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GLOBAL-IN-TIME GEVREY REGULARITY SOLUTIONS FOR THE FUNCTIONALIZED CAHN-HILLIARD EQUATION

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Abstract. The existence and uniqueness of Gevrey regularity solutions for the functionalized Cahn-Hilliard (FCH) and Cahn-Hilliard-Willmore (CHW) equations are established. The energy dissipation law yields a uniform-in-time H^2 bound of the solution, and the polynomial patterns of the nonlinear terms enable one to derive a local-in-time solution with Gevrey regularity. A careful calculation reveals that the existence time interval length depends on the H^3 norm of the initial data. A further detailed estimate for the original PDE system indicates a uniform-in-time H^3 bound. Consequently, a global-in-time solution becomes available with Gevrey regularity.

The Cahn-Hilliard (CH) equation, which describes spinodal decomposition in a binary alloy, has been one of the most well-known gradient flow-type equations. In a bounded domain $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or $d = 3$), the standard Cahn-Hilliard (CH) energy [2, 6, 7] is given by

$$\mathcal{F}_0(\phi) = \int_\Omega \left\{ \frac{1}{4}\phi^4 - \frac{1}{2}\phi^2 + \frac{\varepsilon^2}{2}\left|\nabla\phi\right|^2 \right\} d\vec{x}, \tag{1}$$

for any $\phi \in H^1(\Omega)$. The variable $\phi : \Omega \to \mathbb{R}$ stands for the phase parameter, and ε is the width of interface. Here and throughout the manuscript, we will assume that

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Ω is a cuboid and that ϕ is Ω–periodic. According to $\mathcal{F}_0$, the lowest energy "pure phase states" are $\phi = \pm 1$. The Cahn-Hiliard chemical potential is the variational derivative of $\mathcal{F}_0$,

$$\mu_0 := \delta_\phi \mathcal{F}_0 = \phi^3 - \phi - \varepsilon^2 \Delta\phi, \tag{2}$$

and the standard Cahn-Hilliard equation is

$$\partial_t \phi = \Delta \mu_0. \tag{3}$$

The Cahn-Hilliard equation (3) has been extensively studied in the existing literature, at both the theoretical and numerical levels. In particular, the Gevrey regularity solution has been proven by [35] for the Cahn-Hilliard equation with dimensions $d = 1$ to $d = 5$; a more recent work [40] gives a further analysis with a rough initial data. On the other hand, it is observed that the standard CH energy (1) is most appropriate for single layer interfaces, with an essential feature that two dissimilar phases are separated and can not be merged. Therefore, if one uses single layers to model open vesicles, an additional order parameter has to be introduced to indicate the inside and outside of the vesicle [39], since single layers can not be punctured. On the other hand, bilayer interfaces separate two identical phases by a thin region of a second phase, so that they can be punctured, and can have free edges, forming open structures.

To address this well-known difficulty, the Functionalized Cahn-Hilliard (FCH) model has been used to model phase separation of an amphiphilic mixture in [22]; also see related works [13, 14, 20, 21, 36, 37]. In particular, the FCH equations were extended to describe membrane bilayers [13, 14], membranes and networks undergoing pearling bifurcations [14, 37], the formation of pore-like and micelle network structures [20, 21, 37], *et cetera*. In more details, a dimensionless energy of a binary mixture is considered, with the following expansion:

$$\begin{aligned}\mathcal{F}(\phi) &= \frac{\varepsilon^{-2}}{2}\int_\Omega \mu_0^2 d\vec{x} - \eta\mathcal{F}_0(\phi)\\ &= \int_\Omega \Bigg(\frac{\varepsilon^{-2}}{2}\phi^6 - \left(\varepsilon^{-2} + \frac{\eta}{4}\right)\phi^4 + \frac{\varepsilon^{-2}+\eta}{2}\phi^2 + \frac{\varepsilon^2}{2}(\Delta\phi)^2 \\ &\qquad - \left(1 + \frac{\eta}{2}\varepsilon^2\right)|\nabla\phi|^2 + 3\phi^2|\nabla\phi|^2\Bigg) d\vec{x},\end{aligned} \tag{4}$$

where $\eta \in \mathbb{R}$ is a parameter. For $\eta > 0$, (4) represents the FCH energy [14, 25, 36]; when $\eta \le 0$, (4) is the Cahn-Hilliard-Willmore (CHW) energy [41, 42, 44]. In particular, (4) represents the strong FCH energy when $\eta = O(\varepsilon^{-1})$ and weak FCH energy when $\eta = O(1)$ [14]. We will assume that $\eta > 0$ for the following presentation. The FCH chemical potential is the variational derivative of $\mathcal{F}$:

$$\mu := \delta_\phi \mathcal{F} = a_6\phi^5 - a_4\phi^3 + a_2\phi + \varepsilon^2\Delta^2\phi + a_{1,2}\Delta\phi + 6\phi\,|\nabla\phi|^2 - 6\nabla\cdot\left(\phi^2\nabla\phi\right),$$

with the positive constants

$$a_6 = 3\varepsilon^{-2},\ a_4 = 4\varepsilon^{-2} + \eta,\ a_2 = \varepsilon^{-2} + \eta,\ a_{1,2} = 2 + \eta\varepsilon^2.$$

This corresponds to the energy

$$\mathcal{F}(\phi) = \int_\Omega \left(\frac{a_6}{6}\phi^6 - \frac{a_4}{4}\phi^4 + \frac{a_2}{2}\phi^2 + \frac{\varepsilon^2}{2}(\Delta\phi)^2 - \frac{a_{1,2}}{2}|\nabla\phi|^2 + 3\phi^2|\nabla\phi|^2 \right) d\vec{x}. \tag{5}$$

Finally, the FCH equation is the conserved H^{-1} gradient flow with respect to the energy (5) [14, 36]:

$$\begin{aligned}\partial_t\phi &= \Delta\mu \\ &= \Delta\big(a_6\phi^5 - a_4\phi^3 + a_2\phi + a_{1,2}\Delta\phi + 6\phi|\nabla\phi|^2 - 6\nabla\cdot(\phi^2\nabla\phi) + \varepsilon^2\Delta^2\phi\big).\end{aligned} \quad (6)$$

For simplicity of presentation, we assume that $\Omega = (0,1)^3$; recall that ϕ is Ω-periodic. It is obvious that the FCH equation (6) is mass conservative, i.e.,

$$d_t\int_\Omega \phi(\mathbf{x},t)\,d\vec{x} = 0.$$

In addition, the FCH energy is dissipated at the rate

$$d_t\mathcal{F} = -\int_\Omega |\nabla\mu|^2 d\vec{x} \le 0.$$

Herein we analyze only the FCH equation. The CHW equation is obtained when $\eta \le 0$ and does not add any serious difficulties. Similar results will hold for the CHW equation.

The FCH equation (6) is a sixth-order, highly nonlinear parabolic equation. While there have been extensive numerical works for the given model [9, 11, 12, 16, 17, 24, 43], a theoretical justification of the smoothness and analyticity for the PDE solution has been limited. To obtain a PDE solution with real analytic regularity, the Gevrey norm has been a widely-used tool for the analysis for many time-dependent nonlinear PDEs; see the related works for 2-D and 3-D incompressible Navier-Stokes equation [4, 19], Kuramoto-Sivashinsky equation [3], nonlinear parabolic equation [8, 18], 3-D Navier-Stokes-Voigt equation [26], porous media flow [34]. Other than the Gevrey regularity solutions, a more general class of analytic solutions for different models of incompressible fluid have been discussed in [5, 23, 27, 28, 29, 30, 31, 32], etc. For the gradient flows with variational energy formulation, the Gevrey regularity solution has been proven for Cahn-Hilliard equation [35, 40], and certain extensions to the Cahn-Hilliard-fluid models have been reported in [15, 33]. In addition to these Cahn-Hilliard type problems, equations with p-Laplacian type nonlinearities has been analyzed in a more recent article [10], with an establishment of a global-in-time well-posedness.

Meanwhile, it is observed that, the physical energy (5) greatly differs from the standard Cahn-Hilliard one (1), due to the highly nonlinear nature in the expansion. And also, such an energy could not be classified in the p-Laplacian type gradient equations, since the last term in the energy expansion (5), namely, $\int_\Omega 3\phi^2|\nabla\phi|^2 d\mathbf{x}$, is neither convex nor concave. All these features have made the analysis for the FCH equation (6) highly challenging.

