The terms involving γ_m are 'turned on' only when $m \geqslant 1$. Expression (3.12) is the key error equation from which we will define our error estimates.

LEMMA 3.1 Suppose that (ϕ, μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then for all $t_m \in [0, T]$ and for any $h, \tau > 0$, there exists a constant $C > 0$, independent of h and τ and T, such that

$$\left\|\sigma_1^{m+1/2}\right\|_{L^2}^2 \leqslant C\frac{h^{2q+2}}{\tau} \int_{t_m}^{t_{m+1}} \|\partial_s \phi(s)\|_{H^{q+1}}^2 \,\mathrm{d}s, \quad 0 \leqslant m \leqslant M-1, \tag{3.13}$$

$$\left\|\sigma_2^{m+1/2}\right\|_{L^2}^2 \leqslant \frac{\tau^3}{640} \int_{t_m}^{t_{m+1}} \|\partial_{sss} \phi(s)\|_{L^2}^2 \,\mathrm{d}s, \quad 0 \leqslant m \leqslant M-1, \tag{3.14}$$

$$\left\|\nabla\Delta\sigma_3^{m+1/2}\right\|_{L^2}^2 \leqslant \frac{\tau^3}{96}\int_{t_m}^{t_{m+1}} \|\nabla\Delta\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s, \quad 0 \leqslant m \leqslant M-1, \tag{3.15}$$

$$\left\|\nabla\sigma_3^{m+1/2}\right\|_{L^2}^2 \leqslant \frac{\tau^3}{96}\int_{t_m}^{t_{m+1}} \|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s, \quad 0 \leqslant m \leqslant M-1, \tag{3.16}$$

$$\begin{aligned}&\left\|\frac{1}{2}\left(\phi^{m+1}\right)^2 + \frac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right\|_{H^1}^2 \\ &\quad\leqslant \frac{\tau^3}{96}\int_{t_m}^{t_{m+1}} \left\|\partial_{ss}\phi^2(s)\right\|_{H^1}^2\,\mathrm{d}s, \quad 0 \leqslant m \leqslant M-1,\end{aligned} \tag{3.17}$$

$$\left\|\tau^2\nabla\Delta\delta_\tau^2\phi^m\right\|_{L^2}^2 \leqslant \frac{\tau^3}{3}\int_{t_{m-1}}^{t_{m+1}} \|\nabla\Delta\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s, \quad 1 \leqslant m \leqslant M-1, \tag{3.18}$$

$$\left\|\tau^2\nabla\delta_\tau^2\phi^m\right\|_{L^2}^2 \leqslant \frac{\tau^3}{3}\int_{t_{m-1}}^{t_{m+1}} \|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s, \quad 1 \leqslant m \leqslant M-1, \tag{3.19}$$

$$\left\|\nabla\left(\phi^{m+1/2} - \frac{3}{2}\phi^m + \frac{1}{2}\phi^{m-1}\right)\right\|_{L^2}^2 \leqslant \frac{\tau^3}{12}\int_{t_{m-1}}^{t_{m+1}} \|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s, \quad 1 \leqslant m \leqslant M-1, \tag{3.20}$$

$$\left\|\nabla\left(\phi^{1/2} - \phi^0 - \frac{\tau}{2}\partial_t\phi^0\right)\right\|_{L^2} \leqslant \frac{\tau^3}{24}\int_{t_0}^{t_{1/2}} \|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s. \tag{3.21}$$

Proof. The proof of each of the inequalities above is a direct application of Taylor's theorem with integral remainder. We suppress the details for the sake of brevity. □

LEMMA 3.2 Suppose that (ϕ,μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, there exists a constant $C > 0$ independent of h and τ—but possibly dependent upon T through the regularity estimates—such that, for any $h, \tau > 0$,

$$\left\|\nabla\sigma_4^{m+1/2}\right\|_{L^2}^2 \leqslant C\tau^3\int_{t_m}^{t_{m+1}} \|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s + C\tau^3\int_{t_m}^{t_{m+1}} \left\|\partial_{ss}\phi^2(s)\right\|_{H^1}^2\,\mathrm{d}s. \tag{3.22}$$

Proof. We begin with the expansion

$$\begin{aligned}\nabla\sigma_4^{m+1/2} = {}& \left(\tfrac{1}{2}\phi^{m+1} + \tfrac{1}{2}\phi^m - \phi^{m+1/2}\right)\nabla\left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2\right) \\ &+ \left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2\right)\nabla\left(\tfrac{1}{2}\phi^{m+1} + \tfrac{1}{2}\phi^m - \phi^{m+1/2}\right) \\ &+ \phi^{m+1/2}\nabla\left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right) \\ &+ \left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right)\nabla\phi^{m+1/2}.\end{aligned} \tag{3.23}$$

