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LONG TIME STABILITY OF HIGH ORDER MULTISTEP NUMERICAL SCHEMES FOR TWO-DIMENSIONAL INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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Abstract. The long-time stability properties of a few multistep numerical schemes for the two-dimensional incompressible Navier–Stokes equations (formulated in vorticity-stream function) are investigated in this article. These semi-implicit numerical schemes use a combination of explicit Adams–Bashforth extrapolation for the nonlinear convection term and implicit Adams–Moulton interpolation for the viscous diffusion term, up to fourth order accuracy in time. As a result, only two Poisson solvers are needed at each time step to achieve the desired temporal accuracy. The fully discrete schemes, with Fourier pseudospectral approximation in space, are analyzed in detail. With the help of a priori analysis and aliasing error control techniques, we prove uniform in time bounds for these high order schemes in both L^2 and H^m norms, for $m \geq 1$, provided that the time step is bounded by a given constant. Such a long time stability is also demonstrated by the numerical experiments.

Key words. two-dimensional incompressible Navier–Stokes equations, Fourier pseudospectral approximation, Adams–Bashforth extrapolation, Adams–Moulton interpolation, multistep schemes, uniform in time estimate

AMS subject classifications. 65M12, 65M70, 76D06

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1. Introduction.

The two-dimensional (2D) incompressible Navier–Stokes equation (NSE) in the vorticity-stream function formulation takes the form

$$\partial_t \omega + \boldsymbol{u} \cdot \nabla \omega = \nu \Delta \omega + \boldsymbol{f}, \tag{1}$$

$$\Delta \psi = \omega, \tag{2}$$

$$\boldsymbol{u} = \nabla^{\perp} \psi = (-\partial_y \psi, \partial_x \psi) \,. \tag{3}$$

Here $\boldsymbol{u} = (u, v)^T$ is the velocity field, $\omega = -u_y + v_x$ is the scalar vorticity, ψ is the scalar stream function, and $\boldsymbol{f}$ represents (given) external forcing. For simplicity we assume periodic boundary condition, i.e., the domain is a 2D torus $\mathbb{T}^2$, and that all functions have mean zero over the torus. We also assume that $\boldsymbol{f} \in L^\infty(0, T; H^m)$ uniformly for any $t > 0$ and $\|\boldsymbol{f}(\cdot, t)\|_{H^m} \leq M^{(m)}$.

The 2D incompressible flows may become very complicated in long time behavior [9, 16, 18, 33, 34, 40]. As a result, long time stability and accuracy are extremely important for a given numerical method to obtain a better understanding of these complicated phenomena. In particular, a well-known long time stable (uniformly bounded in time) quantity for the NSE (1)–(3) is the enstrophy variable, $\frac{1}{2}\|\omega\|_{L^2}^2$, so that the dynamics possesses a global attractor and invariant measures [9, 16, 40]. As a consequence, one would naturally require that the numerical scheme preserve the

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long time stability of certain physical variables in the numerical simulation of NSE, in order to capture the long time dynamics in a correct way.

On the numerical side, there has been a long list of works on time discretization of the dissipative systems that preserve the dissipativity in various forms [8, 14, 15, 28, 29, 36, 37, 38, 41, 42]; the long time statistical convergence properties have also been addressed in [44, 45]. For the 2D NSE (1)–(3), the simplest example is the following semi-implicit numerical scheme, which treats the viscous term implicitly and the nonlinear advection term explicitly:

$$\frac{\omega^{n+1}-\omega^n}{\Delta t} + \nabla^{\perp}\psi^n \cdot \nabla\omega^n - \nu\Delta\omega^{n+1} = \boldsymbol{f}^n, \tag{4}$$

in which Δt is the time step size, and ω^k corresponds to the approximation of the vorticity at t^k; see the related references [1, 5, 35]. The efficiency of this scheme comes from the fact that only two Poisson solvers are needed at each time step, one for the vorticity and one for the stream function. The convergence of this scheme on any fixed time interval is standard; see the derivations in earlier literature [22, 23, 24, 25, 27, 39]. The long time stability of the above scheme (4) was analyzed in a recent article [20]; it was proven that (4) is long time stable in L^2 and H^1 and that the global attractor as well as the invariant measures of the scheme converge to those of the NSE at a vanishing time step.

For the spatial discretization, the Fourier spectral approach is an obvious choice [5, 35], due to the fact that the NSE solution is analytic in space (in fact, Gevrey class regular [17]). The uniform in time L^2 and H^1 bound for the fully discrete scheme, with pseudospectral approximation in space, was also established in [20], so that a statistical convergence could be derived as a consequence. Also see the related reference works [11, 26, 32] on the spectral schemes for incompressible NSEs.

Of course, the first order temporal accuracy of the scheme (4) is not satisfactory in practical computations; instead, higher order accurate, long time stable numerical algorithms are highly desired in scientific computing. For example, the second order two-step method

$$\frac{\frac{3}{2}\omega^{n+1} - 2\omega^n + \frac{1}{2}\omega^{n-1}}{\Delta t} + \nabla^{\perp}(2\psi^n - \psi^{n-1}) \cdot \nabla(2\omega^n - \omega^{n-1}) - \nu\Delta\omega^{n+1} = \boldsymbol{f}^{n+1/2} \tag{5}$$

was analyzed in a more recent article [46]. This scheme falls into the category of implicit-explicit schemes (IMEX) [1, 10], which combine a second order backward-differentiation for the diffusion term and an explicit second order Adams–Bashforth treatment for the nonlinear convection part. A uniform in time L^2, H^1, and H^2 bound of the numerical solution was established in [46], so that a statistical convergence becomes available.

In turn, the third and fourth order schemes of the IMEX family have been applied to the incompressible NSEs with various spatial approximations [2, 3, 30, 31]. However, only the linear stability analysis was covered in these existing works, and the nonlinear analysis is not available, even for local in time estimates; see the relevant discussions in [35].

In this work we provide a novel long time stability analysis for a family of multistep numerical schemes, up to fourth order temporal accuracy, combined with Fourier pseudospectral approximation in space, applied to 2D NSE (1)–(3). These multistep schemes adopt an explicit Adams–Bashforth approach for the nonlinear convection term and an implicit Adams–Moulton method for the diffusion term; as a result, the nonlinear terms are treated in an inexpensive way and the stability is preserved associated with implicit methods. More importantly, a necessary condition on the Adams–Moulton coefficients has to be satisfied for the nonlinear numerical stability. This necessary condition was first reported in a recent article [21], and a local in time convergence was proven for its application to the 3D viscous Burgers' equation. In this article, we apply this multistep approach to 2D NSE (1)–(3) and establish the uniform in time L^2 and H^m bounds of the fully discrete pseudospectral numerical solution for $m \geq 1$.

This paper is organized as follows. A general description of Fourier pseudospectral differentiation is recalled in section 2. The fully discrete multistep schemes (up to fourth order temporal accuracy) are outlined in section 3, and the main results are stated there. A detailed analyses is presented in section 4. Some numerical results are provided in section 5, which verifies the stability and convergence analysis for the proposed third and fourth order schemes. Finally, some concluding remarks are given by section 6.

2. Review of Fourier pseudospectral differentiation. We review Fourier collocation spectral differentiation in this section. Assume that $L_x = N_x \cdot h_x$, $L_y = N_y \cdot h_y$ for some mesh sizes $h_x, h_y > 0$ and some positive integers N_x and N_y. For simplicity of presentation, a uniform mesh size $h_x = h_y = h$, $N_x = N_y = 2N+1$ is taken. All the variables are evaluated at the regular numerical grid (x_i, y_j) with $x_i = ih$, $y_j = jh$, $0 \leq i, j \leq 2N+1$.

Without loss of generality, we assume that $L_x = L_y = 1$. For a periodic function f over the given 2D numerical grid, we assume its discrete Fourier collocation expansion as

$$f_{i,j} = \sum_{k_1,l_1=-N}^{N} \hat{f}^c_{k_1,l_1} \mathrm{e}^{2\pi\mathrm{i}(k_1 x_i + l_1 y_j)}, \tag{6}$$

in which the collocation coefficients are given by the following backward transformation formula:

$$\hat{f}^c_{k_1,l_1} = \frac{1}{(2N+1)^2} \sum_{i,j=0}^{2N} f_{i,j} \mathrm{e}^{-2\pi\mathrm{i}(k_1 x_i + l_1 y_j)}. \tag{7}$$

In turn, the collocation Fourier spectral approximations to first and second order partial derivatives become

$$\left(\mathcal{D}_{Nx} f\right)_{i,j} = \sum_{k_1,l_1=-N}^{N} (2k_1\pi\mathrm{i})\, \hat{f}^c_{k_1,l_1} \mathrm{e}^{2\pi\mathrm{i}(k_1 x_i + l_1 y_j)}, \tag{8}$$

$$\left(\mathcal{D}^2_{Nx} f\right)_{i,j} = \sum_{k_1,l_1=-N}^{N} \left(-4\pi^2 k_1^2\right) \hat{f}^c_{k_1,l_1} \mathrm{e}^{2\pi\mathrm{i}(k_1 x_i + l_1 y_j)}. \tag{9}$$

The differentiation operators $\mathcal{D}_{Ny}$ and $\mathcal{D}^2_{Ny}$ could be defined in the same fashion. In turn, the discrete Laplacian, gradient, and divergence become

$$\Delta_N f = \left(\mathcal{D}^2_{Nx} + \mathcal{D}^2_{Ny}\right) f,$$
$$\nabla_N f = \begin{pmatrix} \mathcal{D}_{Nx} f \\ \mathcal{D}_{Ny} f \end{pmatrix}, \quad \nabla_N \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathcal{D}_{Nx} f_1 + \mathcal{D}_{Ny} f_2 \tag{10}$$

at the pointwise level. It is also straightforward to verify that $\nabla_N \cdot \nabla_N f = \Delta_N f$. See the derivations in the related references [4, 6, 19], etc.