In this paper, we prove a global-in-time existence of Gevrey regularity solution for (6). The paper is organized as follows. Some notations associated with Gevrey space and some preliminary inequalities are outlined in Sec. 1. In Sec. 2 we construct the approximate solution, using the standard Galerkin procedure, and give the leading order H^2 estimate. In Sec. 3 we prove the existence and uniqueness of a local in time Gevrey regularity solution for (6), with the existence time interval length dependent on the initial data through $A^{3/2}\phi^0$, where $A = -\Delta$ with periodic boundary conditions on Ω. Finally, a uniform in time H^3 bound of the solution is presented in Sec. 4, so that a global in time Gevrey regularity solution is obtained. Finally, some concluding remarks are given in Sec. 5.

1. **Notation and preliminaries.** We use the standard symbols for Lebesgue and Sobolev spaces of complex-valued functions and their norms. To begin, for $u, v \in L^2(\Omega, \mathbb{C}) = L^2(\Omega)$, we set $(u, v) := \int_\Omega u(\vec{x}) v^*(\vec{x})\, d\vec{x}$, where $z^* = a - ib$ is the complex conjugate of $z = a + ib$. The $L^2(\Omega)$ norm is denoted $\|u\| = \sqrt{(u, u)}$. Let us also define the following function spaces:

$$
\begin{aligned}
\mathring{L}^2(\Omega) &:= \left\{ u \in L^2(\Omega) \;\middle|\; (u, 1) = 0 \right\}, \\
C^m_{\rm per}(\Omega) &:= \left\{ u \in C^m(\mathbb{R}^d) \;\middle|\; u \text{ is } \Omega\text{-periodic} \right\}, \\
\mathring{C}^m_{\rm per}(\Omega) &:= C^m_{\rm per}(\Omega) \cap \mathring{L}^2(\Omega), \\
W^{m,p}_{\rm per}(\Omega) &:= \left\{ u \in W^{m,p}_{\rm loc}(\mathbb{R}^d) \;\middle|\; u \text{ is } \Omega\text{-periodic} \right\}, \\
\mathring{W}^{m,p}_{\rm per}(\Omega) &:= W^{m,p}_{\rm per}(\Omega) \cap \mathring{L}^2(\Omega), \\
H^m_{\rm per}(\Omega) &:= W^{m,2}_{\rm per}(\Omega), \\
\mathring{H}^m_{\rm per}(\Omega) &:= \mathring{W}^{m,2}_{\rm per}(\Omega), \\
H^{-m}_{\rm per}(\Omega) &:= \left(H^m_{\rm per}(\Omega) \right)^*, \\
\mathring{H}^{-m}_{\rm per}(\Omega) &:= \left\{ v \in H^{-m}_{\rm per}(\Omega) \;\middle|\; \langle v, 1 \rangle = 0 \right\},
\end{aligned}
$$

where $\langle \cdot\, , \cdot \rangle$ is the duality pairing between $H^{-m}_{\rm per}$ and $H^m_{\rm per}$. Specifically, for $v \in H^{-m}_{\rm per}(\Omega)$, $u_k \in H^m_{\rm per}(\Omega)$,

$$
\left\langle v, \sum_{k=1}^n c_k u_k \right\rangle := \sum_{k=1}^n c_k^* v(u_k^*) = \sum_{k=1}^n c_k^* \langle v, u_k \rangle.
$$

We denote the standard semi-norm and norm on $W^{m,p}(\Omega)$ by $|\cdot|_{m,p,\Omega} = |\cdot|_{m,p}$ and $\|\cdot\|_{m,p,\Omega} = \|\cdot\|_{m,p}$, respectively, dropping the subscript m whenever $m = 0$. Since the domain $\Omega = (0,1)^d$ is understood in our discussion, we usually also drop the subscript Ω in referencing the (semi-)norms.

Define the operator A to be $-\Delta$ paired with Ω–periodic boundary conditions. We define the range of A as $R(A) := \mathring{L}^2(\Omega)$. The domain of A is simply $D(A) = \mathring{H}^2_{\rm per}(\Omega)$, and $A : D(A) \to R(A)$ is a positive, self-adjoint linear operator that admits a compact inverse. The eigenfunctions of A may be chosen as $\Phi_{\vec{\alpha}}(\vec{x}) = \exp(2\pi i\, \vec{\alpha} \cdot \vec{x}) \in \mathring{C}^\infty_{\rm per}(\Omega)$, for all $\vec{\alpha} \in \mathbb{Z}^d \setminus \{\vec{0}\} =: \mathbb{Z}^d_\star$, in which case the eigenvalues are $\lambda_{\vec{\alpha}} = (2\pi)^2 |\vec{\alpha}|^2 > 0$. Set $\mathring{B} := \left\{ \Phi_{\vec{\alpha}} \;\middle|\; \vec{\alpha} \in \mathbb{Z}^d_\star \right\}$; this is an orthonormal basis for $\mathring{L}^2(\Omega)$. We can increase $\mathring{B}$ so the resulting set is an orthonormal basis for all of $L^2(\Omega)$; in particular, $B := \mathring{B} \cup \left\{ \Phi_{\vec{0}} \equiv 1 \right\}$ serves this purpose.

Since A is symmetric and positive, we can define the following Hilbert spaces: for any $s \geq 0$, define

$$
D(A^s) := \left\{ u \in \mathring{L}^2(\Omega) \;\middle|\; \sum_{\vec{\alpha} \in \mathbb{Z}^d_\star} (2\pi)^{4s} |\vec{\alpha}|^{4s} |\hat{u}_{\vec{\alpha}}|^2 < \infty, \right\},
$$

and equip this space with the inner product

$$
(u, v)_{D(A^s)} := \sum_{\vec{\alpha} \in \mathbb{Z}^d_\star} (2\pi)^{4s} |\vec{\alpha}|^{4s} \hat{u}_{\vec{\alpha}}\, \hat{v}^*_{\vec{\alpha}},
$$

where $\hat{u}_{\vec{\alpha}} := (u, \Phi_{\vec{\alpha}}) = \int_\Omega u(\vec{x})\mathrm{e}^{-2\pi i\,\vec{\alpha}\cdot\vec{x}}d\vec{x}$ are the Fourier coefficients of u. For $u \in D(A^s)$, we define

$$A^s u := \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} (2\pi)^{2s}|\vec{\alpha}|^{2s}\hat{u}_{\vec{\alpha}}\Phi_{\vec{\alpha}}.$$

Then, of course, $(u,v)_{D(A^s)} = (A^s u, A^s v)$ and $\|u\|_{D(A^s)} = \|A^s u\|$, and it is not difficult to show that, in general, $D(A^s) = \mathring{H}^{2s}_{\mathrm{per}}(\Omega)$. It is possible to define the exponential operator $\exp(\tau A^s) = \mathrm{e}^{\tau A^s}$, for any $\tau, s \geq 0$. To do so we introduce the Hilbert space

$$D\big(\mathrm{e}^{\tau A^s}\big) := \Big\{u \in \mathring{L}^2(\Omega) \,\Big|\, \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{2\tau(2\pi)^{2s}|\vec{\alpha}|^{2s}}|\hat{u}_{\vec{\alpha}}|^2 < \infty\Big\}.$$

For any $u \in D\big(\mathrm{e}^{\tau A^s}\big)$, define

$$\mathrm{e}^{\tau A^s}u := \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{\tau(2\pi)^{2s}|\vec{\alpha}|^{2s}}\hat{u}_{\vec{\alpha}}\,\Phi_{\vec{\alpha}}.$$

We introduce the Gevrey space $G_\tau := D\big(\mathrm{e}^{\tau A^{1/2}}\big)$. This is a Hilbert space with the inner product and norm denoted by

$$(u,v)_\tau := \big(\mathrm{e}^{\tau A^{1/2}}u, \mathrm{e}^{\tau A^{1/2}}v\big) = \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{2\tau 2\pi|\vec{\alpha}|}\hat{u}_{\vec{\alpha}}\,\hat{v}^*_{\vec{\alpha}}, \quad |u|_\tau := \sqrt{(u,u)_\tau}.$$

Observe that, for any $u \in G_\tau$,

$$|u|^2_\tau = \sum_{m=0}^\infty \frac{(2\tau)^m}{m!}\sum_{\vec{\alpha}\in\mathbb{Z}^d_\star}(2\pi)^m|\vec{\alpha}|^m|\hat{u}_{\vec{\alpha}}|^2 = \sum_{m=0}^\infty \frac{(2\tau)^m}{m!}\|u\|^2_{D(A^{m/4})}.$$

Since $|u|_\tau$ is finite, it follows that every H^k norm of u is also finite. These spaces can be increased trivially to contain functions that are not of mean zero, in which case, the sums are taken over $\mathbb{Z}^d$.