By the triangle inequality, Young's inequality, and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned}
\left\|\nabla\sigma_4^{m+1/2}\right\|_{L^2} \leqslant{}& \left\|\tfrac{1}{2}\phi^{m+1} + \tfrac{1}{2}\phi^m - \phi^{m+1/2}\right\|_{L^6} \left\|\nabla\left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2\right)\right\|_{L^3} \\
&+ \left\|\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2\right\|_{L^\infty} \left\|\nabla\left(\tfrac{1}{2}\phi^{m+1} + \tfrac{1}{2}\phi^m - \phi^{m+1/2}\right)\right\|_{L^2} \\
&+ \left\|\phi^{m+1/2}\right\|_{L^\infty} \left\|\nabla\left(\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right)\right\|_{L^2} \\
&+ \left\|\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right\|_{L^6} \left\|\nabla\phi^{m+1/2}\right\|_{L^3} \\
\leqslant{}& C\left\{\left\|\phi^{m+1}\right\|_{L^\infty}^2 + \left\|\phi^m\right\|_{L^\infty}^2 + \left\|\phi^{m+1}\right\|_{L^6}\left\|\nabla\phi^{m+1}\right\|_{L^6} + \left\|\phi^m\right\|_{L^6}\left\|\nabla\phi^m\right\|_{L^6}\right\} \\
&\times \left\|\nabla\left(\tfrac{1}{2}\phi^{m+1} + \tfrac{1}{2}\phi^m - \phi^{m+1/2}\right)\right\|_{L^2} \\
&+ C\left\{\left\|\phi^{m+1/2}\right\|_{L^\infty} + \left\|\nabla\phi^{m+1/2}\right\|_{L^3}\right\} \times \left\|\tfrac{1}{2}\left(\phi^{m+1}\right)^2 + \tfrac{1}{2}\left(\phi^m\right)^2 - \left(\phi^{m+1/2}\right)^2\right\|_{H^1}.
\end{aligned} \tag{3.24}$$

Using the assumed regularities (3.1) of the PDE solution, and appealing to the truncation error estimates (3.16) and (3.17), the result follows. □

Lemma 3.3 Suppose that (ϕ, μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, there exists a constant $C > 0$ independent of h and τ, but possibly dependent upon T, such that, for any $h, \tau > 0$,

$$\left\|\nabla\sigma_5^{m+1/2}\right\|_{L^2}^2 \leqslant C\left\|\nabla\mathcal{E}^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}^2, \tag{3.25}$$

where $\mathcal{E}^{\phi,m} := \phi^m - \phi_h^m$.

Proof. We begin with the detailed expansion

$$\begin{aligned}
4\nabla\sigma_5^{m+1/2} ={}& \left\{\left(\phi_h^{m+1}\right)^2 + \left(\phi_h^m\right)^2 + 2\phi_h^{m+1}\left(\phi_h^{m+1} + \phi_h^m\right)\right\}\nabla\left(\phi_h^{m+1} - \phi^{m+1}\right) \\
&+ \left\{\left(\phi_h^{m+1}\right)^2 + \left(\phi_h^m\right)^2 + 2\phi_h^m\left(\phi_h^{m+1} + \phi_h^m\right)\right\}\nabla\left(\phi_h^m - \phi^m\right) \\
&+ \left\{\nabla\left(\phi^{m+1} + \phi^m\right)\cdot\left(\phi_h^{m+1} + \phi^{m+1}\right) + 2\nabla\phi^{m+1}\left(\phi_h^{m+1} + \phi_h^m\right)\right. \\
&\left.+ 2\phi^{m+1}\nabla\phi^{m+1} + 2\phi^m\nabla\phi^m\right\}\left(\phi_h^{m+1} - \phi^{m+1}\right) \\
&+ \left\{\nabla\left(\phi^{m+1} + \phi^m\right)\cdot\left(\phi_h^m + \phi^m\right) + 2\nabla\phi^m\left(\phi_h^{m+1} + \phi_h^m\right)\right. \\
&\left.+ 2\phi^{m+1}\nabla\phi^{m+1} + 2\phi^m\nabla\phi^m\right\}\left(\phi_h^m - \phi^m\right).
\end{aligned} \tag{3.26}$$