Moreover, given any periodic grid functions f and g (over the 2D numerical grid), the spectral approximations to the L^2 inner product and L^2 norm are introduced as

$$\|f\|_2 = \sqrt{\langle f, f \rangle} \quad \text{with} \quad \langle f, g \rangle = h^2 \sum_{i,j=0}^{2N} f_{i,j} g_{i,j}. \tag{11}$$

A detailed calculation reveals that the following formulas of summation by parts are also valid at the discrete level (see the related discussions [7, 21]):

$$\left\langle f, \nabla_N \cdot \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = - \left\langle \nabla_N f, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle, \qquad \langle f, \Delta_N g \rangle = - \langle \nabla_N f, \nabla_N g \rangle, \tag{12}$$

$$\langle f, \Delta_N^2 g \rangle = \langle \Delta_N f, \Delta_N g \rangle. \tag{13}$$

To facilitate the long time stability analysis, we introduce the following notation for differential operators, at both the continuous and discrete levels:

$$\nabla^m f = \begin{cases} \Delta^k f & \text{if } m = 2k, \\ \nabla \Delta^k f & \text{if } m = 2k+1, \end{cases} \qquad \nabla_N^m f = \begin{cases} \Delta_N^k f & \text{if } m = 2k, \\ \nabla_N \Delta_N^k f & \text{if } m = 2k+1. \end{cases} \tag{14}$$

Also, we denote $\| \cdot \|$ as the standard L^2 norm for a continuous function.

2.1. A preliminary estimate in Fourier collocation spectral space. It is well-known that the existence of an aliasing error in the nonlinear term poses a serious challenge in the numerical analysis of Fourier pseudospectral scheme. We introduce a continuous extension of a grid function and a Fourier collocation interpolation operator.

DEFINITION 1. *For any periodic grid function f defined over a uniform 2D numerical grid, we denote $\boldsymbol{f}_N$ as its continuous extension. In more detail, assume that the grid function f has a discrete Fourier expansion as* (6)*; its continuous extension (projection) into the trigonometric polynomial space P^N (with trigonometric polynomial up to degree N) is given by*

$$\boldsymbol{f}_N(x, y) = \sum_{k_1, l_1 = -N}^{N} \hat{f}^c_{k_1, l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x + l_1 y)}. \tag{15}$$

Moreover, for any periodic continuous function $\boldsymbol{f}$, which may contain larger wave length, we define its collocation interpolation operator as

$$P_c^N \boldsymbol{f}(x, y) = \sum_{k_1, l_1 = -N}^{N} (\hat{f}_c)_{k_1, l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x + l_1 y)} \tag{16}$$

with the Fourier collocation coefficients given by the following discrete expansion:

$$(\hat{f}_c)_{k_1, l_1} = \frac{1}{(2N+1)^2} \sum_{i,j=0}^{2N} \boldsymbol{f}_{i,j} \mathrm{e}^{-2\pi \mathrm{i}(k_1 x_i + l_1 y_j)}$$

$$\text{so that } \boldsymbol{f}_{i,j} = \sum_{k_1, l_1 = -N}^{N} (\hat{f}_c)_{k_1, l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x_i + l_1 y_j)}. \tag{17}$$

Note that $\hat{f}_c$ may not be the Fourier coefficients of $\boldsymbol{f}$, due to the truncation and aliasing errors.

To overcome a key difficulty associated with the H^m bound of the nonlinear term obtained by collocation interpolation, the following lemma is introduced. In fact, the case of $k_0 = 0$ was proven in E's earlier works [12, 13]. The case of $k_0 \geq 1$ was analyzed in a recent article by Gottlieb and Wang [21]; we cite the result here.

LEMMA 1. *For any $\varphi \in P^{2N}$ in dimension d, we have*

$$\left\| P_c^N \varphi \right\|_{H^{k_0}} \leq \left(\sqrt{2} \right)^d \| \varphi \|_{H^{k_0}} \quad \forall k_0 \in \mathbb{Z},\, k_0 \geq 0. \tag{18}$$

3. The formulation of the high order multistep schemes and the main results.

3.1. Description of the fully discrete schemes. For the 2D incompressible NSEs (1)–(3), we treat the nonlinear convection term explicitly for the sake of convenience and the diffusion term implicitly to preserve a numerical stability. This semi-implicit approach leads to a Poisson-like equation at each time step, which makes the numerical algorithm extremely efficient. After the scalar vorticity is updated, the stream function can be determined through the kinematic equation, a Poisson equation. In turn, the velocity is computed as the perpendicular gradient of the stream function.

This semi-implicit idea can be applied to derive a second order in time method for (1)–(3). The nonlinear convection term is updated explicitly, using a standard second order Adams–Bashforth extrapolation formula, which involves the numerical solutions at time node points t^n, t^{n-1}, with well-known coefficients $3/2$ and $-1/2$. Meanwhile, an implicit treatment of the diffusion term is based on a second order Adams–Moulton interpolation. However, the standard second order formula yields the Crank–Nicholson scheme for the diffusion term, which leads to a difficulty in the stability analysis if the nonlinear convection term is taken into consideration. Instead, we look for an Adams–Moulton interpolation so that the diffusion term is more focused on the time step t^{n+1}, i.e., the coefficient at time step t^{n+1} dominates the sum of the rest of diffusion coefficients. It is discovered that the Adams–Moulton interpolation which involves the time node points t^{n+1} and t^{n-1} gives the corresponding coefficients as $3/4$, $1/4$, respectively, which satisfies the unconditional stability condition. Therefore, the fully discrete scheme is formulated as

$$\begin{aligned} &\frac{\omega^{n+1} - \omega^n}{\Delta t} + \frac{3}{4} \left(\boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n) \right) \\ &\qquad - \frac{1}{4} \left(\boldsymbol{u}^{n-1} \cdot \nabla_N \omega^{n-1} + \nabla_N \cdot (\boldsymbol{u}^{n-1} \omega^{n-1}) \right) \\ &= \nu \Delta_N \left(\frac{3}{4} \omega^{n+1} + \frac{1}{4} \omega^{n-1} \right) + \boldsymbol{f}^{n+1/2}, \end{aligned} \tag{19}$$

$$\Delta_N \psi^{n+1} = \omega^{n+1}, \tag{20}$$

$$\boldsymbol{u}^{n+1} = \nabla_N^{\perp} \psi^{n+1} = \left(-\mathcal{D}_{Ny} \psi^{n+1}, \mathcal{D}_{Nx} \psi^{n+1} \right). \tag{21}$$

Similar ideas can be applied to derive third and fourth order in time schemes for (1)–(3). The nonlinear convection term is updated by an explicit Adams–Bashforth extrapolation formula, with the time node points $t^n, t^{n-1}, \ldots, t^{n-k+1}$ involved and an order of accuracy k. The diffusion term is computed by an implicit Adams–Moulton interpolation with the given accuracy order in time. To ensure an unconditional numerical stability for a fixed time, we have to derive an Adams–Moulton formula so that the coefficient at time step t^{n+1} dominates the sum of the other diffusion coefficients. In more detail, a kth order (in time) scheme takes the form of

$$\frac{\omega^{n+1}-\omega^n}{\Delta t}+\frac{1}{2}\sum_{i=0}^{k-1}B_i\left(\boldsymbol{u}^{n-i}\cdot\nabla_N\omega^{n-i}+\nabla_N\cdot(\boldsymbol{u}^{n-i}\omega^{n-i})\right)$$
$$=\nu\Delta_N\left(D_0\omega^{n+1}+\sum_{i=0}^{k-1}D_{j(i)}\omega^{n-j(i)}\right)+\tilde{\boldsymbol{f}}^n, \quad (22)$$

in which $B_i\mid_{i=0}^{k-1}$ are the standard Adams–Bashforth coefficients with extrapolation points t^n, $t^{n-1},\dots,t^{n-k+1}$, $j(i)\mid_{i=0}^{k-1}$ are a set of (distinct) indices with $j(i)\geq 0$, and D_0, $D_{j(i)}\mid_{i=0}^{k-1}$ correspond to the Adams–Moulton coefficients to achieve kth order accuracy. A necessary condition for unconditional numerical stability is given by [21]

$$D_0>\sum_{i=0}^{k-1}\left|D_{j(i)}\right|. \quad (23)$$

In other words, a stretched stencil is needed to satisfy this condition. For the third order scheme, a careful calculation shows that a stencil comprising the node points t^{n+1}, t^{n-1}, and t^{n-3} is adequate. The fully discrete scheme can be formulated as