Set $\mathcal{G}_M := \operatorname{span}(\{\Phi_{\vec{\alpha}} \mid |\vec{\alpha}| \leq M\})$. The operator $\mathcal{P}_M : L^2(\Omega) \to \mathcal{G}_M$ is the canonical orthogonal projection:

$$\mathcal{P}_M u := \sum_{|\vec{\alpha}|\leq M} \hat{u}_{\vec{\alpha}}\,\Phi_{\vec{\alpha}}. \tag{7}$$

Of course, if $u \in \mathring{L}^2(\Omega)$, then $\hat{u}_{\vec{0}} = 0$. One can extend the domain of definition $\mathcal{P}_M$ to $\mathring{H}^{-r}_{\mathrm{per}}(\Omega)$, for any $r \in (0,\infty)$, as follows: if $u \in \mathring{H}^{-r}_{\mathrm{per}}(\Omega)$, then

$$\mathcal{P}_M u := \sum_{|\vec{\alpha}|\leq M} u\,(\Phi^*_{\vec{\alpha}})\,\Phi_{\vec{\alpha}} = \sum_{|\vec{\alpha}|\leq M} \langle u, \Phi_{\vec{\alpha}}\rangle\Phi_{\vec{\alpha}},$$

which implies that

$$(\mathcal{P}_M u, v) := \langle u, \mathcal{P}_M v\rangle, \quad \forall v \in \mathring{H}^r_{\mathrm{per}}(\Omega).$$

Recall that $\Big(\mathring{H}^{-r}_{\mathrm{per}}(\Omega), \|\cdot\|_{\mathring{H}^{-r}_{\mathrm{per}}}\Big)$ is a Hilbert space using the standard operator norm. We have the following basic properties of the orthogonal projection that we state without proof [38]:

Lemma 1.1. *Let $X = \mathring{H}^{-r}_{\mathrm{per}}(\Omega)$, or $D(A^s)$, for any $r, s \geq 0$. Then, for any $u \in X$,*

$$\|\mathcal{P}_M u\|_X \leq \|u\|_X, \quad \textit{and} \quad \|u - \mathcal{P}_M u\|_X \xrightarrow{M\to\infty} 0. \tag{8}$$

The results can be modified in a trivial way to accommodate functions that are not of mean zero.

We have the following interpolation inequalities [1]:

Lemma 1.2. *Let $r, k, j \in \mathbb{R}$, with $0 \le k < j < r$. Then, for any $\psi \in \mathring{H}^r_{\rm per}(\Omega) = D(A^{r/2})$,*

$$\left\| A^{j/2}\psi \right\| \le C \left\| A^{k/2}\psi \right\|^{\frac{r-j}{r-k}} \left\| A^{r/2}\psi \right\|^{\frac{j-k}{r-k}}. \tag{9}$$

For integer values of the indices, we have

$$\left\| \nabla^j\psi \right\| \le \left\| \nabla^k\psi \right\|^{\frac{r-j}{r-k}} \left\| \nabla^r\psi \right\|^{\frac{j-k}{r-k}}, \tag{10}$$

where a constant of 1 suffices.

Frequent use will be made of following Gagliardo-Nirenberg-type interpolation inequality [1]:

Theorem 1.3. *Let $j, m \in \mathbb{N}$, $q, r, \theta \in \mathbb{R}$. Suppose $1 \le q, r \le \infty$, $\frac{j}{m} \le \theta \le 1$, and*

$$\frac{1}{p} - \frac{j}{d} = \left(\frac{1}{r} - \frac{m}{d} \right)\theta + \frac{1-\theta}{q}. \tag{11}$$

If $\psi \in L^q(\Omega) \cap W^{m,r}_{\rm per}(\Omega)$, then $\psi \in W^{j,p}_{\rm per}(\Omega)$, and there exists a constant $C = C(d, j, m, p, q, r, \Omega) > 0$ such that

$$|\psi|_{j,p} \le C \left(|\psi|^{\theta}_{m,r} \|\psi\|^{1-\theta}_q + \|\psi\|_q \right). \tag{12}$$

2. **Approximate solutions and an energy estimate.** Consider the following Galerkin approximation problem: Suppose that $\phi^0 \in L^2(\Omega)$, and $\phi^0_M = \mathcal{P}_M(\phi^0)$. Find coefficients $\tilde{\phi}_{\vec{\alpha},M} : [0,T] \to \mathbb{R}$, for $|\vec{\alpha}| \le M$, in the representation

$$\phi_M(t) = \sum_{|\vec{\alpha}| \le M} \tilde{\phi}_{\vec{\alpha},M}(t)\Phi_{\vec{\alpha}}$$

such that $\phi_M(0) = \phi^0_M$, and for $t \in (0, T]$,

$$\frac{\partial \phi_M}{\partial t} = \Delta \mu_M \tag{13}$$

where

$$\begin{aligned}\mu_M := {}& \mathcal{P}_M \left(6\phi_M |\nabla\phi_M|^2 - 6\nabla\cdot(\phi_M^2 \nabla\phi_M) + a_6\phi_M^5 - a_4\phi_M^3 \right) \\ & + a_2\phi_M + a_{1,2}\Delta\phi_M + \varepsilon^2\Delta^2\phi_M.\end{aligned}$$

Lemma 2.1. *Let $\phi^0 \in L^2(\Omega)$. The solution to the Galerkin approximation problem exists for some $T_\star = T_\star(M, \phi^0) > 0$, such that $\tilde{\phi}_{\vec{\alpha},M} \in C^1([0, T_\star])$, for all $|\vec{\alpha}| \le M$, and $\int_\Omega \left(\phi_M(\vec{x}, t) - \phi^0(\vec{x}) \right) d\vec{x} = 0$, for all $t \in [0, T_\star]$. Furthermore, the following energy stability is valid: $\mathcal{F}(\phi_M(t)) \le \mathcal{F}(\phi_M(0))$, for any $t \in [0, T_\star]$.*

Proof. The approximation problem can be recast as a system of nonlinear ODE's; it has a unique solution up to some finite time $T_\star$, such that $\tilde{\phi}_{\vec{\alpha},M} \in C^\infty([0, T_\star])$, for all $|\vec{\alpha}| \le M$. It is clear that $\int_\Omega \left(\phi_M(\vec{x}, t) - \phi^0(\vec{x}) \right) d\vec{x} = 0$, for all $t \in [0, T_\star]$. Observe $\mu_M \in \mathcal{G}_M$. Testing this with the Equation (13) and integrating, we arrive (after a standard energy variation calculation) at the result

$$(\partial_t\phi_M, \mu_M) = d_t\mathcal{F}(\phi_M(t)) = -\int_\Omega |\nabla\mu_M(t)|^2 d\vec{x}.$$

Integrating this in time, we have, for any $T \in [0, T_\star]$,

$$\mathcal{F}(\phi_M(T)) + \int_0^T \|\nabla \mu_M(t)\|^2 \, dt = \mathcal{F}(\phi_M(0)). \tag{14}$$

$\square$

Meanwhile, the following energy estimate is valid.

Proposition 1. *Let $\mathcal{F}$ be the energy given by (5). There is a constant $C_0 > 0$, such that for all $\phi \in H^2_{\text{per}}(\Omega)$,*

$$\mathcal{F}(\phi) \geq \frac{\varepsilon^{-2}}{4}\|\phi\|_6^6 + \frac{\varepsilon^2}{4}\|\Delta\phi\|^2 - C_0 = \frac{a_6}{12}\|\phi\|_6^6 + \frac{\varepsilon^2}{4}\|\Delta\phi\|^2 - C_0. \tag{15}$$

The constant C_0 only depends on ε, η and $|\Omega|$.

Proof. We begin with an application of integration-by-parts and the Cauchy-Schwarz inequality:

$$\begin{aligned}\int_\Omega \frac{a_{1,2}}{2}|\nabla\phi|^2 d\mathbf{x} = -\frac{a_{1,2}}{2}\int_\Omega \phi \cdot \Delta\phi d\mathbf{x} &\leq \frac{a_{1,2}}{2}\|\phi\| \cdot \|\Delta\phi\| \\ &\leq \frac{\varepsilon^2}{4}\|\Delta\phi\|^2 + \frac{a_{1,2}^2\varepsilon^{-2}}{4}\|\phi\|^2.\end{aligned} \tag{16}$$

Define

$$C_1 := \frac{a_{1,2}^2\varepsilon^{-2}}{4} - \frac{a_2}{2} = \frac{\varepsilon^{-2}+\eta}{2} + \frac{\eta^2\varepsilon^2}{4} > 0. \tag{17}$$

A careful application of Young's inequality reveals that, at a point-wise level,

$$\frac{a_4}{4}\phi^4 \leq \frac{a_6}{24}\phi^6 + C_2 = \frac{\varepsilon^{-2}}{8}\phi^6 + C_2, \quad \text{with} \quad C_2 := \frac{1}{3}\left(\left(\frac{3}{16}\right)^{-\frac{2}{3}}\varepsilon^{\frac{4}{3}}\frac{a_4}{4}\right)^3, \tag{18}$$

and

$$C_1\phi^2 \leq \frac{a_6}{24}\phi^6 + C_3 = \frac{\varepsilon^{-2}}{8}\phi^6 + C_3, \quad \text{with} \quad C_3 := \frac{2}{3}\left(\left(\frac{3}{8}\right)^{-\frac{1}{3}}\varepsilon^{\frac{2}{3}}C_1\right)^{\frac{3}{2}}. \tag{19}$$

Therefore, a substitution of (16)-(19) into (5) leads to (15), with $C_0 = (C_2+C_3)|\Omega|$. The proof is complete. $\square$

As a consequence of Lemma 2.1 and Proposition 1, the following result is valid.