Then, using the unconditional *a priori* estimates in Lemmas 2.6 and 2.11, the assumption that $\phi \in L^\infty(0,T;W^{1,6}(\Omega))$, and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we have, for any $0 \leqslant m \leqslant M-1$,

$$
\begin{aligned}
\left\|\nabla\sigma_5^{m+1/2}\right\|_{L^2} &\leqslant C\left\{\left\|\phi_h^{m+1}\right\|_{L^\infty}^2 + \left\|\phi_h^m\right\|_{L^\infty}^2\right\}\left(\left\|\nabla\mathcal{E}^{\phi,m+1}\right\|_{L^2} + \left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}\right) \\
&\quad + C\left\{\left(\left\|\nabla\phi^{m+1}\right\|_{L^6} + \|\nabla\phi^m\|_{L^6}\right)\right. \\
&\quad \cdot \left.\left(\left\|\phi^{m+1}\right\|_{L^6} + \|\phi^m\|_{L^6} + \left\|\phi_h^{m+1}\right\|_{L^6} + \left\|\phi_h^m\right\|_{L^6}\right)\right\} \\
&\quad \times \left(\left\|\mathcal{E}^{\phi,m+1}\right\|_{L^6} + \left\|\mathcal{E}^{\phi,m}\right\|_{L^6}\right) \\
&\leqslant C\left\|\nabla\mathcal{E}^{\phi,m+1}\right\|_{L^2} + C\left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}.
\end{aligned} \tag{3.27}
$$

□

LEMMA 3.4 Suppose that (ϕ,μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, there exists a constant $C > 0$ independent of h and τ such that, for any $h,\tau > 0$,

$$
\begin{aligned}
\left\|\nabla\sigma_6^{m+1/2}\right\|_{L^2}^2 &\leqslant \gamma_m C\tau^3\int_{t_{m-1}}^{t_m}\|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s + C\tau^3\int_{t_m}^{t_{m+1}}\|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s \\
&\quad + C\left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}^2 + \gamma_m C\left\|\nabla\mathcal{E}^{\phi,m-1}\right\|_{L^2}^2 + \delta_{0,m}Ch^{2q}\,|\phi_0|_{H^{q+1}}^2,
\end{aligned} \tag{3.28}
$$

where $\mathcal{E}^{\phi,m} := \phi^m - \phi_h^m$ and $\delta_{k,\ell}$ is the Kronecker delta.

Proof. For $m = 0$, using the truncation error estimate (3.21) and a standard finite element estimate for the Ritz projection, we have

$$
\begin{aligned}
\left\|\nabla\sigma_6^{1/2}\right\|_{L^2}^2 &\leqslant 2\left\|\nabla\left(\phi^{1/2} - \phi_0 - \frac{\tau}{2}\partial_t\phi(0)\right)\right\|_{L^2}^2 + 2\left\|\nabla\left(\phi_0 - \phi_h^0\right)\right\|_{L^2}^2 \\
&\leqslant 2\frac{\tau^3}{24}\int_{t_0}^{t_{1/2}}\|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s + Ch^{2q}\,|\phi_0|_{H^{q+1}}^2,
\end{aligned} \tag{3.29}
$$

with the observation that $\phi_h^0 := R_h\phi_0$. For $1 \leqslant m \leqslant M-1$, using the truncation error estimate (3.20), we obtain

$$
\left\|\nabla\sigma_6^{m+1/2}\right\|_{L^2}^2 \leqslant 3\frac{\tau^3}{6}\int_{t_{m-1}}^{t_{m+1}}\|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s + \frac{27}{4}\left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}^2 + \frac{3}{4}\left\|\nabla\mathcal{E}^{\phi,m-1}\right\|_{L^2}^2. \tag{3.30}
$$

□

We now proceed to estimate the terms on the right-hand-side of (3.12). We will need the following technical lemmas. The proof of the next result can be found in Diegel *et al.* (2015).

LEMMA 3.5 Suppose $g \in H^1(\Omega)$, and $v \in S_h$. Then

$$
|(g,v)| \leqslant C\,\|\nabla g\|_{L^2}\,\|v\|_{-1,h}, \tag{3.31}
$$

for some $C > 0$ that is independent of h.

LEMMA 3.6 Suppose that (ϕ, μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, for any h, $\tau > 0$ and any $\alpha > 0$, there exists a constant $C = C(\alpha, T) > 0$, independent of h and τ, such that, for $0 \leqslant m \leqslant M-1$,

$$\frac{\varepsilon}{2} a\left(\mathcal{E}_h^{\phi,m+1} + \mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\gamma_m \varepsilon \tau^2}{4} a\left(\delta_\tau^2 \mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2$$
$$\leqslant C\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + \gamma_m C\left\|\nabla \mathcal{E}_h^{\phi,m-1}\right\|_{L^2}^2 + \alpha\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 + C\mathcal{R}^{m+1/2}, \quad (3.32)$$