$$\begin{aligned}&\frac{\omega^{n+1}-\omega^n}{\Delta t}+\frac{23}{24}\left(\boldsymbol{u}^n\cdot\nabla_N\omega^n+\nabla_N\cdot(\boldsymbol{u}^n\omega^n)\right)\\&\quad-\frac{2}{3}\left(\boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1}+\nabla_N\cdot(\boldsymbol{u}^{n-1}\omega^{n-1})\right)\\&\quad+\frac{5}{24}\left(\boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2}+\nabla_N\cdot(\boldsymbol{u}^{n-2}\omega^{n-2})\right)\\&=\nu\Delta_N\left(\frac{2}{3}\omega^{n+1}+\frac{5}{12}\omega^{n-1}-\frac{1}{12}\omega^{n-3}\right)+\tilde{\boldsymbol{f}}^n, &(24)\\&\Delta_N\psi^{n+1}=\omega^{n+1}, &(25)\\&\boldsymbol{u}^{n+1}=\nabla_N^{\perp}\psi^{n+1}=\left(-\mathcal{D}_{Ny}\psi^{n+1},\mathcal{D}_{Nx}\psi^{n+1}\right) &(26)\end{aligned}$$

with $\tilde{\boldsymbol{f}}^n=\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\boldsymbol{f}(\cdot,\tau)d\tau$. For the fourth order scheme, we use an Adams–Moulton interpolation at node points t^{n+1}, t^{n-1}, t^{n-5}, and t^{n-7} for the diffusion term. Combined with the Adams–Bashforth extrapolation for the nonlinear convection term, the scheme is given by

$$\begin{aligned}&\frac{\omega^{n+1}-\omega^n}{\Delta t}+\frac{55}{48}\left(\boldsymbol{u}^n\cdot\nabla_N\omega^n+\nabla_N\cdot(\boldsymbol{u}^n\omega^n)\right)\\&\quad-\frac{59}{48}\left(\boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1}+\nabla_N\cdot(\boldsymbol{u}^{n-1}\omega^{n-1})\right)\\&\quad+\frac{37}{48}\left(\boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2}+\nabla_N\cdot(\boldsymbol{u}^{n-2}\omega^{n-2})\right)\\&\quad-\frac{3}{16}\left(\boldsymbol{u}^{n-3}\cdot\nabla_N\omega^{n-3}+\nabla_N\cdot(\boldsymbol{u}^{n-3}\omega^{n-3})\right)\\&=\nu\Delta_N\left(\frac{757}{1152}\omega^{n+1}+\frac{470}{1152}\omega^{n-1}-\frac{118}{1152}\omega^{n-5}+\frac{43}{1152}\omega^{n-7}\right)+\tilde{\boldsymbol{f}}^n, &(27)\\&\Delta_N\psi^{n+1}=\omega^{n+1}, &(28)\\&\boldsymbol{u}^{n+1}=\nabla_N^{\perp}\psi^{n+1}=\left(-\mathcal{D}_{Ny}\psi^{n+1},\mathcal{D}_{Nx}\psi^{n+1}\right). &(29)\end{aligned}$$

Remark 1. The second order scheme (19)–(21) is a two-step method, so that an initial guess for ω^{-1} is needed. Similarly, the third order scheme (24)–(26) is a four-step method, so that an initial guess for ω^{-j}, $1 \le j \le 3$, is needed; the fourth order scheme (27)–(29) is an eight-step method, so that an initial guess for ω^{-j}, $1 \le j \le 7$, is needed.

Note that the nonlinear term at time step t^k is a spectral approximation to $\frac{1}{2}\left(\boldsymbol{u}\cdot\nabla\omega + \nabla\cdot(\boldsymbol{u}\omega)\right)$ at the same time step. Also, only two Poisson type equations need to be solved at each time step.

Meanwhile, it is observed that the numerical velocity $\boldsymbol{u}^k = \nabla_N^{\perp}\psi^k$ is automatically divergence-free at any time step:

$$\nabla_N \cdot \boldsymbol{u}^k = \mathcal{D}_{Nx} u^k + \mathcal{D}_{Ny} v^k = -\mathcal{D}_{Nx}(\mathcal{D}_{Ny}\psi^k) + \mathcal{D}_{Ny}(\mathcal{D}_{Nx}\psi^k) = 0. \tag{30}$$

Moreover, a careful application of summation by parts formula (12) implies that

$$\langle \omega, \boldsymbol{u}\cdot\nabla_N\omega + \nabla_N\cdot(\boldsymbol{u}\omega)\rangle = \langle \omega, \boldsymbol{u}\cdot\nabla_N\omega\rangle - \langle \nabla_N\omega, \boldsymbol{u}\omega\rangle = 0. \tag{31}$$

In other words, the nonlinear convection term appearing in the numerical scheme, the so-called Temam technique, makes the nonlinear term orthogonal to the vorticity field in the discrete L^2 space, without considering the temporal discretization. This property is crucial in the stability analysis for the Fourier collocation spectral scheme; see the related references [20, 46].

In addition, we denote $\boldsymbol{U}^k = (U^k, V^k)$, $\boldsymbol{\omega}^k$, and $\boldsymbol{\psi}^k$ as the continuous projection of $\boldsymbol{u}^k$, ω^k, and ψ^k, respectively, with the projection formula given by (6), (15). It is clear that $\boldsymbol{U}^k, \boldsymbol{\omega}^k, \boldsymbol{\psi}^k \in P^N$ and the kinematic equation $\Delta\boldsymbol{\psi}^k = \boldsymbol{\omega}^k$, $\boldsymbol{U}^k = \nabla^{\perp}\boldsymbol{\psi}^k$ is satisfied at the continuous level. Because of these kinematic equations, an application of elliptic regularity shows that

$$\begin{aligned}
&\|\psi^k\|_{H_h^{m+2}} \le C\|\omega^k\|_{H_h^m}, \quad \|\psi^k\|_{H_h^{m+2+\alpha}} \le C\|\omega^k\|_{H_h^{m+\alpha}},\\
&\|\boldsymbol{\psi}^k\|_{H^{m+2}} \le C\|\boldsymbol{\omega}^k\|_{H^m}, \quad \|\boldsymbol{\psi}^k\|_{H^{m+2+\alpha}} \le C\|\boldsymbol{\omega}^k\|_{H^{m+\alpha}},
\end{aligned} \tag{32}$$

in which we normalize the stream function with $\int_\Omega \psi^k d\boldsymbol{x} = 0$. Meanwhile, since all the profiles have mean zero over the domain,

$$\begin{aligned}
&\overline{\psi^k} = 0, \quad \overline{\boldsymbol{u}^k} = \left(-\overline{\mathcal{D}_{Ny}\psi^k}, \overline{\mathcal{D}_{Nx}\psi^k}\right) = 0, \quad \overline{\omega^k} = \overline{\Delta_N\psi^k} = 0,\\
&\overline{\boldsymbol{\psi}^k} = 0, \quad \overline{\boldsymbol{U}^k} = \left(-\overline{\partial_y\boldsymbol{\psi}^k}, \overline{\partial_x\boldsymbol{\psi}^k}\right) = 0, \quad \overline{\boldsymbol{\omega}^k} = \overline{\Delta\boldsymbol{\psi}^k} = 0,
\end{aligned} \tag{33}$$

all the Poincaré inequality and elliptic regularity can be applied.

3.2. The main results. The multistep schemes (22) were proposed and analyzed in [21]; the local in time convergence analysis (up to the fourth order temporal accuracy) of its application to the 3D viscous Burgers' equation, combined with pseudospectral spatial approximation, was provided. A similar estimate of this multistep approach was presented in a more recent work [43], with the local discontinuous Galerkin algorithm in space.

Using similar techniques, we are able to derive the following convergence results for the proposed multistep methods applied to the 2D NSEs; the details are skipped for the sake of brevity.

THEOREM 1. *For any final time $T^* > 0$, assume the exact solution (ω_e, ψ_e) to the 2D NSE (1)–(2) has a regularity of $\omega_e \in H^k(0, T^*; H^{m+3})$ with $m \ge 2$. Denote*

$\omega_{\Delta t,h}$ as the continuous (in space) extension of the fully discrete numerical solution given by the kth order multistep scheme, either the second order one (19)–(21), the third order one (24)–(26), or the fourth order one (27)–(29). As $\Delta t, h \to 0$, we have the following convergence result:

$$\|\omega_{\Delta t,h} - \omega_e\|_{l^\infty(0,T^*;L^2)} + \sqrt{\nu}\,\|\omega_{\Delta t,h} - \omega_e\|_{l^2(0,T^*;H^1)} \le C\left(\Delta t^k + h^m\right) \tag{34}$$

with k the order of temporal accuracy, provided that the time step Δt and the space grid size h are independently bounded by given constants which depend only on T^* and ν.

For the long time stability analysis of these multistep algorithms, we focus on the third order scheme (24)–(26); the second and fourth order ones could be analyzed in the same manner. Here is the main result.