Corollary 1. *Suppose that $\phi^0 \in H^2_{\text{per}}(\Omega)$. Then ϕ_M and μ_M, defined as in Lemma 2.1, exist for all time, and, moreover, for any $T > 0$,*

$$\max_{0\leq t\leq T}\|\phi_M(t)\|_{H^2}^2 + \int_0^T \|\nabla\mu_M(t)\|^2 \, dt \leq C_4, \tag{20}$$

where C_4 depends on the initial data and the equation parameters, but is independent of M and T.

Proof. A combination of Lemma 2.1 and Proposition 1 indicates that, for any $0 < t \leq T_\star$,

$$
\begin{aligned}
\frac{\varepsilon^2}{4}\left\|\Delta\phi_M(t)\right\|^2 - C_0 + \int_0^t \left\|\nabla\mu_M(\tau)\right\|^2 d\tau &\le \mathcal{F}(\phi_M(t)) + \int_0^t \left\|\nabla\mu_M(\tau)\right\|^2 d\tau \\
&= \mathcal{F}(\phi_M(0)) \\
&\le C_5 \left\|\phi_M(0)\right\|_{H^2}^6 + C_6 \\
&\le C_7 \left\|\phi^0\right\|_{H^2}^6 + C_6,
\end{aligned} \tag{21}
$$

where Lemma 1.1 (stability of the projection) was employed in the last step. By regularity, there is a constant, $C_8 > 0$ such that

$$
\|\psi\|_{H^2} \le \left\|\psi - \overline{\psi}\right\|_{H^2} + \left\|\overline{\psi}\right\|_{H^2} \le C_8 \left\|\Delta\psi\right\| + |\overline{\psi}|\sqrt{|\Omega|}, \tag{22}
$$

for any $\psi \in H_{\rm per}^2(\Omega)$, where $\overline{\psi}$ is the Ω–average of ψ. Motivated by the fact that $\overline{\phi_M(t)} = \overline{\phi^0}$, for all $t \in [0, T_\star]$, we are able to prove estimate (20) for $T = T_\star$. But, since C_4 is independent of the final time T, the Galerkin approximate solutions do not blow-up and can be extended up to any final time $T > 0$ [38]. □

Definition 2.2. Suppose $T > 0$ and $\phi, \mu : \Omega \times [0, T] \to \mathbb{R}$ are Ω–periodic in space. We say that the pair (ϕ, μ) is a weak solution of (6) on the time interval $[0, T]$ if and only if

$$
\begin{aligned}
\phi &\in L^\infty\left(0, T; H_{\rm per}^2(\Omega)\right) \cap C^0\left(0, T; L^2(\Omega)\right), \\
\mu &\in L^2\left(0, T; H_{\rm per}^1(\Omega)\right), \\
\partial_t\phi &\in L^2\left(0, T; \mathring{H}_{\rm per}^{-1}(\Omega)\right),
\end{aligned} \tag{23}
$$

and, for almost all $t \in [0, T]$,

$$
\langle \partial_t\phi, \nu\rangle + (\nabla\mu, \nabla\nu) = 0, \quad \forall\, \nu \in H_{\rm per}^1(\Omega), \tag{24}
$$

$$
\begin{aligned}
&a_6\left(\phi^5, \psi\right) - a_4\left(\phi^3, \psi\right) + a_2\left(\phi, \psi\right) \\
&\quad + \varepsilon^2\left(\Delta\phi, \Delta\psi\right) - a_{1,2}\left(\nabla\phi, \nabla\psi\right) \\
&\quad + 6\left(\phi|\nabla\phi|^2, \psi\right) + 6\left(\phi^2\nabla\phi, \nabla\psi\right) - (\mu, \psi) = 0, \quad \forall\, \psi \in H_{\rm per}^2(\Omega),
\end{aligned} \tag{25}
$$

with $\phi(0) = \phi^0 \in L^2(\Omega)$.

Theorem 2.3. *Suppose that $\phi^0 \in H_{\rm per}^2(\Omega)$. Then a weak solution to (6) exists on any time interval $[0, T]$, however large the final time T may be.*

Proof. Since the bound (20) is uniform in M, there exist subsequences ϕ_{M_m} and μ_{M_m} and limit points $\phi \in L^\infty\left(0, T; H_{\rm per}^2(\Omega)\right)$ and $\mu \in L^2\left(0, T; H_{\rm per}^1(\Omega)\right)$, such that ϕ_{M_m} converges weakly to ϕ, μ_{M_m} converges weakly to μ, and

$$
\|\phi\|_{L^\infty\left(0,T;H_{\rm per}^2(\Omega)\right)} + \|\mu\|_{L^2\left(0,T;H_{\rm per}^1(\Omega)\right)} \le C_9, \tag{26}
$$

where $C_9 > 0$ is independent of T. Passing to limits, one can prove that the pair (ϕ, μ) is a weak solution to the FCH equation (6). The details are standard and are skipped for brevity. □

3. **A local in time Gevrey regularity solution.**

In this section, we establish an estimate of the Gevrey norm of the solution for (6). We must analyze the Galerkin approximate solution, that is the solution of (13), and pass to limits, because we do not know *a priori* the weak solution is sufficiently regular to perform the computations. However, we will often drop the lower index M for the sake of notational convenience, and we skip the details of the passage to limits on the Galerkin approximations.

3.1. **A preliminary estimate for the nonlinear term.** We choose the highest order part in the nonlinear expansion of the chemical potential

$$\mathcal{N}_1(\phi) := 6\phi\,|\nabla\phi|^2 - 6\nabla\cdot\left(\phi^2\nabla\phi\right) = -6\phi\,|\nabla\phi|^2 - 6\phi^2\Delta\phi,$$

and the corresponding expansion in $\Delta\mu$ becomes

$$\begin{aligned} B_1(\phi) :=& \Delta(\mathcal{N}_1(\phi)) \\ =& -6\Delta\phi\cdot|\nabla\phi|^2 - 12\nabla\phi\cdot\nabla(|\nabla\phi|^2) - 6\phi\Delta(|\nabla\phi|^2) \\ & -6\phi^2\Delta^2\phi - 6\Delta(\phi^2)\cdot\Delta\phi - 12\nabla(\phi^2)\cdot\nabla\Delta\phi \\ =& -6\Delta\phi\cdot|\nabla\phi|^2 - 12\nabla\phi\cdot(2\nabla\phi\cdot\nabla\nabla\phi) - 6\phi(2\nabla\phi\cdot\nabla\Delta\phi + 2|\nabla\nabla\phi|^2) \\ & -6\phi^2\Delta^2\phi - 6(2\phi\Delta\phi + 2|\nabla\phi|^2)\cdot\Delta\phi - 12\cdot 2\phi\nabla\phi\cdot\nabla\Delta\phi \\ =& -18B_{11}(\phi) - 24B_{12}(\phi) - 12B_{13}(\phi) - 6B_{14}(\phi) - 12B_{15}(\phi) - 36B_{16}(\phi), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &B_{11}(\phi) := |\nabla\phi|^2\,\Delta\phi, \quad B_{12}(\phi) := \nabla\phi\cdot(\mathsf{H}(\phi)\nabla\phi), \\ &B_{13}(\phi) := \phi\,|\mathsf{H}(\phi)|^2, \quad B_{14}(\phi) := \phi^2\Delta^2\phi, \\ &B_{15}(\phi) := \phi(\Delta\phi)^2, \quad B_{16}(\phi) := \phi(\nabla\phi\cdot\nabla\Delta\phi), \end{aligned} \tag{28}$$

and $\mathsf{H}(\phi)$ is the (symmetric) Hessian matrix of ϕ. We use the notation $|\mathsf{H}(\phi)|^2 := \mathsf{H}(\phi):\mathsf{H}(\phi)$. In the following lemma, we analyze $B_{11}(\phi)$ in the Gevrey space with inner product $(\cdot,\cdot)_\tau$.