where

$$\begin{aligned}\mathcal{R}^{m+1/2} = {} & \frac{h^{2q+2}}{\tau}\int_{t_m}^{t_{m+1}} \|\partial_s \phi(s)\|_{H^{q+1}}^2 \, \mathrm{d}s + h^{2q}\left|\mu^{m+1/2}\right|_{H^{q+1}}^2 \\ & + h^{2q}\left|\phi^{m+1}\right|_{H^{q+1}}^2 + h^{2q}\left|\phi^m\right|_{H^{q+1}}^2 + \gamma_m h^{2q}\left|\phi^{m-1}\right|_{H^{q+1}}^2 \\ & + \tau^3 \int_{t_m}^{t_{m+1}} \|\partial_{sss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \tau^3 \int_{t_m}^{t_{m+1}} \left\|\partial_{ss}\phi^2(s)\right\|_{H^1}^2 \, \mathrm{d}s \\ & + \gamma_m \tau^3 \int_{t_{m-1}}^{t_m} \|\nabla \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \tau^3 \int_{t_m}^{t_{m+1}} \|\nabla \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s \\ & + \gamma_m \tau^3 \int_{t_{m-1}}^{t_m} \|\nabla \Delta \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \tau^3 \int_{t_m}^{t_{m+1}} \|\nabla \Delta \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s. \end{aligned} \quad (3.33)$$

Proof. Define, for $0 \leqslant m \leqslant M-1$, time-dependent spatial mass average

$$\overline{\mathcal{E}_h^{\mu,m+1/2}} := |\Omega|^{-1}\left(\mathcal{E}_h^{\mu,m+1/2}, 1\right). \quad (3.34)$$

Using the Cauchy–Schwarz inequality, the Poincaré inequality, with the fact that

$$\left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, 1\right) = 0,$$

and the local truncation error estimates (3.13) and (3.14), we get the following estimate:

$$\begin{aligned}\left|\left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, \mathcal{E}_h^{\mu,m+1/2}\right)\right| & = \left|\left(\sigma_1^{m+1/2} + \sigma_2^{m+1/2}, \mathcal{E}_h^{\mu,m+1/2} - \overline{\mathcal{E}_h^{\mu,m+1/2}}\right)\right| \\ & \leqslant \left\|\sigma_1^{m+1/2} + \sigma_2^{m+1/2}\right\|_{L^2}\left\|\mathcal{E}_h^{\mu,m+1/2} - \overline{\mathcal{E}_h^{\mu,m+1/2}}\right\|_{L^2} \\ & \leqslant C\left\|\sigma_1^{m+1/2} + \sigma_2^{m+1/2}\right\|_{L^2}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2} \\ & \leqslant C\left\|\sigma_1^{m+1/2}\right\|_{L^2}^2 + C\left\|\sigma_2^{m+1/2}\right\|_{L^2}^2 + \frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 \\ & \leqslant C\frac{h^{2q+2}}{\tau}\int_{t_m}^{t_{m+1}} \|\partial_s \phi(s)\|_{H^{q+1}}^2 \, \mathrm{d}s \\ & \quad + C\frac{\tau^3}{640}\int_{t_m}^{t_{m+1}} \|\partial_{sss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \frac{\varepsilon}{2}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2. \end{aligned} \quad (3.35)$$

Standard finite element approximation theory shows that

$$\left\|\nabla \mathcal{E}_a^{\mu,m+1/2}\right\|_{L^2} = \left\|\nabla\left(R_h\mu^{m+1/2} - \mu^{m+1/2}\right)\right\|_{L^2} \leqslant Ch^q \left|\mu^{m+1/2}\right|_{H^{q+1}}.$$

Applying Lemma 3.5 and the last estimate, we have

$$\begin{aligned}\left|\left(\mathcal{E}_a^{\mu,m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right| &\leqslant C\left\|\nabla \mathcal{E}_a^{\mu,m+1/2}\right\|_{L^2}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h} \\ &\leqslant Ch^q\left|\mu^{m+1/2}\right|_{H^{q+1}}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h} \\ &\leqslant Ch^{2q}\left|\mu^{m+1/2}\right|_{H^{q+1}}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2. \end{aligned} \tag{3.36}$$

Using Lemma 3.5 and estimate (3.15), we find

$$\begin{aligned}\varepsilon a\left(\sigma_3^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) &= -\varepsilon\left(\Delta\sigma_3^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) \\ &\leqslant C\left\|\nabla\Delta\sigma_3^{m+1/2}\right\|_{L^2}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h} \\ &\leqslant C\frac{\tau^3}{96}\int_{t_m}^{t_{m+1}}\|\nabla\Delta\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2. \end{aligned} \tag{3.37}$$