THEOREM 2. *Let $\omega_0 \in H^2$ and let ω^n be the discrete solution of the fully discrete third order numerical scheme (24)–(26), with initial guess data ω^{-j} being bounded in H^2 norm $(1 \le j \le 3)$. Denote $\boldsymbol{\omega}^n$ as the continuous extension of ω^n in space, given by (16). Also, let $\boldsymbol{f} \in L^\infty(\mathbf{R}_+; L^2)$ and set $\|\boldsymbol{f}\|_\infty := \|\boldsymbol{f}\|_{L^\infty(\mathbf{R}+;L^2)} = M$. Then there exists $M_0 = M_0(\|\omega_0\|_{H^2}, \nu, \|\boldsymbol{f}\|_\infty)$ such that if*

$$\Delta t \le \frac{\nu}{C_w M_0^2}, \quad C_w \text{ is a constant only dependent on } \Omega, \tag{35}$$

then

$$\|\boldsymbol{\omega}^n\|_{H^1} \le M_1 \,\forall\, n \ge 0, \tag{36}$$

with M_1 a uniform in time constant. In more detail, we have

$$\|\boldsymbol{\omega}^n\| \le (1 + \gamma_0 \nu \Delta t)^{-\frac{n+1}{2}} (\tilde{E}_0^0)^{1/2} + Q^{(0)} \quad \forall\, n \ge 0, \tag{37}$$

$$\|\nabla \boldsymbol{\omega}^n\| \le (1 + \gamma_1 \nu \Delta t)^{-\frac{n+1}{2}} (\tilde{E}_1^0)^{1/2} + Q^{(1)} \quad \forall\, n \ge 0, \tag{38}$$

$$\begin{aligned} \tilde{E}_j^0 = {} & \|\nabla_N^j \omega^0\|_2^2 + \left(\frac{1}{4} + B_0\right) \|\nabla_N^j(\omega^0 - \omega^{-1})\|_2^2 + \frac{1}{4}\|\nabla_N^j(\omega^{-1} - \omega^{-2})\|_2^2 \\ & + \beta_0 \nu \Delta t \|\nabla_N^{j+1}\omega^0\|_2^2 + \left(\frac{5}{12} + \beta_1\right) \nu \Delta t \|\nabla_N^{j+1}\omega^{-1}\|_2^2 \\ & + \beta_2 \nu \Delta t \|\nabla_N^{j+1}\omega^{-2}\|_2^2 + \frac{1}{12}\nu\Delta t \|\nabla_N^{j+1}\omega^{-3}\|_2^2, \end{aligned} \tag{39}$$

in which γ_0 and γ_1 are constants associated with elliptic regularity, $Q^{(0)}$ and $Q^{(1)}$ are uniform in time constants only dependent on $\|\omega^0\|_{H^2}$, $\|\boldsymbol{f}\|_\infty$, Ω, and ν, and the coefficients β_1, β_2, and B_0 satisfy condition (68), which will be specified below.

In addition, if we assume that $\omega_0 \in H^{m+1}$, the initial guess data ω^{-j} are bounded in H^{m+1} $(1 \le j \le 3)$, $\boldsymbol{f} \in L^\infty(\mathbf{R}_+; H^{m-1})$ and set $\|\boldsymbol{f}\|_\infty^{(m-1)} := \|\boldsymbol{f}\|_{L^\infty(\mathbf{R}+;H^{m-1})} = M^{(m-1)}$. Then there exists $M_0^{(m)} = M_0(\|\omega_0\|_{H^{m+1}}, \nu, \|\boldsymbol{f}\|_\infty^{(m-1)})$ such that if

$$\Delta t \le \frac{\nu}{C_w (M_0^{(m)})^2}, \quad C_w \text{ is a constant only dependent on } \Omega, \tag{40}$$

then

$$\|\boldsymbol{\omega}^n\|_{H^m} \le M_1^{(m)} \,\forall\, n \ge 0, \tag{41}$$

with $M_1^{(m)}$ a uniform in time constant. In more detail, we have

$$\|\nabla^m \boldsymbol{\omega}^n\| \leq (1+\gamma_m \nu \Delta t)^{-\frac{n+1}{2}} (\tilde{E}_m^0)^{1/2} + Q^{(m)} \quad \forall\, n \geq 0, \tag{42}$$

in which γ_m is a constant associated with elliptic regularity, and $Q^{(m)}$ is a uniform in time constant only dependent on $\|\omega^0\|_{H^{m+1}}$, $\|\boldsymbol{f}\|_\infty^{(m-1)}$, Ω, and ν.

Remark 2. To the authors' knowledge, the third and fourth order (local in time) convergence analysis reported in [21] (for the viscous Burgers' equation) is the first theoretical result for third order time numerical stepping (or higher order) applied to nonlinear PDEs, and Theorem 2 is the first long time stability analysis for such schemes.

4. Proof of Theorem 2: Long time stability analysis for the third order multistep scheme (24)–(26). Motivated by the techniques used in recent works [21, 46], we organize the proof of Theorem 2 in the following way. First, an H^δ a priori assumption for the continuous version of the numerical solution is made. Subsequently, a uniform in time L^2 bound could be derived based on this assumption, with repeated applications of Sobolev embedding and Hölder's inequality. On the other hand, this L^2 bound is not sufficient to recover the a priori assumption, due to the fact that H^δ is a norm stronger than the standard L^2 one. To remedy this analysis, we derive a uniform in time H^1 estimate for the numerical solution, with the help of the leading L^2 bound. More importantly, both the global in time L^2 and H^1 bound constants are independent of the a priori constant. Therefore, the a priori assumption can be recovered so that an induction can be applied to establish the desired result.

4.1. Leading estimate: $\boldsymbol{\ell^\infty(0,T;L^2)\cap \ell^2(0,T;H^1)}$ estimate for $\boldsymbol{\omega}$. To deal with the multistep method (24)–(26), an H^δ a priori bound is assumed for the numerical solution at all previous time steps:

$$\|\boldsymbol{\omega}^n\|_{H^\delta} \leq \tilde{C}_1 \quad \forall 1 \leq k \leq n \tag{43}$$

for some $\delta > 0$. Note that $\tilde{C}_1$ is a global constant in time. We are going to prove that such a bound for the numerical solution is also available at time step t^{n+1}. Consequently, an application of induction could justify the a priori estimate.

Taking a discrete inner product with (24) by $2\omega^{n+1}$ gives

$$\begin{aligned}
&\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \|\omega^{n+1}-\omega^n\|_2^2 \\
&+\nu\Delta t \left\langle \nabla_N \left(\frac{4}{3}\omega^{n+1} + \frac{5}{6}\omega^{n-1} - \frac{1}{6}\omega^{n-3}\right), \nabla_N \omega^{n+1} \right\rangle \\
= &-\frac{23}{12}\Delta t \left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n), \omega^{n+1} \right\rangle \\
&+\frac{4}{3}\Delta t \left\langle \boldsymbol{u}^{n-1} \cdot \nabla_N \omega^{n-1} + \nabla_N \cdot \left(\boldsymbol{u}^{n-1}\omega^{n-1}\right), \omega^{n+1} \right\rangle \\
&-\frac{5}{12}\Delta t \left\langle \boldsymbol{u}^{n-2} \cdot \nabla_N \omega^{n-2} + \nabla_N \cdot \left(\boldsymbol{u}^{n-2}\omega^{n-2}\right), \omega^{n+1} \right\rangle + 2\Delta t \left\langle \tilde{\boldsymbol{f}}^n, \omega^{n+1} \right\rangle
\end{aligned} \tag{44}$$

with the summation by parts formula (12) applied to the diffusion term. A bound for the external force term is straightforward:

$$\begin{aligned}
2\left\langle \tilde{\boldsymbol{f}}^n, \omega^{n+1}\right\rangle &\leq 2\left\|\tilde{\boldsymbol{f}}^n\right\|_2 \cdot \left\|\omega^{n+1}\right\|_2 \leq 2C_2 \left\|\tilde{\boldsymbol{f}}^n\right\|_2 \cdot \left\|\nabla_N \omega^{n+1}\right\|_2 \\
&\leq \frac{1}{24}\nu \left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{24C_2^2}{\nu}\left\|\tilde{\boldsymbol{f}}^n\right\|_2^2 \leq \frac{1}{24}\nu\left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{24C_2^2M^2}{\nu},
\end{aligned} \tag{45}$$

in which the Poincaré inequality

$$\left\|\omega^{n+1}\right\|_2 \le C_2 \left\|\nabla_N \omega^{n+1}\right\|_2 \tag{46}$$

was used in the third step. Since the diffusion coefficient at t^{n+1} dominates the other ones, the viscosity term can be analyzed as follows:

$$\begin{aligned} &\left\langle \nabla_N \left(\frac{4}{3}\omega^{n+1} + \frac{5}{6}\omega^{n-1} - \frac{1}{6}\omega^{n-3}\right), \nabla_N \omega^{n+1} \right\rangle \\ &= \frac{4}{3}\left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{5}{6}\left\langle \nabla_N \omega^{n-1}, \nabla_N \omega^{n+1}\right\rangle - \frac{1}{6}\left\langle \nabla_N \omega^{n-3}, \nabla_N \omega^{n+1}\right\rangle \\ &\ge \frac{5}{6}\left\|\nabla_N \omega^{n+1}\right\|_2^2 - \frac{5}{12}\left\|\nabla_N \omega^{n-1}\right\|_2^2 - \frac{1}{12}\left\|\nabla_N \omega^{n-3}\right\|_2^2. \end{aligned} \tag{47}$$

More details have to be involved in the treatment of the nonlinear terms. For the one at time step t^n, we start with the following rewritten form:

$$\begin{aligned} &-\Delta t \left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n), \omega^{n+1} \right\rangle \\ &= -\Delta t \left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} + \nabla_N \cdot \left(\boldsymbol{u}^n \omega^{n+1}\right), \omega^{n+1} \right\rangle \\ &\quad + \Delta t \left\langle \boldsymbol{u}^n \cdot \nabla_N (\omega^{n+1} - \omega^n) + \nabla_N \cdot \left(\boldsymbol{u}^n (\omega^{n+1} - \omega^n)\right), \omega^{n+1} \right\rangle. \end{aligned} \tag{48}$$

The first term disappears on the right-hand side, using a similar estimate as (31):

$$\begin{aligned} &\left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} + \nabla_N \cdot \left(\boldsymbol{u}^n \omega^{n+1}\right), \omega^{n+1} \right\rangle \\ &= \left\langle \omega^{n+1}, \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} \right\rangle - \left\langle \nabla_N \omega^{n+1}, \boldsymbol{u}^n \omega^{n+1} \right\rangle = 0. \end{aligned} \tag{49}$$