Lemma 3.1. *Let $u^{(1)}, u^{(2)}, v, w \in D(A^3 \mathrm{e}^{\tau A^{1/2}})$ be given, with $\tau > 0$. Define the trilinear operator*

$$D_{11}(u^{(1)}, u^{(2)}, v) := (\nabla u^{(1)}\cdot\nabla u^{(2)})\Delta v.$$

Then the following inequality is valid in space dimension $d = 3$: for some constant $C > 0$, independent of the input functions,

$$\left|\left(\mathrm{e}^{\tau A^{1/2}} D_{11}(u^{(1)}, u^{(2)}, v), \mathrm{e}^{\tau A^{1/2}} A^3 w\right)\right| \le C|A^{\frac{3}{2}}u^{(1)}|_\tau\cdot|A^{\frac{3}{2}}u^{(2)}|_\tau\cdot|A^{\frac{3}{2}}v|_\tau\cdot|A^3 w|_\tau. \tag{29}$$

As a direct consequence, by setting $u^{(1)} = u^{(2)} = v = w = \phi$, we obtain the following estimate:

$$\left|\left(\mathrm{e}^{\tau A^{1/2}} B_{11}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| \le C|A^{\frac{3}{2}}\phi|_\tau^3\cdot|A^3\phi|_\tau. \tag{30}$$

Proof. The following notation is introduced:

$$v(\vec{x}) = \sum_{\vec{p}\in\mathbb{Z}^3} \hat{v}_{\vec{p}}\,\mathrm{e}^{2\pi i\vec{p}\cdot\vec{x}}, \qquad \check{\hat{v}}_{\vec{p}} := e^{\tau 2\pi|\vec{p}|}\hat{v}_{\vec{p}}, \qquad \check{v}(\vec{x}) := \sum_{\vec{p}\in\mathbb{Z}^3} \check{\hat{v}}_{\vec{p}}\,\mathrm{e}^{2\pi i\vec{p}\cdot\vec{x}}$$

with analogous notation for $u^{(1)}$, $u^{(2)}$, and w. Observe that $(u,v)_\tau = (\check{u},\check{v})$. Then we see that

$$\left(\mathrm{e}^{\tau A^{1/2}} D_{11}(u^{(1)}, u^{(2)}, v), \mathrm{e}^{\tau A^{1/2}} A^3 w\right)$$

$$
\begin{aligned}
&= (2\pi)^{10} \sum_{\vec{j}_1+\vec{j}_2+\vec{p}=\vec{\ell}} \hat{u}^{(1)}_{\vec{j}_1} \hat{u}^{(2)}_{\vec{j}_2} (\vec{j}_1 \cdot \vec{j}_2)\, \hat{v}_{\vec{p}} |\vec{p}|^2 \hat{w}^*_{\vec{\ell}} |\vec{\ell}|^6 \mathrm{e}^{2\tau 2\pi |\vec{\ell}|} \\
&= (2\pi)^{10} \sum_{\vec{j}_1+\vec{j}_2+\vec{p}=\vec{\ell}} \check{\hat{u}}^{(1)}_{\vec{j}_1} \check{\hat{u}}^{(2)}_{\vec{j}_2} (\vec{j}_1 \cdot \vec{j}_2)\, \check{\hat{v}}_{\vec{p}} |\vec{p}|^2 \check{\hat{w}}^*_{\vec{\ell}} |\vec{\ell}|^6 \cdot \mathrm{e}^{\tau 2\pi (|\vec{\ell}| - |\vec{j}_1| - |\vec{j}_2| - |\vec{p}|)},
\end{aligned}
$$

where $*$ indicates complex conjugation, and $|\vec{p}| = \sqrt{\vec{p}\cdot\vec{p}}$. For the summation indices, due to the simple vector inequality

$$
|\vec{\ell}| - |\vec{j}_1| - |\vec{j}_2| - |\vec{p}| = |\vec{j}_1 + \vec{j}_2 + \vec{p}| - |\vec{j}_1| - |\vec{j}_2| - |\vec{p}| \le 0,
$$

we get

$$
\begin{aligned}
&\left| \left(\mathrm{e}^{\tau A^{1/2}} D_{11}(u^{(1)}, u^{(2)}, v), \mathrm{e}^{\tau A^{1/2}} A^3 w \right) \right| \\
&\quad \le (2\pi)^{10} \sum_{\vec{j}_1+\vec{j}_2+\vec{p}=\vec{\ell}} |\check{\hat{u}}^{(1)}_{\vec{j}_1}|\, |\vec{j}_1|\, |\check{\hat{u}}^{(2)}_{\vec{j}_2}|\, |\vec{j}_2|\, |\check{\hat{v}}_{\vec{p}}|\, |\vec{p}|^2\, |\check{\hat{w}}_{\vec{\ell}}|\, |\vec{\ell}|^6 \\
&\quad = \int_\Omega \xi_1(\vec{x}) \xi_2(\vec{x}) \psi(\vec{x}) \theta(\vec{x})\, d\vec{x},
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_r(\vec{x}) &:= \sum_{\vec{j}_r \in \mathbb{Z}^d} |\check{\hat{u}}^{(r)}_{\vec{j}_r}|\, |\vec{j}_r| \mathrm{e}^{2\pi i \vec{j}_r \cdot \vec{x}}, \\
\psi(\vec{x}) &:= \sum_{\vec{p} \in \mathbb{Z}^d} |\check{\hat{v}}_{\vec{p}}|\, |\vec{p}|^2 \mathrm{e}^{2\pi i \vec{p} \cdot \vec{x}}, \\
\theta(\vec{x}) &:= (2\pi)^{10} \sum_{\vec{\ell} \in \mathbb{Z}^d} |\check{\hat{w}}_{\vec{\ell}}|\, |\vec{\ell}|^6 \mathrm{e}^{-2\pi i \vec{\ell} \cdot \vec{x}}.
\end{aligned}
$$

Careful applications of Sobolev inequalities imply that

$$
\begin{aligned}
\|\xi_r\|_\infty &\le C \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} u^{(r)}\|, \quad r = 1, 2, \\
\|\psi\| &= (2\pi)^{-2} \|A \mathrm{e}^{\tau A^{1/2}} v\| \le C \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} v\|, \\
\|\theta\| &= (2\pi)^4 \|A^3 \mathrm{e}^{\tau A^{1/2}} w\|.
\end{aligned} \tag{31}
$$

A substitution of (31) into (31) yields the estimate (29). Subsequently, (30) is a direct consequence of (29), by setting $u^{(1)} = u^{(2)} = v = \phi$. This completes the proof of the Lemma. □

Using similar arguments, we are able to derive the following estimates for the other nonlinear terms: for B_{12},

$$
\begin{aligned}
&\left| \left(\mathrm{e}^{\tau A^{1/2}} B_{12}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3 \phi \right) \right| = \left| \left(\mathrm{e}^{\tau A^{1/2}} (\nabla\phi \cdot (\mathsf{H}(\phi) \nabla\phi)), \mathrm{e}^{\tau A^{1/2}} A^3 \phi \right) \right| \\
&\le C \|\nabla(\mathrm{e}^{\tau A^{1/2}} \phi)\|_\infty \cdot \|\nabla(\mathrm{e}^{\tau A^{1/2}} \phi)\|_\infty \cdot \|\mathsf{H}(\mathrm{e}^{\tau A^{1/2}} \phi)\| \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi\| \\
&\le C \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} \phi\|^3 \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi\|,
\end{aligned} \tag{32}
$$

for B_{13},

$$
\begin{aligned}
&\left| \left(\mathrm{e}^{\tau A^{1/2}} B_{13}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3 \phi \right) \right| = \left| \left(\mathrm{e}^{\tau A^{1/2}} (\phi |\mathsf{H}(\phi)|^2), \mathrm{e}^{\tau A^{1/2}} A^3 \phi \right) \right| \\
&\le C \|\mathrm{e}^{\tau A^{1/2}} \phi\|_\infty \cdot \|\mathsf{H}(\mathrm{e}^{\tau A^{1/2}} \phi)\|_4 \cdot \|\mathsf{H}(\mathrm{e}^{\tau A^{1/2}} \phi)\|_4 \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi\| \\
&\le C \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} \phi\|^3 \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi\|,
\end{aligned} \tag{33}
$$

for B_{14},

$$\begin{aligned}
&\left|\left(\mathrm{e}^{\tau A^{1/2}} B_{14}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| = \left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi^2\Delta^2\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| \\
&\quad \le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty^2 \cdot \|A^2\mathrm{e}^{\tau A^{1/2}}\phi\| \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\quad \le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^2 \cdot \|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{2}{3}} \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{1}{3}} \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\quad \le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{8}{3}} \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{4}{3}},
\end{aligned} \tag{34}$$

for B_{15},

$$\begin{aligned}
&\left|\left(\mathrm{e}^{\tau A^{1/2}} B_{15}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| = \left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi(\Delta\phi)^2), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| \\
&\quad \le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty \cdot \|A\mathrm{e}^{\tau A^{1/2}}\phi\|_4 \cdot \|A\mathrm{e}^{\tau A^{1/2}}\phi\|_4 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\quad \le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|,
\end{aligned} \tag{35}$$

and for B_{16},

$$\begin{aligned}
&\left|\left(\mathrm{e}^{\tau A^{1/2}} B_{16}(\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| = \left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi(\nabla\phi\cdot\nabla\Delta\phi)), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| \\
&\quad \le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty \cdot \|\mathrm{e}^{\tau A^{1/2}}(\nabla\phi)\|_\infty \cdot \|\mathrm{e}^{\tau A^{1/2}}(\nabla\Delta\phi)\| \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\quad \le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|.
\end{aligned} \tag{36}$$

Here we have made repeated use of 3-D Sobolev embedding and interpolation inequalities. A combination of inequality (30) and inequalities (32) – (36) gives the following estimate.