Now, using Lemmas 3.2 and 3.5, we obtain

$$\begin{aligned}\varepsilon^{-1}\left|\left(\sigma_4^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right| &\leqslant C\left\|\nabla\sigma_4^{m+1/2}\right\|_{L^2}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h} \\ &\leqslant C\left\|\nabla\sigma_4^{m+1/2}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \\ &\leqslant C\tau^3\int_{t_m}^{t_{m+1}}\|\nabla\partial_{ss}\phi(s)\|_{L^2}^2\,\mathrm{d}s \\ &\quad + C\tau^3\int_{t_m}^{t_{m+1}}\left\|\partial_{ss}\phi^2(s)\right\|_{H^1}^2\,\mathrm{d}s + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2. \end{aligned} \tag{3.38}$$

Similarly, using Lemmas 3.3 and 3.5, the relation $\mathcal{E}^{\phi,m+1} = \mathcal{E}_a^{\phi,m+1} + \mathcal{E}_h^{\phi,m+1}$, and a standard finite element error estimate, we arrive at

$$\begin{aligned}\varepsilon^{-1}\left|\left(\sigma_5^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right| &\leqslant C\left\|\nabla\sigma_5^{m+1/2}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \\ &\leqslant C\left\|\nabla\mathcal{E}^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla\mathcal{E}^{\phi,m}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \end{aligned}$$

$$
\begin{aligned}
&\leqslant C\left\|\nabla \mathcal{E}_a^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla \mathcal{E}_a^{\phi,m}\right\|_{L^2}^2 \\
&\quad + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \\
&\leqslant Ch^{2q}\left|\phi^{m+1}\right|_{H^{q+1}}^2 + C\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + Ch^{2q}\left|\phi^{m}\right|_{H^{q+1}}^2 \\
&\quad + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 .
\end{aligned} \tag{3.39}
$$

Applying Lemmas 3.4 and 3.5, the relation $\mathcal{E}^{\phi,m+1} = \mathcal{E}_a^{\phi,m+1} + \mathcal{E}_h^{\phi,m+1}$, and a standard finite element error estimate,

$$
\begin{aligned}
\varepsilon^{-1}\left|\left(\sigma_6^{m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right| &\leqslant C\left\|\nabla \sigma_6^{m+1/2}\right\|_{L^2}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \\
&\leqslant C\tau^3\left(\gamma_m \int_{t_{m-1}}^{t_m} \|\nabla \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \int_{t_m}^{t_{m+1}} \|\nabla \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s\right) \\
&\quad + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + C\gamma_m\left\|\nabla \mathcal{E}_h^{\phi,m-1}\right\|_{L^2}^2 \\
&\quad + Ch^{2q}\left|\phi^m\right|_{H^{q+1}}^2 + C\gamma_m h^{2q}\left|\phi^{m-1}\right|_{H^{q+1}}^2 + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 .
\end{aligned} \tag{3.40}
$$

To finish up, using (3.16),

$$
\frac{\gamma_m \varepsilon \tau^2}{4} a\left(\delta_\tau^2 \phi^m, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) \leqslant C\gamma_m \frac{\tau^3}{3}\int_{t_{m-1}}^{t_m} \|\nabla \Delta \partial_{ss}\phi(s)\|_{L^2}^2 \, \mathrm{d}s + \frac{\alpha}{6}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 . \tag{3.41}
$$

Combining the estimates (3.35–3.41) with the error equation (3.12), the result follows. □

LEMMA 3.7 Suppose that (ϕ, μ) is a weak solution to (3.3a–3.3b), with the additional regularities (3.1). Then, for any $h, \tau > 0$, there exists a constant $C > 0$, independent of h and τ, such that

$$
\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 \leqslant 2\varepsilon^2\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 + C\mathcal{R}^{m+1/2}, \tag{3.42}
$$

where $\mathcal{R}^{m+1/2}$ is the consistency term given in (3.33).

Proof. Define $\mathsf{T}_h : S_h \to S_h$ via the variational problem: given $\zeta \in S_h$, find $\xi \in S_h$ such that $a(\mathsf{T}_h(\zeta), \xi) = (\zeta, \xi)$ for all $\xi \in S_h$. Then, setting $\nu = \mathsf{T}_h(\delta_\tau \mathcal{E}_h^{\phi,1/2})$ in (3.8a) and $\nu = \mathsf{T}_h(\delta_\tau \mathcal{E}_h^{\phi,m+1/2})$ in