The summation by parts formula (12) can be applied to the second term:

$$\left\langle \boldsymbol{u}^n \cdot \nabla_N (\omega^{n+1} - \omega^n), \omega^{n+1} \right\rangle = -\left\langle \omega^{n+1} - \omega^n, \nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1}) \right\rangle, \tag{50}$$

$$\left\langle \nabla_N \cdot \left(\boldsymbol{u}^n (\omega^{n+1} - \omega^n)\right), \omega^{n+1} \right\rangle = -\left\langle \omega^{n+1} - \omega^n, \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} \right\rangle. \tag{51}$$

Furthermore, the term $\nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1})$ cannot be expanded as $\boldsymbol{u}^n \cdot \nabla_N \omega^{n+1}$, due to the aliasing error in the pseudospectral approximation, even though $\boldsymbol{u}^n$ is divergence-free at the discrete level (30). In the Fourier collocation space, the following expansion has to be applied:

$$\nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1}) = \mathcal{D}_{Nx}(u^n \omega^{n+1}) + \mathcal{D}_{Ny}(v^n \omega^{n+1}). \tag{52}$$

To obtain an estimate of these nonlinear expansions, we recall $\boldsymbol{U}^n = (U^n, V^n)$, $\boldsymbol{\omega}^{n+1}$, and $\boldsymbol{\psi}^{n+1}$ as the continuous projection of $\boldsymbol{u}^n$, ω^{n+1}, and ψ^{n+1}, respectively. Since $\boldsymbol{U}^n, \boldsymbol{\omega}^{n+1} \in P^N$, we have $\boldsymbol{U}^n \boldsymbol{\omega}^{n+1} \in P^{2N}$ and an application of Lemma 1 indicates that

$$\begin{aligned} \left\|\mathcal{D}_{Nx}(u^n \omega^{n+1})\right\|_2 &= \left\|\partial_x P_c^N (U^n \boldsymbol{\omega}^{n+1})\right\| \le 2\left\|\partial_x (U^n \boldsymbol{\omega}^{n+1})\right\|, \\ \left\|\mathcal{D}_{Ny}(v^n \omega^{n+1})\right\|_2 &= \left\|\partial_y P_c^N (V^n \boldsymbol{\omega}^{n+1})\right\| \le 2\left\|\partial_y (V^n \boldsymbol{\omega}^{n+1})\right\|. \end{aligned} \tag{53}$$

A detailed expansion in the continuous space and application of Hölder's inequality show that

$$\begin{aligned} \left\|\partial_x (U^n \boldsymbol{\omega}^{n+1})\right\| &= \left\|U_x^n \boldsymbol{\omega}^{n+1} + U^n \boldsymbol{\omega}_x^{n+1}\right\| \le \left\|U_x^n \boldsymbol{\omega}^{n+1}\right\| + \left\|U^n \boldsymbol{\omega}_x^{n+1}\right\| \\ &\le \left\|U_x^n\right\|_{L^{2/(1-\delta)}} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{L^{2/\delta}} + \left\|U^n\right\|_{L^\infty} \cdot \left\|\boldsymbol{\omega}_x^{n+1}\right\| \\ &\le C(\left\|U_x^n\right\|_{H^\delta} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^1} + \left\|U^n\right\|_{H^{1+\delta}} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|) \\ &\le C\left\|\boldsymbol{\omega}^n\right\|_{H^\delta} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|, \end{aligned} \tag{54}$$

in which the elliptic regularity (32) and Poincaré inequality have been repeatedly utilized in the derivation. Similar estimates can be derived for $\left\|\partial_y(V^n\boldsymbol{\omega}^{n+1})\right\|$. Going back to (52), we arrive at

$$\left\|\nabla_N\cdot(\boldsymbol{u}^n\omega^{n+1})\right\|_2 \le C\left\|\boldsymbol{\omega}^n\right\|_{H^\delta}\cdot\left\|\nabla\boldsymbol{\omega}^{n+1}\right\|_{L^2} \le C\left\|\boldsymbol{\omega}^n\right\|_{H^\delta}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2, \tag{55}$$

in which the second step is based on the fact that $\|\nabla\boldsymbol{\omega}^{n+1}\| = \|\nabla_N\omega^{n+1}\|_2$ (since $\boldsymbol{\omega}^{n+1}\in P^N$). In addition, the nonlinear term in (51) can be controlled in a similar way:

$$\left\|\boldsymbol{u}^n\cdot\nabla_N\omega^{n+1}\right\|_2 \le \left\|\boldsymbol{u}^n\right\|_\infty\cdot\left\|\nabla_N\omega^{n+1}\right\|_2 \le C\left\|\boldsymbol{\omega}^n\right\|_{H^\delta}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2. \tag{56}$$

Therefore, a substitution of (55)–(56) into (48), (49), (50)–(51) results in

$$\begin{aligned}
&-\frac{23}{12}\Delta t\left\langle \boldsymbol{u}^n\cdot\nabla_N\omega^n + \nabla_N\cdot(\boldsymbol{u}^n\omega^n), \omega^{n+1}\right\rangle\\
&\le C\Delta t\|\boldsymbol{\omega}^n\|_{H^\delta}\cdot\left\|\omega^{n+1}-\omega^n\right\|_2\cdot\left\|\nabla_N\omega^{n+1}\right\|_2\\
&\le C\tilde{C}_1\Delta t\left\|\omega^{n+1}-\omega^n\right\|_2\cdot\left\|\nabla_N\omega^{n+1}\right\|_2\\
&\le \frac{1}{24}\nu\Delta t\left\|\nabla_N\omega^{n+1}\right\|_2^2 + \frac{C_3\tilde{C}_1^2}{\nu}\Delta t\left\|\omega^{n+1}-\omega^n\right\|_2^2.
\end{aligned} \tag{57}$$

The nonlinear term at time step t^{n-1} could be treated in a similar fashion. We start from the equality

$$\begin{aligned}
&\Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1} + \nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n-1}\right), \omega^{n+1}\right\rangle\\
&= \Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n+1} + \nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n+1}\right), \omega^{n+1}\right\rangle\\
&\quad - \Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N(\omega^{n+1}-\omega^{n-1}) + \nabla_N\cdot\left(\boldsymbol{u}^{n-1}(\omega^{n+1}-\omega^{n-1})\right), \omega^{n+1}\right\rangle\\
&= -\Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N(\omega^{n+1}-\omega^{n-1}) + \nabla_N\cdot\left(\boldsymbol{u}^{n-1}(\omega^{n+1}-\omega^{n-1})\right), \omega^{n+1}\right\rangle,\\
&= \Delta t\left(\left\langle\omega^{n+1}-\omega^{n-1}, \nabla_N\cdot(\boldsymbol{u}^{n-1}\omega^{n+1})\right\rangle + \left\langle\omega^{n+1}-\omega^{n-1}, \boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n+1}\right\rangle\right)
\end{aligned} \tag{58}$$

with the summation by parts formula applied in the last step. Using similar derivations as in (52)–(56), we obtain the following estimates:

$$\begin{aligned}
&\left\|\nabla_N\cdot(\boldsymbol{u}^{n-1}\omega^{n+1})\right\|_2 \le C\left\|\boldsymbol{\omega}^{n-1}\right\|_{H^\delta}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2,\\
&\left\|\boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n+1}\right\|_2 \le C\left\|\boldsymbol{\omega}^{n-1}\right\|_{H^\delta}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2.
\end{aligned} \tag{59}$$

This in turn implies that

$$\begin{aligned}
&\frac{4}{3}\Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1} + \nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n-1}\right), \omega^{n+1}\right\rangle\\
&\le C\Delta t\|\boldsymbol{\omega}^{n-1}\|_{H^\delta}\cdot\left\|\omega^{n+1}-\omega^{n-1}\right\|_2\cdot\left\|\nabla_N\omega^{n+1}\right\|_2\\
&\le C\tilde{C}_1\Delta t\left\|\omega^{n+1}-\omega^{n-1}\right\|_2\cdot\left\|\nabla_N\omega^{n+1}\right\|_2\\
&\le \frac{1}{24}\nu\Delta t\left\|\nabla_N\omega^{n+1}\right\|_2^2 + \frac{C_4\tilde{C}_1^2}{\nu}\Delta t\left\|\omega^{n+1}-\omega^{n-1}\right\|_2^2\\
&\le \frac{1}{24}\nu\Delta t\left\|\nabla_N\omega^{n+1}\right\|_2^2 + \frac{2C_4\tilde{C}_1^2}{\nu}\Delta t\left(\left\|\omega^{n+1}-\omega^n\right\|_2^2 + \left\|\omega^n-\omega^{n-1}\right\|_2^2\right),
\end{aligned} \tag{60}$$

in which the H^δ a priori assumption (43) was applied in the second step and the last step comes from the following inequality:

$$\left\|\omega^{n+1}-\omega^{n-1}\right\|_2^2 \le 2\left(\left\|\omega^{n+1}-\omega^n\right\|_2^2 + \left\|\omega^n-\omega^{n-1}\right\|_2^2\right). \tag{61}$$