Lemma 3.2. *Suppose that $\phi \in D(A^3\mathrm{e}^{\tau A^{1/2}})$, $\tau > 0$. Then there is some constant $C > 0$, independent of ϕ, such that*

$$\begin{aligned}
\left|\left(\mathrm{e}^{\tau A^{1/2}} B_1(\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| &\le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\quad + C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{8}{3}} \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|^{\frac{4}{3}}.
\end{aligned} \tag{37}$$

In addition, there are two other nonlinear terms in the chemical potential, namely $a_6\phi^5$ and $-a_4\phi^3$, and the corresponding nonlinear terms that must be analyzed in (6) are

$$\begin{aligned}
B_2(\phi) &:= \Delta(a_6\phi^5) = a_6(5\phi^4\Delta\phi + 20\phi^3|\nabla\phi|^2), \\
B_3(\phi) &:= \ -\Delta(a_4\phi^3) = -a_4(3\phi^2\Delta\phi + 6\phi|\nabla\phi|^2).
\end{aligned} \tag{38}$$

For these nonlinear terms, we are able to obtain the following estimates in a style similar to that used before:

$$\begin{aligned}
\left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi^4\Delta\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| &\le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty^4 \cdot \|A\mathrm{e}^{\tau A^{1/2}}\phi\| \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^5 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|,
\end{aligned} \tag{39}$$

$$\begin{aligned}
\left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi^3|\nabla\phi|^2), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| &\le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty^3 \cdot \|\nabla(\mathrm{e}^{\tau A^{1/2}}\phi)\|_4^2 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^5 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|,
\end{aligned} \tag{40}$$

$$\begin{aligned}
\left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi^2\Delta\phi), \mathrm{e}^{\tau A^{1/2}} A^3\phi\right)\right| &\le C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty^2 \cdot \|A\mathrm{e}^{\tau A^{1/2}}\phi\| \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\
&\le C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|,
\end{aligned} \tag{41}$$

$$\begin{aligned}\left|\left(\mathrm{e}^{\tau A^{1/2}}(\phi|\nabla\phi|^2), \mathrm{e}^{\tau A^{1/2}}A^3\phi\right)\right| &\leq C\|\mathrm{e}^{\tau A^{1/2}}\phi\|_\infty \cdot \|\nabla(\mathrm{e}^{\tau A^{1/2}}\phi)\|_4^2 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\| \\ &\leq C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|.\end{aligned} \tag{42}$$

Therefore, the following results become available.

Lemma 3.3. *Let $\phi \in D(A^3\mathrm{e}^{\tau A^{1/2}})$, $\tau > 0$. Then there is a constant $C > 0$, independent of ϕ, such that*

$$\begin{aligned}\left|\left(\mathrm{e}^{\tau A^{1/2}}B_2(\phi), \mathrm{e}^{\tau A^{1/2}}A^3\phi\right)\right| &\leq C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^5 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|, \\ \left|\left(\mathrm{e}^{\tau A^{1/2}}B_3(\phi), \mathrm{e}^{\tau A^{1/2}}A^3\phi\right)\right| &\leq C\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi\|^3 \cdot \|A^3\mathrm{e}^{\tau A^{1/2}}\phi\|.\end{aligned} \tag{43}$$

3.2. **A local-in-time solution with Gevrey regularity.** The following theorem is the main result of this section.

Theorem 3.4. *Assume that $\phi^0 = \psi + \overline{\phi^0}$, where $\psi \in D(A^{\frac{3}{2}})$ and $\overline{\phi^0} \in \mathbb{R}$. Then there exists $T_\star$ that depends upon the $D(A^{\frac{3}{2}})$ norm of ϕ^0 and the parameters, such that (6) possesses a unique, regular solution on $(0, T_\star)$, such that $\tau \to e^{\tau A^{1/2}}\phi(\tau)$ is analytic on $(0, T_\star)$.*

Proof. Consider the Galerkin approximate solution ϕ_M of (13). To simplify the calculations, let us assume that $\overline{\phi^0} = 0$. The more general case is easily handled. In this case, $\overline{\phi_M(t)} = 0$, for all $t \geq 0$. At time τ we test (13) with $A^3\phi_M(\tau)$ in $D(\mathrm{e}^{\tau A^{1/2}})$ inner product:

$$\begin{aligned}&\left(\mathrm{e}^{\tau A^{1/2}}\phi_M'(\tau), A^3\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) + \varepsilon^2\left(\mathrm{e}^{\tau A^{1/2}}A^3\phi_M(\tau), A^3\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\ &\quad= -a_2\left(\mathrm{e}^{\tau A^{1/2}}A\phi_M(\tau), A^3 e^{\tau A^{1/2}}\phi_M(\tau)\right) \\ &\qquad + a_{1,2}\left(\mathrm{e}^{\tau A^{1/2}}A^2\phi_M(\tau), A^3\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\ &\qquad + \sum_{j=1}^{3}\left(\mathrm{e}^{\tau A^{1/2}}B_j(\phi_M), A^3\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right).\end{aligned} \tag{44}$$

The above terms are evaluated as follows: the time-derivative term,

$$\begin{aligned}&\left(\mathrm{e}^{\tau A^{1/2}}\phi_M'(\tau), A^3\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\ &\quad= \left(A^{\frac{3}{2}}\left(\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right)' - A^2\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau), \mathrm{e}^{\tau A^{1/2}}A^{\frac{3}{2}}\phi_M(\tau)\right) \\ &\quad= \frac{1}{2}\frac{d}{d\tau}\|A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\|^2 - \left(A^2\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau), A^{\frac{3}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\ &\quad= \frac{1}{2}\frac{d}{d\tau}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 - (A^2\phi_M(\tau), A^{\frac{3}{2}}\phi_M(\tau))_\tau \\ &\quad\geq \frac{1}{2}\frac{d}{d\tau}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 - \frac{\varepsilon^2}{4}|A^2\phi_M(\tau)|_\tau^2 - \frac{1}{\varepsilon^2}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 \\ &\quad\geq \frac{1}{2}\frac{d}{d\tau}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 - \frac{\varepsilon^2}{4}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^{\frac{4}{3}} \cdot |A^3\phi_M(\tau)|_\tau^{\frac{2}{3}} - \frac{1}{\varepsilon^2}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 \\ &\quad\geq \frac{1}{2}\frac{d}{d\tau}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 - \left(\frac{\varepsilon^2}{6} + \frac{1}{\varepsilon^2}\right)|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 - \frac{\varepsilon^2}{12}|A^3\phi_M(\tau)|_\tau^2;\end{aligned} \tag{45}$$

the surface diffusion term,

$$\varepsilon^2 \left(\mathrm{e}^{\tau A^{1/2}} A^3 \phi_M(\tau), A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\right) = \varepsilon^2 |A^3 \phi_M(\tau)|_\tau^2; \tag{46}$$

the first lower-order linear term,

$$\begin{aligned} a_2 &\left|\left(\mathrm{e}^{\tau A^{1/2}} A \phi_M(\tau), A^3 e^{\tau A^{1/2}} \phi_M(\tau)\right)\right| \\ &\leq a_2 \|\mathrm{e}^{\tau A^{1/2}} A^{\frac{3}{2}} \phi_M(\tau)\| \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\| \\ &\leq 3a_2^2 \varepsilon^{-2} |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^2 + \frac{\varepsilon^2}{12} |A^3 \phi_M(\tau)|_\tau^2; \end{aligned} \tag{47}$$

and the second lower-order linear term,

$$\begin{aligned} a_{1,2} &\left|\left(\mathrm{e}^{\tau A^{1/2}} A^2 \phi_M(\tau), A^3 e^{\tau A^{1/2}} \phi_M(\tau)\right)\right| \\ &\leq a_{1,2} \|\mathrm{e}^{\tau A^{1/2}} A^2 \phi_M(\tau)\| \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\| \\ &\leq a_{1,2} \|\mathrm{e}^{\tau A^{1/2}} A^{\frac{3}{2}} \phi_M(\tau)\|^{\frac{2}{3}} \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^{\frac{1}{3}} \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\| \\ &= a_{1,2} \|\mathrm{e}^{\tau A^{1/2}} A^{\frac{3}{2}} \phi_M(\tau)\|^{\frac{2}{3}} \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^{\frac{4}{3}} \\ &= a_{1,2} |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^{\frac{2}{3}} \cdot |A^3 \phi_M(\tau)|_\tau^{\frac{4}{3}} \\ &\leq \frac{1}{3} \cdot 8^2 \varepsilon^{-4} a_{1,2}^3 |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^2 + \frac{\varepsilon^2}{12} |A^3 \phi_M(\tau)|_\tau^2. \end{aligned} \tag{48}$$

Notice that we have used Lemma 1.2 in the last estimate and Young's inequality in several places.