(3.9a) and combining, we have

$$
\begin{aligned}
\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 &= -\varepsilon a\left(\mathcal{E}_h^{\mu,m+1/2}, \mathsf{T}_h\left(\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right) + \left(\sigma_1^{m+1/2}+\sigma_2^{m+1/2}, \mathsf{T}_h\left(\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right)\\
&= -\varepsilon\left(\mathcal{E}_h^{\mu,m+1/2}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \left(\sigma_1^{m+1/2}+\sigma_2^{m+1/2}, \mathsf{T}_h\left(\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right)\\
&\leqslant \varepsilon\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h} + \left\|\sigma_1^{m+1/2}+\sigma_2^{m+1/2}\right\|_{L^2}\left\|\mathsf{T}_h\left(\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right)\right\|_{L^2}\\
&\leqslant \varepsilon^2\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 + \frac{1}{4}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2\\
&\quad + C\left\|\sigma_2^{m+1/2}+\sigma_1^{m+1/2}\right\|_{L^2}^2 + \frac{1}{4}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2\\
&\leqslant \varepsilon^2\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 + \frac{1}{2}\left\|\delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right\|_{-1,h}^2 + C\mathcal{R}^{m+1/2},
\end{aligned} \tag{3.43}
$$

for $0 \leqslant m \leqslant M-1$ and where we have used Lemma 3.1. The result now follows. □

LEMMA 3.8 Suppose that (ϕ, μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, for any h, $\tau > 0$, there exists a constant $C > 0$, independent of h and τ, but possibly dependent upon T, such that

$$
\begin{aligned}
&\frac{\varepsilon}{2} a\left(\mathcal{E}_h^{\phi,m+1}+\mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\gamma_m \tau^2 \varepsilon}{4} a\left(\delta_\tau^2 \mathcal{E}_h^{\phi,m}, \delta_\tau \mathcal{E}_h^{\phi,m+1/2}\right) + \frac{\varepsilon}{4}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2\\
&\quad \leqslant C\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + \gamma_m C\left\|\nabla \mathcal{E}_h^{\phi,m-1}\right\|_{L^2}^2 + C\mathcal{R}^{m+1/2}.
\end{aligned} \tag{3.44}
$$

Proof. This follows upon combining the last two lemmas and choosing α in (3.32) appropriately. □

Using the last lemma, we are ready to show the main convergence result for our second-order convex-splitting scheme.

THEOREM 3.9 Suppose (ϕ, μ) is a weak solution to (3.3a) and (3.3b), with the additional regularities (3.1). Then, provided $0 < \tau < \tau_0$, for some τ_0 sufficiently small,

$$
\max_{0 \leqslant m \leqslant M-1}\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + \tau \sum_{m=0}^{M-1}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2 \leqslant C(T)(\tau^4 + h^{2q}) \tag{3.45}
$$

for some $C(T) > 0$ that is independent of τ and h.

Proof. Using Lemma 3.8, we have

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 - \left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2\right) + \frac{1}{4}\left\|\nabla \mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2\\
&\quad + \frac{\gamma_m}{8\tau}\left(\left\|\nabla \mathcal{E}_h^{\phi,m+1} - \nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 - \left\|\nabla \mathcal{E}_h^{\phi,m} - \nabla \mathcal{E}_h^{\phi,m-1}\right\|_{L^2}^2\right)\\
&\leqslant C\left\|\nabla \mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2 + C\left\|\nabla \mathcal{E}_h^{\phi,m}\right\|_{L^2}^2 + \gamma_m C\left\|\nabla \mathcal{E}_h^{\phi,m-1}\right\|_{L^2}^2 + C\mathcal{R}^{m+1/2}.
\end{aligned} \tag{3.46}
$$

Letting $m=0$ in the previous equation and noting that $\mathcal{E}_h^{\phi,0}\equiv 0$ and $\gamma_0=0$, then

$$\frac{1}{2\tau}\left\|\nabla\mathcal{E}_h^{\phi,1}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla\mathcal{E}_h^{\mu,1/2}\right\|_{L^2}^2\leqslant C_1\left\|\nabla\mathcal{E}_h^{\phi,1}\right\|_{L^2}^2+C\mathcal{R}^{1/2}. \tag{3.47}$$

If $0<\tau\leqslant\tau_0:=1/2C_1<1/C_1$, it follows from the last estimate that

$$\left\|\nabla\mathcal{E}_h^{\phi,1}\right\|_{L^2}^2+\frac{\tau}{2}\left\|\nabla\mathcal{E}_h^{\mu,1/2}\right\|_{L^2}^2\leqslant\tau C\mathcal{R}^{1/2}\leqslant C(\tau^4+h^{2q}), \tag{3.48}$$

where we have used the regularity assumptions to conclude $\tau C\mathcal{R}^{1/2}\leqslant C(\tau^4+h^{2q})$. Now, applying $\tau\sum_{m=0}^{\ell}$ to (3.46),

$$\left\|\nabla\mathcal{E}_h^{\phi,\ell+1}\right\|_{L^2}^2+\frac{\tau}{2}\sum_{m=0}^{\ell}\left\|\nabla\mathcal{E}_h^{\mu,m+1/2}\right\|_{L^2}^2\leqslant C\tau\sum_{m=0}^{\ell}\mathcal{R}^{m+1/2}+C_2\tau\sum_{m=0}^{\ell}\left\|\nabla\mathcal{E}_h^{\phi,m+1}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla\mathcal{E}_h^{\phi,1}\right\|_{L^2}^2. \tag{3.49}$$