Similar bounds can be derived for the nonlinear convection term at time step t^{n-2}:

$$
\begin{aligned}
&\left\langle \boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2} + \nabla_N\cdot\left(\boldsymbol{u}^{n-2}\omega^{n-2}\right), \omega^{n+1}\right\rangle \\
&= \left(\left\langle \omega^{n+1}-\omega^{n-2}, \nabla_N\cdot(\boldsymbol{u}^{n-2}\omega^{n+1})\right\rangle + \left\langle \omega^{n+1}-\omega^{n-2}, \boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n+1}\right\rangle\right), \qquad (62)
\end{aligned}
$$

$$
\left\|\nabla_N\cdot(\boldsymbol{u}^{n-2}\omega^{n+1})\right\|_2, \left\|\boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n+1}\right\|_2 \le C\left\|\boldsymbol{\omega}^{n-2}\right\|_{H^\delta}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2, \qquad (63)
$$

$$
\begin{aligned}
&-\frac{5}{12}\Delta t\left\langle \boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2} + \nabla_N\cdot\left(\boldsymbol{u}^{n-2}\omega^{n-2}\right), \omega^{n+1}\right\rangle \\
&\le \frac{1}{24}\nu\Delta t\left\|\nabla_N\omega^{n+1}\right\|_2^2 + \frac{C_5\tilde{C}_1^2}{\nu}\Delta t\left\|\omega^{n+1}-\omega^{n-2}\right\|_2^2 \\
&\le \frac{1}{24}\nu\Delta t\left\|\nabla_N\omega^{n+1}\right\|_2^2 + \frac{3C_5\tilde{C}_1^2}{\nu}\Delta t\left(\left\|\omega^{n+1}-\omega^n\right\|_2^2 + \left\|\omega^n-\omega^{n-1}\right\|_2^2\right. \\
&\qquad\qquad\left. + \left\|\omega^{n-1}-\omega^{n-2}\right\|_2^2\right). \qquad (64)
\end{aligned}
$$

As a result, a substitution of (45), (47), (57), (60), and (64) into (44) leads to

$$
\begin{aligned}
&\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \left(1 - \frac{(C_3+2C_4+3C_5)\tilde{C}_1^2}{\nu}\Delta t\right)\|\omega^{n+1}-\omega^n\|_2^2 \\
&\quad - \frac{(2C_4+3C_5)\tilde{C}_1^2}{\nu}\Delta t\|\omega^n-\omega^{n-1}\|_2^2 - \frac{3C_5\tilde{C}_1^2}{\nu}\Delta t\|\omega^{n-1}-\omega^{n-2}\|_2^2 \\
&\quad + \frac{2}{3}\nu\Delta t\|\nabla_N\omega^{n+1}\|_2^2 \\
&\le \frac{24C_2^2M^2}{\nu}\Delta t + \frac{5}{12}\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2 + \frac{1}{12}\nu\Delta t\|\nabla_N\omega^{n-3}\|_2^2. \qquad (65)
\end{aligned}
$$

Under a constraint for the time step

$$
\frac{(C_3+2C_4+3C_5)\tilde{C}_1^2}{\nu}\Delta t \le \frac{1}{4}, \quad \text{i.e.,} \quad \Delta t \le \frac{\nu}{4(C_3+2C_4+3C_5)\tilde{C}_1^2}, \qquad (66)
$$

we arrive at

$$
\begin{aligned}
&\|\omega^{n+1}\|_2^2 + \frac{3}{4}\|\omega^{n+1}-\omega^n\|_2^2 + \frac{2}{3}\nu\Delta t\|\nabla_N\omega^{n+1}\|_2^2 \\
&\le \|\omega^n\|_2^2 + \frac{1}{4}\|\omega^n-\omega^{n-1}\|_2^2 + \frac{1}{4}\|\omega^{n-1}-\omega^{n-2}\|_2^2 \\
&\quad + \frac{5}{12}\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2 + \frac{1}{12}\nu\Delta t\|\nabla_N\omega^{n-3}\|_2^2 + C_6\Delta t \qquad (67)
\end{aligned}
$$

with $C_6 = \frac{24C_2^2M^2}{\nu}$. Next, we choose three positive values $\beta_0, \beta_1, \beta_2$, and one more value B_0, which satisfy

$$
\frac{1}{12} < \beta_2 < \beta_1, \quad \frac{5}{12} + \beta_1 < \beta_0 < \frac{2}{3}, \quad \frac{1}{4} < B_0 < \frac{1}{2}. \qquad (68)
$$

Since $\frac{1}{12} + \frac{5}{12} = \frac{1}{2} < \frac{2}{3}$, so condition (68) is possible. This in turn shows the key point of the unconditional stability analysis: the diffusion term coefficient at time step t^{n+1} dominates the sum of the other diffusion coefficients. Subsequently, an additional term $\beta_0\nu\Delta t\|\nabla_N\omega^n\|_2^2 + \beta_1\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2 + \beta_2\nu\Delta t\|\nabla_N\omega^{n-2}\|_2^2 + B_0\|\omega^n-\omega^{n-1}\|_2^2$ is added to both sides of (67):

(69)

$$\begin{aligned}
&\|\omega^{n+1}\|_2^2+\frac{3}{4}\|\omega^{n+1}-\omega^n\|_2^2+B_0\|\omega^n-\omega^{n-1}\|_2^2\\
&\quad+\frac{2}{3}\nu\Delta t\|\nabla_N\omega^{n+1}\|_2^2+\beta_0\nu\Delta t\|\nabla_N\omega^n\|_2^2+\beta_1\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2+\beta_2\nu\Delta t\|\nabla_N\omega^{n-2}\|_2^2\\
&\le\|\omega^n\|_2^2+\left(\frac{1}{4}+B_0\right)\|\omega^n-\omega^{n-1}\|_2^2+\frac{1}{4}\|\omega^{n-1}-\omega^{n-2}\|_2^2+\beta_0\nu\Delta t\|\nabla_N\omega^n\|_2^2\\
&\quad+\left(\frac{5}{12}+\beta_1\right)\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2+\beta_2\nu\Delta t\|\nabla_N\omega^{n-2}\|_2^2+\frac{1}{12}\nu\Delta t\|\nabla_N\omega^{n-3}\|_2^2+C_6\Delta t.
\end{aligned}$$

The following modified energy is defined:

$$\begin{aligned}
\tilde{E}_j^n=&\|\nabla_N^j\omega^n\|_2^2+\left(\frac{1}{4}+B_0\right)\|\nabla_N^j(\omega^n-\omega^{n-1})\|_2^2+\frac{1}{4}\|\nabla_N^j(\omega^{n-1}-\omega^{n-2})\|_2^2\\
&+\beta_0\nu\Delta t\|\nabla_N^{j+1}\omega^n)\|_2^2\\
&+\left(\frac{5}{12}+\beta_1\right)\nu\Delta t\|\nabla_N^{j+1}\omega^{n-1}\|_2^2+\beta_2\nu\Delta t\|\nabla_N^{j+1}\omega^{n-2}\|_2^2+\frac{1}{12}\nu\Delta t\|\nabla_N^{j+1}\omega^{n-3}\|_2^2.
\end{aligned}\tag{70}$$

Note that this definition is compatible with the initial energy quantity E_j^0 given by (39). Then we arrive at

$$\begin{aligned}
&\tilde{E}_0^{n+1}+\left(\frac{1}{2}-B_0\right)\|\omega^{n+1}-\omega^n\|_2^2+(B_0-\frac{1}{4})\|\omega^n-\omega^{n-1}\|_2^2\\
&+\left(\frac{2}{3}-\beta_0\right)\nu\Delta t\|\nabla_N\omega^{n+1}\|_2^2+(\beta_0-\frac{5}{12}-\beta_1)\nu\Delta t\|\nabla_N\omega^n\|_2^2\\
&+(\beta_1-\beta_2)\nu\Delta t\|\nabla_N\omega^{n-1}\|_2^2+(\beta_2-\frac{1}{12})\nu\Delta t\|\nabla_N\omega^{n-2}\|_2^2\le\tilde{E}_0^n+C_6\Delta t.
\end{aligned}\tag{71}$$

Furthermore, because of condition (68), there exists a constant C_7 with

$$\begin{aligned}
&C_7\nu\Delta t\left(\frac{1}{4}+B_0\right)\le\frac{1}{2}-B_0,\quad \frac{1}{4}C_7\nu\Delta t\le B_0-\frac{1}{4},\\
&\beta_0-\frac{5}{12}-\beta_1\ge C_7\left(\frac{5}{12}+\beta_1\right),\quad \beta_1-\beta_2\ge C_7\beta_2,\quad \beta_2-\frac{1}{12}\ge\frac{C_7}{12},\\
&\left(\frac{2}{3}-\beta_0\right)\|\nabla_N\omega^{n+1}\|_2^2\ge C_7\left(\|\omega^{n+1}\|_2^2+\beta_0\nu\Delta t\|\nabla_N\omega^{n+1}\|_2^2\right)
\end{aligned}\tag{72}$$

with the Poincaré inequality applied. Therefore, we arrive at

$$(1+C_7\nu\Delta t)\,\tilde{E}_0^{n+1}\le\tilde{E}_0^n+C_6\Delta t.\tag{73}$$

Applying an induction to the above inequality yields

(74)

$$\begin{aligned}
&\|\omega^{n+1}\|_2^2\le\tilde{E}_0^{n+1}\le(1+C_7\nu\Delta t)^{-(n+1)}\,\tilde{E}_0^0+\frac{C_6}{C_7\nu},\\
&\text{i.e.,}\quad \|\omega^{n+1}\|_2\le(1+C_7\nu\Delta t)^{-\frac{n+1}{2}}\,(\tilde{E}_0^0)^{1/2}+\sqrt{\frac{C_6}{C_7\nu}}\le C_8:=(\tilde{E}_0^0)^{1/2}+\sqrt{\frac{C_6}{C_7\nu}},
\end{aligned}$$

so that the leading L^2 estimate (37) is available, by taking $\gamma_0=C_7$, $Q^{(0)}=\sqrt{\frac{C_6}{C_7\nu}}$. Note that C_8 is a time-independent value; and also, this constant is independent on the a priori constant $\tilde{C}_1$ in (43).