For the nonlinear terms, a combination of Lemmas 3.2 and 3.3 yields

$$\begin{aligned} \sum_{j=1}^{3} &\left(\mathrm{e}^{\tau A^{1/2}} B_j(\phi_M(\tau)), A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\right) \\ &\leq C(\|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^5 + \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^3) \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\| \\ &\quad + C \|A^{\frac{3}{2}} \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^{\frac{8}{3}} \cdot \|A^3 \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\|^{\frac{4}{3}} \\ &= C(|A^{\frac{3}{2}} \phi_M(\tau)|_\tau^5 + |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^3) |A^3 \phi_M(\tau)|_\tau \\ &\quad + C |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^{\frac{8}{3}} \cdot |A^3 \phi_M(\tau)|_\tau^{\frac{4}{3}} \\ &\leq C_{10} \varepsilon^{-2} (|A^{\frac{3}{2}} \phi_M(\tau)|_\tau^{10} + |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^6) + C_{11} \varepsilon^{-4} |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^8 \\ &\quad + \frac{\varepsilon^2}{4} |A^3 \phi_M(\tau)|_\tau^2, \end{aligned} \tag{49}$$

for some constants $C_{10}, C_{11} > 0$, using Young's inequality in the last step.

Subsequently, a substitution of (45) – (49) into (44) leads to

$$\begin{aligned} \frac{1}{2} \frac{d}{d\tau} &|A^{\frac{3}{2}} \phi_M(\tau)|_\tau^2 + \frac{\varepsilon^2}{2} |A^3 \phi_M(\tau)|_\tau^2 \\ &\leq C_{10} \varepsilon^{-2} (|A^{\frac{3}{2}} \phi_M(\tau)|_\tau^{10} + |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^6) \\ &\quad + C_{11} \varepsilon^{-4} |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^8 + C_{12} |A^{\frac{3}{2}} \phi_M(\tau)|_\tau^2, \end{aligned} \tag{50}$$

where

$$C_{12} := (3a_2^2 + 1)\varepsilon^{-2} + \frac{1}{3} \cdot 8^2 \varepsilon^{-4} a_{1,2}^3 + \frac{\varepsilon^2}{6}.$$

This in turn gives

$$\frac{1}{2}\frac{d}{d\tau}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 \le C_{13}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^{10} + C_{12}|A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2, \tag{51}$$

with $C_{13} = C(C_{11}\varepsilon^{-2} + C_{12}\varepsilon^{-4} + 1)$. By setting $y = 1 + |A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2$, we obtain the differential inequality

$$y(\tau)' \le C_{14}y(\tau)^5,$$

where $C_{14} > 0$ depends on C_{12} and C_{13}. The following estimate is then valid:

$$y(\tau) \le \left(\frac{1}{1 - 4C_{14}\tau y^4(0)}\right)^{\frac{1}{4}} y(0), \quad \text{for all} \quad \tau \in [0, T_1), \tag{52}$$

where $T_1 := (4C_{14}y^4(0))^{-1}$. This implies that

$$\begin{aligned} 1 + |A^{\frac{3}{2}}\phi_M(\tau)|_\tau^2 &= y(\tau) \le 2y(0) \\ &= 2 + 2\|A^{\frac{3}{2}}\phi_M(0)\|^2 \\ &\le 2 + 2\|A^{\frac{3}{2}}\phi^0\|^2, \quad \text{for all} \quad \tau \in [0, T_2], \end{aligned}$$

using stability of the L^2 projection, where

$$T_2 := \frac{15}{64C_{14}}(1 + \|A^{3/2}\phi^0\|^2)^{-4} \ge \frac{15}{64C_{14}}(1 + \|A^{3/2}\phi_M(0)\|^2)^{-4}.$$

We can now extract a further subsequence of ϕ_M and pass to limits to obtain our estimates for the limit point ϕ, which is observed to be Gevrey regular on the time interval $(0, T_2)$. The uniqueness analysis of the Gevrey regularity solution is straightforward, due to the high order regularity. The details are left to interested readers. The theorem is proven with $T_\star = T_2$. □

4. **Existence of a global-in-time solution with Gevrey regularity.** Note that the existence time interval length $T_\star$ in Theorem 3.4 for the Gevrey regularity solution depends on the initial data through $A^{\frac{3}{2}}\phi^0$. To obtain a global in time solution with Gevrey regularity, we have to establish a uniform-in-time bound for $\|A^{\frac{3}{2}}\phi(t)\|$, i.e., $\|\nabla\Delta\phi(t)\|$, so that the constructed solution can be extended to any time.

We have already obtained a global in time bound of $\|A\phi\|$, as given by (20). The global in time H^3 estimate is stated in the following proposition.

Proposition 2. *Assume that $\phi^0 = \psi + \overline{\phi^0}$, where $\psi \in D(A^{\frac{3}{2}})$ and $\overline{\phi^0} \in \mathbb{R}$. For the FCH equation (6), the following a-priori estimate is valid:*

$$\|\nabla\Delta\phi(t)\| \le C_{15}, \quad \textit{for all} \quad t > 0, \tag{53}$$

where the constant $C_{15} > 0$ is independent of t.

Proof. Once again, we assume, for simplicity of notation, that $\overline{\phi^0} = 0$. Also for simplicity, let us work directly with the limit point, ϕ rather the smooth Galerkin approximation, ϕ_M. Strictly speaking, we must prove the estimate for ϕ_M before passing to the limit to show that the calculations hold true for the limit function. We skip these steps for brevity.

Taking an inner product with (6) by $-\Delta^3\phi$ gives

$$\begin{aligned}\left(\phi_t, -\Delta^3\phi\right) + \varepsilon^2\|\Delta^3\phi\|^2 &= -a_2\left(\Delta\phi, \Delta^3\phi\right) - a_{1,2}\left(\Delta^2\phi, \Delta^3\phi\right)\\ &\quad -\sum_{j=1}^{3}\left(B_j(\phi), \Delta^3\phi\right). \qquad (54)\end{aligned}$$

The time derivative term can be treated using integration by parts:

$$\left(\partial_t\phi, -\Delta^3\phi\right) = \left(\partial_t\nabla\Delta\phi, \nabla\Delta\phi\right) = \frac{1}{2}\partial_t\|\nabla\Delta\phi\|^2. \qquad (55)$$

The lower order linear diffusion terms could be bounded as follows:

$$\begin{aligned}-a_2\left(\Delta\phi, \Delta^3\phi\right) &\le a_2\|\Delta\phi\|\cdot\|\Delta^3\phi\|\\ &\le a_2C_9\|\Delta^3\phi\|\\ &\le 2a_2^2C_9^2\varepsilon^{-2} + \frac{\varepsilon^2}{8}\|\Delta^3\phi\|^2, \qquad (56)\\ -a_{1,2}\left(\Delta^2\phi, \Delta^3\phi\right) &\le a_{1,2}\|\Delta^2\phi\|\cdot\|\Delta^3\phi\|\\ &\le a_{1,2}\|\Delta\phi\|^{1/2}\cdot\|\Delta^3\phi\|^{1/2}\cdot\|\Delta^3\phi\|\\ &\le a_{1,2}C_9^{1/2}\cdot\|\Delta^3\phi\|^{3/2}\\ &\le Ca_{1,2}^4C_9^2\varepsilon^{-6} + \frac{\varepsilon^2}{8}\|\Delta^3\phi\|^2, \qquad (57)\end{aligned}$$

in which a Sobolev interpolation inequality and Young's inequality have been repeatedly applied. Note that C_9 is given by (26). (If we were working with the Galerkin approximation, we would use C_4 from (20).)