If $0<\tau\leqslant\tau_0:=1/2C_2<1/C_2$, it follows from the last estimate that

$$\begin{aligned}\left\|\nabla\mathcal{E}_h^{\phi,\ell+1}\right\|_{L^2}^2&\leqslant C\tau\sum_{m=0}^{\ell}\mathcal{R}^{m+1/2}+\frac{C_2\tau}{1-C_2\tau}\sum_{m=0}^{\ell}\left\|\nabla\mathcal{E}_h^{\phi,m}\right\|_{L^2}^2+\frac{1}{4}\left\|\nabla\mathcal{E}_h^{\phi,1}\right\|_{L^2}^2\\&\leqslant C(\tau^4+h^{2q})+C\tau\sum_{m=0}^{\ell}\left\|\nabla\mathcal{E}_h^{\phi,m}\right\|_{L^2}^2,\end{aligned} \tag{3.50}$$

where we have used (3.48) and the regularity assumptions to conclude $\tau\sum_{m=0}^{M-1}\mathcal{R}^{m+1/2}\leqslant C(\tau^4+h^{2q})$. Appealing to the discrete Gronwall inequality, it follows that, for any $0<\ell\leqslant M-1$,

$$\left\|\nabla\mathcal{E}_h^{\phi,\ell+1}\right\|_{L^2}^2\leqslant C(T)(\tau^4+h^{2q}). \tag{3.51}$$

Considering estimates (3.48), (3.49) and (3.51), we get the desired result. □

REMARK 3.10 From here it is straightforward to establish an optimal error estimate of the form

$$\max_{0\leqslant m\leqslant M-1}\left\|\nabla\mathcal{E}^{\phi,\ell+1}\right\|_{L^2}^2+\tau\sum_{m=0}^{M-1}\left\|\nabla\mathcal{E}^{\mu,m+1/2}\right\|_{L^2}^2\leqslant C(T)(\tau^4+h^{2q}), \tag{3.52}$$

using the error splittings (for example, $\mathcal{E}^{\phi}=\mathcal{E}_a^{\phi}+\mathcal{E}_h^{\phi}$) the triangle inequality, and the standard spatial approximations. We omit the details for the sake of brevity. Furthermore, one can show similar results for the full H^1-norm on the phase field variable by using a Poincaré inequality.

4. Numerical experiments

In this section, we provide some numerical experiments to gauge the accuracy and reliability of the fully discrete finite element method developed in the previous sections. We use a square domain

TABLE 1 H^1 *Cauchy convergence test. The final time is* $T = 4.0 \times 10^{-1}$, *and the refinement path is taken to be* $\tau = 0.001\sqrt{2}h$ *with* $\varepsilon = 6.25 \times 10^{-2}$. *The Cauchy difference is defined via* $\delta_\phi := \phi_{h_f} - \phi_{h_c}$, *where the approximations are evaluated at time* $t = T$, *and analogously for* δ_μ. (*See the discussion in the text.*) *Since* $q = 2$, *i.e., we use* $\mathcal{P}_2$ *elements for these variables, the norm of the Cauchy difference at* T *is expected to be* $\mathcal{O}(\tau_f^2) + \mathcal{O}(h_f^2) = \mathcal{O}(h_f^2)$

h_c	h_f	$\lVert\delta_\phi\rVert_{H^1}$	Rate	$\lVert\delta_\mu\rVert_{H^1}$	Rate
$\sqrt{2}/16$	$\sqrt{2}/32$	1.148×10^{-1}	—	1.307×10^{-1}	—
$\sqrt{2}/32$	$\sqrt{2}/64$	2.939×10^{-2}	1.95	3.299×10^{-2}	1.98
$\sqrt{2}/64$	$\sqrt{2}/128$	7.468×10^{-3}	1.97	8.295×10^{-3}	1.99
$\sqrt{2}/128$	$\sqrt{2}/256$	1.913×10^{-3}	1.95	2.087×10^{-3}	1.99

$\Omega = (0,1)^2 \subset \mathbb{R}^2$ and take $\mathcal{T}_h$ to be a regular triangulation of Ω consisting of right isosceles triangles. To refine the mesh, we assume that $\mathcal{T}_\ell$, $\ell = 0, 1, \ldots, L$, is an hierarchy of nested triangulations of Ω, where $\mathcal{T}_\ell$ is obtained by subdividing the triangles of $\mathcal{T}_{\ell-1}$ into four congruent sub-triangles. Note that $h_{\ell-1} = 2h_\ell$, $\ell = 1, \ldots, L$, and that $\{\mathcal{T}_\ell\}$ is a quasi-uniform family. (We use a family of meshes $\mathcal{T}_h$ such that no triangle in the mesh has more than one edge on the boundary.) We use the $\mathcal{P}_2$ finite element space for the phase field and chemical potential. In short, we take $q = 2$.