Taking a summation (in time) of (71), we get the $\ell^2(0,T;H^1)$ bound for the numerical solution:

$$\nu\Delta t\sum_{k=1}^{N_k}\left\|\nabla_N\omega^k\right\|_2^2\le C(\tilde{E}_0^0+C_6\,T^*). \tag{75}$$

However, it is observed that the a priori estimate (74) is not sufficient to bound the H^δ norm (43) of the vorticity field. In turn, we perform a higher order energy estimate $\ell^\infty(0,T;H^1)\cap\ell^2(0,T;H^2)$ for the numerical solution of vorticity field.

4.2. $\boldsymbol{\ell^\infty(0,T;H^1)\cap\ell^2(0,T;H^2)}$ estimate for $\boldsymbol{\omega}$. Taking a discrete inner product with (24) by $-2\Delta_N\omega^{n+1}$ gives

$$\begin{aligned}
&\|\nabla_N\omega^{n+1}\|_2^2-\|\nabla_N\omega^n\|_2^2+\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\|_2^2\\
&\quad+\nu\Delta t\left\langle\Delta_N\left(\frac{4}{3}\omega^{n+1}+\frac{5}{6}\omega^{n-1}-\frac{1}{6}\omega^{n-3}\right),\Delta_N\omega^{n+1}\right\rangle\\
&=\frac{23}{12}\Delta t\left\langle\boldsymbol{u}^n\cdot\nabla_N\omega^n+\nabla_N\cdot\left(\boldsymbol{u}^n\omega^n\right),\Delta_N\omega^{n+1}\right\rangle\\
&\quad-\frac{4}{3}\Delta t\left\langle\boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1}+\nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n-1}\right),\Delta_N\omega^{n+1}\right\rangle\\
&\quad+\frac{5}{12}\Delta t\left\langle\boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2}+\nabla_N\cdot\left(\boldsymbol{u}^{n-2}\omega^{n-2}\right),\Delta_N\omega^{n+1}\right\rangle-2\Delta t\left\langle\tilde{\boldsymbol{f}}^n,\Delta_N\omega^{n+1}\right\rangle.
\end{aligned}\tag{76}$$

The external force term is bounded by the standard Cauchy inequality:

$$\begin{aligned}
-2\left\langle\boldsymbol{f}^n,\Delta_N\omega^{n+1}\right\rangle&\le\frac{1}{24}\nu\left\|\Delta_N\omega^{n+1}\right\|_2^2+\frac{24}{\nu}\left\|\boldsymbol{f}^n\right\|_2^2\\
&\le\frac{1}{24}\nu\left\|\Delta_N\omega^{n+1}\right\|_2^2+\frac{24M^2}{\nu}.
\end{aligned}\tag{77}$$

The diffusion term can be analyzed in the same way as (47):

$$\begin{aligned}
&\left\langle\Delta_N\left(\frac{4}{3}\omega^{n+1}+\frac{5}{6}\omega^{n-1}-\frac{1}{6}\omega^{n-3}\right),\Delta_N\omega^{n+1}\right\rangle\\
&\ge\frac{5}{6}\left\|\Delta_N\omega^{n+1}\right\|_2^2-\frac{5}{12}\left\|\Delta_N\omega^{n-1}\right\|_2^2-\frac{1}{12}\left\|\Delta_N\omega^{n-3}\right\|_2^2.
\end{aligned}\tag{78}$$

For the nonlinear term at t^n, we have the following decomposition:

$$\boldsymbol{u}^n\cdot\nabla_N\omega^n=-\boldsymbol{u}^n\cdot\nabla_N\left(\omega^{n+1}-\omega^n\right)-\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\right)\cdot\nabla_N\omega^{n+1}+\boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1},\tag{79}$$

$$\nabla_N\cdot\left(\boldsymbol{u}^n\omega^n\right)=\nabla_N\cdot\left(-\boldsymbol{u}^n(\omega^{n+1}-\omega^n)-(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\omega^{n+1}+\boldsymbol{u}^{n+1}\omega^{n+1}\right).\tag{80}$$

The following estimates have been derived in section 4.4, proof of Lemma 4.2, in the recent article [20]. We recall these estimates; the details are skipped for simplicity of presentation.

Lemma 2. *We have*

$$\begin{aligned}
&\frac{23}{12}\Delta t\left\langle-\boldsymbol{u}^n\cdot\nabla_N(\omega^{n+1}-\omega^n)-\nabla_N\cdot\left(\boldsymbol{u}^n(\omega^{n+1}-\omega^n)\right),\Delta_N\omega^{n+1}\right\rangle\\
&\le\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+\frac{C\tilde{C}_1^2}{\nu}\Delta t\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2,
\end{aligned}\tag{81}$$

$$\frac{23}{12}\Delta t\left\langle -\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n}\right)\cdot\nabla_N\omega^{n+1}-\nabla_N\cdot\left((\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n})\omega^{n+1}\right),\Delta_N\omega^{n+1}\right\rangle$$
$$\leq \Delta t\left\|\nabla_N\left(\omega^{n+1}-\omega^{n}\right)\right\|_2^2+\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+C_9\Delta t, \tag{82}$$
$$\frac{23}{12}\Delta t\left\langle \boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1}+\nabla_N\cdot\left(\boldsymbol{u}^{n+1}\omega^{n+1}\right),\Delta_N\omega^{n+1}\right\rangle$$
$$\leq \frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+C_{10}\Delta t \tag{83}$$

with C_9 and C_{10} only dependent on C_8, ν, δ and a few constants associated with the Poincaré inequality and 2D Sobolev embedding, independent on $\tilde{C}_1$ and the final time T.

These bounds in turn indicate that

$$\frac{23}{12}\Delta t\left\langle \boldsymbol{u}^{n}\cdot\nabla_N\omega^{n}+\nabla_N\cdot\left(\boldsymbol{u}^{n}\omega^{n}\right),\Delta_N\omega^{n+1}\right\rangle$$
$$\leq \frac{1}{24}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+\left(\frac{C\tilde{C}_1^2}{\nu}+1\right)\Delta t\left\|\nabla_N\left(\omega^{n+1}-\omega^{n}\right)\right\|_2^2+C_{11}\Delta t \tag{84}$$

with $C_{11}=C_9+C_{10}$.

The nonlinear term at time step t^{n-1} can be treated in a similar way. For simplicity of presentation, we only give the main estimates.

$$\begin{aligned}\boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1}&=-\boldsymbol{u}^{n-1}\cdot\nabla_N\left(\omega^{n+1}-\omega^{n-1}\right)\\&\quad-\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1}\right)\cdot\nabla_N\omega^{n+1}+\boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1},\end{aligned} \tag{85}$$
$$\begin{aligned}\nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n-1}\right)&=\nabla_N\cdot\left(-\boldsymbol{u}^{n-1}(\omega^{n+1}-\omega^{n-1})-(\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1})\omega^{n+1}\right.\\&\quad\left.+\boldsymbol{u}^{n+1}\omega^{n+1}\right),\end{aligned} \tag{86}$$

$$\begin{aligned}&\frac{4}{3}\Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N(\omega^{n+1}-\omega^{n-1})+\nabla_N\cdot\left(\boldsymbol{u}^{n-1}(\omega^{n+1}-\omega^{n-1})\right),\Delta_N\omega^{n+1}\right\rangle\\&\leq\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+\frac{C\tilde{C}_1^2}{\nu}\Delta t\left\|\nabla_N\left(\omega^{n+1}-\omega^{n-1}\right)\right\|_2^2\\&\leq\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+\frac{C\tilde{C}_1^2}{\nu}\Delta t\left(\left\|\nabla_N\left(\omega^{n+1}-\omega^{n}\right)\right\|_2^2\right.\\&\quad\left.+\left\|\nabla_N\left(\omega^{n}-\omega^{n-1}\right)\right\|_2^2\right),\end{aligned} \tag{87}$$
$$\begin{aligned}&\frac{4}{3}\Delta t\left\langle\left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1}\right)\cdot\nabla_N\omega^{n+1}+\nabla_N\cdot\left((\boldsymbol{u}^{n+1}-\boldsymbol{u}^{n-1})\omega^{n+1}\right),\Delta_N\omega^{n+1}\right\rangle\\&\leq\Delta t\left\|\nabla_N\left(\omega^{n+1}-\omega^{n-1}\right)\right\|_2^2+\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+C_{12}\Delta t\\&\leq 2\Delta t\left(\left\|\nabla_N\left(\omega^{n+1}-\omega^{n}\right)\right\|_2^2+\left\|\nabla_N\left(\omega^{n}-\omega^{n-1}\right)\right\|_2^2\right)\\&\quad+\frac{1}{72}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+C_{12}\Delta t.\end{aligned} \tag{88}$$