For the nonlinear terms, we apply regularity estimates, the Gagliardo-Nirenberg inequality, and interpolation inequalities to get

$$\begin{aligned}\|\phi\|_\infty &\le C\|\phi\|_{H^2} \le CC_9, \qquad (58)\\ \|\nabla\phi\|_4 &\le C\|\Delta\phi\| \le CC_9, \qquad (59)\\ \|\nabla\phi\|_\infty &\le C(\|\Delta\phi\| + \|\Delta\phi\|^{7/8}\cdot\|\Delta^3\phi\|^{1/8})\\ &\le C(C_9 + C_9^{7/8}\cdot\|\Delta^3\phi\|^{1/8}), \qquad (60)\\ \|\Delta\phi\|_\infty + \|\mathsf{H}(\phi)\|_\infty &\le C(\|\nabla\Delta\phi\| + \|\nabla\Delta\phi\|^{5/6}\cdot\|\Delta^3\phi\|^{1/6})\\ &\le C(\|\Delta\phi\|^{3/4}\cdot\|\Delta^3\phi\|^{1/4} + \|\Delta\phi\|^{5/8}\cdot\|\Delta^3\phi\|^{3/8})\\ &\le C(C_9^{3/4}\cdot\|\Delta^3\phi\|^{1/4} + C_9^{5/8}\cdot\|\Delta^3\phi\|^{3/8}), \qquad (61)\\ \|\Delta\phi\|_4 + \|\mathsf{H}(\phi)\|_4 &\le C\|\phi\|_{H^{\frac{11}{4}}} \le C\|\Delta\phi\|^{\frac{13}{16}}\cdot\|\Delta^3\phi\|^{\frac{3}{16}}\\ &\le CC_9^{\frac{13}{16}}\cdot\|\Delta^3\phi\|^{\frac{3}{16}}, \qquad (62)\\ \|\nabla\Delta\phi\| &\le \|\Delta\phi\|^{3/4}\cdot\|\Delta^3\phi\|^{1/4} \le C_9^{3/4}\cdot\|\Delta^3\phi\|^{1/4}, \qquad (63)\\ \|\Delta^2\phi\| &\le \|\Delta\phi\|^{1/2}\cdot\|\Delta^3\phi\|^{1/2} \le C_9^{1/2}\cdot\|\Delta^3\phi\|^{1/2}. \qquad (64)\end{aligned}$$

Then we get

$$\begin{aligned}\|B_{11}(\phi)\| &= \|\Delta\phi(\nabla\phi\cdot\nabla\phi)\| \le \|\Delta\phi\|\cdot\|\nabla\phi\|_\infty^2\\ &\le CC_9(C_9^2 + C_9^{7/4}\cdot\|\Delta^3\phi\|^{1/4}), \qquad (65)\\ \|B_{12}(\phi)\| &= \|\nabla\phi\cdot(\mathsf{H}(\phi)\nabla\phi)\|\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathsf{H}(\phi)\| \cdot \|\nabla\phi\|_\infty^2 \leq CC_9(C_9^2 + C_9^{7/4} \cdot \|\Delta^3\phi\|^{1/4}), && (66)\\
\|B_{13}(\phi)\| &= \|\phi|\mathsf{H}(\phi)|^2\| \leq \|\phi\|_\infty \cdot \|\mathsf{H}(\phi)\|_4^2 \leq CC_9^{\frac{21}{8}} \cdot \|\Delta^3\phi\|^{\frac{3}{8}}, && (67)\\
\|B_{14}(\phi)\| &= \|\phi^2\Delta^2\phi\| \leq \|\phi\|_\infty^2 \cdot \|\Delta^2\phi\| \leq CC_9^{5/2} \cdot \|\Delta^3\phi\|^{1/2}, && (68)\\
\|B_{15}(\phi)\| &= \|\phi(\Delta\phi)^2\| \leq \|\phi\|_\infty \cdot \|\Delta\phi\|_4^2 \leq CC_9^{\frac{21}{8}} \cdot \|\Delta^3\phi\|^{\frac{3}{8}}, && (69)\\
\|B_{16}(\phi)\| &= \|\phi(\nabla\phi \cdot \nabla\Delta\phi)\| \leq \|\phi\|_\infty \cdot \|\nabla\phi\|_\infty \cdot \|\nabla\Delta\phi\| \\
&\leq CC_9(C_9 + C_9^{7/8} \cdot \|\Delta^3\phi\|^{1/8})C_9^{3/4} \cdot \|\Delta^3\phi\|^{1/4} \\
&\leq CC_9(C_9^2 + C_9^{\frac{13}{8}} \cdot \|\Delta^3\phi\|^{\frac{3}{8}}). && (70)
\end{aligned}
$$

Then we arrive at

$$
\|B_1(\phi)\| \leq \sum_{j=11}^{16} \|B_j(\phi)\| \leq C_{16}(\|\Delta^3\phi\|^{\frac{1}{4}} + \|\Delta^3\phi\|^{\frac{3}{8}} + \|\Delta^3\phi\|^{\frac{1}{2}} + 1), \tag{71}
$$

where $C_{16} > 0$ depends on C_9 (C_4 in the case of the Galerkin approximation).

The terms in $B_2(\phi)$ and $B_3(\phi)$ can be bounded in an even more straightforward way:

$$
\begin{aligned}
\|\phi^4\Delta\phi\| &\leq \|\phi\|_\infty^4 \cdot \|\Delta\phi\| \leq CC_9^5, && (72)\\
\|\phi^3|\nabla\phi|^2\| &\leq \|\phi\|_\infty^3 \cdot \|\nabla\phi\|_4^2 \leq CC_9^5, && (73)\\
\|\phi^2\Delta\phi\| &\leq \|\phi\|_\infty^2 \cdot \|\Delta\phi\| \leq CC_9^3, && (74)\\
\|\phi|\nabla\phi|^2\| &\leq \|\phi\|_\infty \cdot \|\nabla\phi\|_4^2 \leq CC_9^3. && (75)
\end{aligned}
$$

Consequently, we obtain the following estimate

$$
\sum_{j=1}^{3} \|B_j(\phi)\| \leq C_{17}(\|\Delta^3\phi\|^{\frac{1}{4}} + \|\Delta^3\phi\|^{\frac{1}{2}} + \|\Delta^3\phi\|^{\frac{3}{8}} + 1), \tag{76}
$$

so that the following inequality could be derived:

$$
\begin{aligned}
-\sum_{j=1}^{3} \big(B_j(\phi), \Delta^3\phi\big) &\leq \sum_{j=1}^{3} \|B_j(\phi)\| \cdot \|\Delta^3\phi\| \\
&\leq C_{17}(\|\Delta^3\phi\|^{\frac{5}{4}} + \|\Delta^3\phi\|^{\frac{11}{8}} + \|\Delta^3\phi\|^{\frac{3}{2}} + \|\Delta^3\phi\|) \\
&\leq C_{18} + \frac{\varepsilon^2}{4}\|\Delta^3\phi\|^2,
\end{aligned} \tag{77}
$$

in which the Young's inequality has been applied in the last step, using the fact that all three power indices are less than 2.

Finally, a substitution of (55), (56), (57), (77) into (54) results in

$$
\frac{d}{dt}\|\nabla\Delta\phi\|^2 + \varepsilon^2\|\Delta^3\phi\|^2 \leq C_{19}. \tag{78}
$$

By denoting $\hat{E}(t) = \|\nabla\Delta\phi\|^2$ and making use of the elliptic regularity estimate

$$
\|\nabla\Delta\phi\|^2 \leq C\|\Delta^3\phi\|^2, \tag{79}
$$

we arrive at

$$
\partial_t\hat{E}(t) + C_{20}\hat{E}(t) \leq C_{19}. \tag{80}
$$

This differential inequality implies that

$$
\hat{E}(t) = \|\nabla\Delta\phi(t)\|^2 \leq \mathrm{e}^{-C_{20}t}\hat{E}(0) + \frac{C_{19}}{C_{20}} \leq \hat{E}(0) + \frac{C_{19}}{C_{20}} =: C_{21}. \tag{81}
$$

Therefore, a global bound of $\|A^{\frac{3}{2}}\phi(t)\|$ is available, since $C_{21} > 0$ is time independent. This completes the proof of Prop. 2. $\square$

As a consequence of Theorem 3.4 and Proposition 2, we arrive at the following theorem, the main result of this paper.

Theorem 4.1. *Assume that $\phi^0 \in H^3_{\rm per}$. Then the FCH equation (6) has a unique, global-in-time regular solution, such that $\tau \to e^{\tau A^{1/2}}\phi(\tau)$ is analytic on $(0, T_\star)$, for any $T_\star > 0$.*

5. Concluding remarks.

The Gevrey regularity of solutions for the functionalized Cahn-Hilliard (FCH)/Cahn-Hilliard-Willmore (CHW) equations have been analyzed in this article. It was proved that, for any $H^3_{\rm per}$ initial data, there is a unique Gevrey regularity solution for the highly nonlinear equation, and the existence of such a solution is global-in-time.

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