We solve the scheme (2.1a) and (2.4b) with $\epsilon = 6.25 \times 10^{-2}$. The initial data for the phase field are taken to be

$$\phi_h^0 = \mathcal{I}_h \left\{ \tfrac{1}{2}(1.0 - \cos(4.0\pi x)) \cdot (1.0 - \cos(2.0\pi y)) - 1.0 \right\}, \tag{4.1}$$

where $\mathcal{I}_h : H^2(\Omega) \to S_h$ is the standard nodal interpolation operator. Recall that our analysis does not specifically cover the use of the operator $\mathcal{I}_h$ in the initialization step. But, since the error introduced by its use is optimal, a slight modification of the analysis shows that this will lead to optimal rates of convergence overall. (See Remark 2.3.) To solve the system of equations above numerically, we are using the finite element libraries from the FEniCS Project (Logg *et al.*, 2012).

Note that source terms are not naturally present in the system of equations (1.2a) and (1.2c). Therefore, it is somewhat artificial to add them to the equations in an attempt to manufacture exact solutions. To circumvent the fact that we do not have possession of exact solutions, and therefore cannot compute the exact error, we instead compute the rate at which the Cauchy difference of the field converges to zero. Specifically, Let ζ be a field variable (i.e., $\zeta = \phi, \mu$). The Cauchy difference of ζ is precisely $\delta_\zeta := \zeta_{h_f}^{M_f} - \zeta_{h_c}^{M_c}$, converges to zero, where $h_f = 2h_c$, $\tau_f = 2\tau_c$ and $\tau_f M_f = \tau_c M_c = T$. Then, using a linear refinement path, i.e., $\tau = Ch$, and assuming $q = 2$ (piecewise quadratic approximations), we have

$$\lVert\delta_\zeta\rVert_{H^1} = \left\lVert \zeta_{h_f}^{M_f} - \zeta_{h_c}^{M_c} \right\rVert_{H^1} \leqslant \left\lVert \zeta_{h_f}^{M_f} - \zeta(T) \right\rVert_{H^1} + \left\lVert \zeta_{h_c}^{M_c} - \zeta(T) \right\rVert_{H^1} = \mathcal{O}(h_f^q + \tau_f^2) = \mathcal{O}(h_f^2). \tag{4.2}$$

The results of the H^1 Cauchy error analysis are found in Table 1 and confirm second-order convergence in this case. Additionally, we have proved that (at the theoretical level) the modified energy is non-increasing at each time step. This is observed in our computations and shown in Fig. 1.

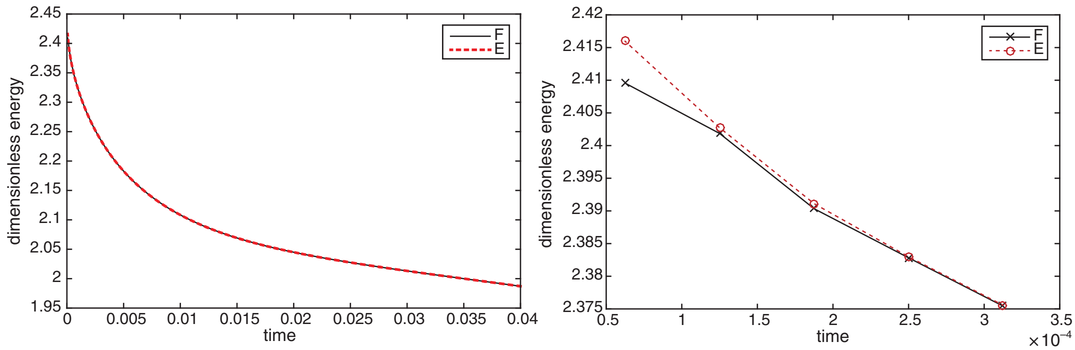

FIG. 1. On the left, we show energy dissipation for the second-order numerical scheme for the Cahn–Hilliard problem, where we have taken $h = \frac{\sqrt{2}}{32}$. The initial data are given in (4.1). The other parameters are as given in the caption of Table 1. We plot both the PDE energy E and the modified (numerical) energy F, though, at this resolution, the plots appear to overlap. On the right, we show the same energy dissipation, magnifying the first five time steps. At the enhanced resolution, we are able to see the difference between the modified energy F and the Cahn–Hilliard energy E.

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