These bounds result in

$$\begin{aligned}&-\frac{4}{3}\Delta t\left\langle \boldsymbol{u}^{n-1}\cdot\nabla_N\omega^{n-1}+\nabla_N\cdot\left(\boldsymbol{u}^{n-1}\omega^{n-1}\right),\Delta_N\omega^{n+1}\right\rangle\\&\leq\frac{1}{24}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2+\left(\frac{C\tilde{C}_1^2}{\nu}+2\right)\Delta t\left(\left\|\nabla_N\left(\omega^{n+1}-\omega^{n}\right)\right\|_2^2\right.\\&\quad\left.+\left\|\nabla_N\left(\omega^{n}-\omega^{n-1}\right)\right\|_2^2\right)+C_{13}\Delta t\quad\text{with } C_{13}=\tfrac{1}{3}C_{10}+C_{12}\ .\end{aligned} \tag{89}$$

For the nonlinear term at time step t^{n-2}, the following estimate can be derived; we only state the results for the sake of brevity.

$$
\begin{aligned}
&\frac{5}{12}\Delta t \left\langle \boldsymbol{u}^{n-2}\cdot\nabla_N\omega^{n-2} + \nabla_N\cdot\left(\boldsymbol{u}^{n-2}\omega^{n-2}\right), \Delta_N\omega^{n+1}\right\rangle \\
&\le \left(\frac{C\tilde{C}_1^2}{\nu}+3\right)\Delta t \left(\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2 + \left\|\nabla_N\left(\omega^n-\omega^{n-1}\right)\right\|_2^2 \right. \\
&\quad \left. + \left\|\nabla_N\left(\omega^{n-1}-\omega^{n-2}\right)\right\|_2^2\right) + \frac{1}{24}\nu\Delta t\left\|\Delta_N\omega^{n+1}\right\|_2^2 + C_{14}\Delta t.
\end{aligned} \tag{90}
$$

Consequently, a substitution of (77)–(90) into (76) shows that

$$
\begin{aligned}
&\|\nabla_N\omega^{n+1}\|_2^2 - \|\nabla_N\omega^n\|_2^2 + \left(1-\left(\frac{C_{15}\tilde{C}_1^2}{\nu}+6\right)\Delta t\right)\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2 \\
&-\left(\frac{C_{16}\tilde{C}_1^2}{\nu}+5\right)\Delta t\left\|\nabla_N\left(\omega^n-\omega^{n-1}\right)\right\|_2^2 - \left(\frac{C_{17}\tilde{C}_1^2}{\nu}+3\right)\Delta t\left\|\nabla_N\left(\omega^{n-1}-\omega^{n-2}\right)\right\|_2^2 \\
&+\frac{2}{3}\nu\Delta t\|\Delta_N\omega^{n+1}\|_2^2 \le \left(\frac{24M^2}{\nu}+C_{11}+C_{13}+C_{14}\right)\Delta t \\
&+\frac{5}{12}\nu\Delta t\left\|\Delta_N\omega^{n-1}\right\|_2^2 + \frac{1}{12}\nu\Delta t\left\|\Delta_N\omega^{n-3}\right\|_2^2.
\end{aligned} \tag{91}
$$

Under a more restrictive constraint for the time step (compared to (66)),

$$
\left(\frac{C_{15}\tilde{C}_1^2}{\nu}+6\right)\Delta t \le \frac{1}{4}, \quad \left(\frac{C_{16}\tilde{C}_1^2}{\nu}+5\right)\Delta t \le \frac{1}{4}, \quad \left(\frac{C_{17}\tilde{C}_1^2}{\nu}+3\right)\Delta t \le \frac{1}{4}, \tag{92}
$$

we have

$$
\begin{aligned}
&\|\nabla_N\omega^{n+1}\|_2^2 + \frac{3}{4}\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\|_2^2 + \frac{2}{3}\nu\Delta t\|\Delta_N\omega^{n+1}\|_2^2 \\
&\le \|\nabla_N\omega^n\|_2^2 + \frac{1}{4}\|\nabla_N\left(\omega^n-\omega^{n-1}\right)\|_2^2 + \frac{1}{4}\|\nabla_N\left(\omega^{n-1}-\omega^{n-2}\right)\|_2^2 \\
&+\frac{5}{12}\nu\Delta t\|\Delta_N\omega^{n-1}\|_2^2 + \frac{1}{12}\nu\Delta t\|\Delta_N\omega^{n-3}\|_2^2 + C_{18}\Delta t
\end{aligned} \tag{93}
$$

with $C_{18} = \frac{24M^2}{\nu}+C_{11}+C_{13}+C_{14}$. Subsequently, the constants $\beta_0, \beta_1, \beta_2$, and B_0 can be chosen as in (68). Hence, an additional term $\beta_0\nu\Delta t\|\Delta_N\omega^n\|_2^2+\beta_1\nu\Delta t\|\Delta_N\omega^{n-1}\|_2^2+\beta_2\nu\Delta t\|\Delta_N\omega^{n-2}\|_2^2 + B_0\|\nabla_N(\omega^n-\omega^{n-1})\|_2^2$ is added to both sides of (93):

$$
\begin{aligned}
&\|\nabla_N\omega^{n+1}\|_2^2 + \frac{3}{4}\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\|_2^2 + B_0\|\nabla_N\left(\omega^n-\omega^{n-1}\right)\|_2^2 \\
&\quad+\frac{2}{3}\nu\Delta t\|\Delta_N\omega^{n+1}\|_2^2 + \beta_0\nu\Delta t\|\Delta_N\omega^n\|_2^2 + \beta_1\nu\Delta t\|\Delta_N\omega^{n-1}\|_2^2 + \beta_2\nu\Delta t\|\Delta_N\omega^{n-2}\|_2^2 \\
&\le \|\nabla_N\omega^n\|_2^2 + \left(\frac{1}{4}+B_0\right)\|\nabla_N\left(\omega^n-\omega^{n-1}\right)\|_2^2 + \frac{1}{4}\|\nabla_N\left(\omega^{n-1}-\omega^{n-2}\right)\|_2^2 \\
&\quad+\beta_0\nu\Delta t\|\Delta_N\omega^n\|_2^2 + \left(\frac{5}{12}+\beta_1\right)\nu\Delta t\|\Delta_N\omega^{n-1}\|_2^2 + \beta_2\nu\Delta t\|\Delta_N\omega^{n-2}\|_2^2 \\
&\quad+\frac{1}{12}\nu\Delta t\|\Delta_N\omega^{n-3}\|_2^2 + C_{18}\Delta t.
\end{aligned} \tag{94}
$$

Similar to the leading L^2 estimate, we recall a modified energy E_1^n as given by (70); as a result, (94) is equivalent to

$$\tilde{E}_1^{n+1} + \left(\frac{1}{2} - B_0\right) \|\nabla_N \left(\omega^{n+1} - \omega^n\right)\|_2^2 + \left(B_0 - \frac{1}{4}\right) \|\nabla_N \left(\omega^n - \omega^{n-1}\right)\|_2^2$$
$$+ \left(\frac{2}{3} - \beta_0\right) \nu\Delta t \|\Delta_N \omega^{n+1}\|_2^2 + \left(\beta_0 - \frac{5}{12} - \beta_1\right) \nu\Delta t \|\Delta_N \omega^n\|_2^2$$
$$+ (\beta_1 - \beta_2)\, \nu\Delta t \|\Delta_N \omega^{n-1}\|_2^2 + (\beta_2 - \frac{1}{12})\nu\Delta t \|\Delta_N \omega^{n-2}\|_2^2 \le \tilde{E}_1^n + C_{18}\Delta t. \tag{95}$$

Using a similar argument as (72), we could find a constant C_{19} to satisfy

$$C_{19}\nu\Delta t \left(\frac{1}{4} + B_0\right) \le \frac{1}{2} - B_0, \quad \frac{1}{4} C_{19}\nu\Delta t \le B_0 - \frac{1}{4},$$
$$\beta_0 - \frac{5}{12} - \beta_1 \ge C_{19}\left(\frac{5}{12} + \beta_1\right), \quad \beta_1 - \beta_2 \ge C_{19}\beta_2, \quad \beta_2 - \frac{1}{12} \ge \frac{C_{19}}{12},$$
$$\left(\frac{2}{3} - \beta_0\right) \|\Delta_N \omega^{n+1}\|_2^2 \ge C_{19}\left(\|\nabla_N \omega^{n+1}\|_2^2 + \beta_0 \nu\Delta t \|\Delta_N \omega^{n+1}\|_2^2\right). \tag{96}$$

That in turn shows that

$$(1 + C_{19}\nu\Delta t)\, \tilde{E}_1^{n+1} \le \tilde{E}_1^n + C_{18}\Delta t. \tag{97}$$

Applying an induction to the above inequality yields

$$\|\nabla_N \omega^{n+1}\|_2^2 \le \tilde{E}_1^{n+1} \le (1 + C_{19}\nu\Delta t)^{-(n+1)}\, \tilde{E}_1^0 + \frac{C_{18}}{C_{19}\nu},$$
$$\text{i.e.,} \quad \|\nabla_N \omega^{n+1}\|_2 \le (1 + C_{19}\nu\Delta t)^{-\frac{n+1}{2}}\, (\tilde{E}_1^0)^{1/2} + \sqrt{\frac{C_{18}}{C_{19}\nu}} \le C_{20} \tag{98}$$
$$:= (\tilde{E}_1^0)^{1/2} + \sqrt{\frac{C_{18}}{C_{19}\nu}},$$

so that the H^1 estimate (38) is available, by taking $\gamma_1 = C_{19}$, $Q^{(1)} = \sqrt{\frac{C_{18}}{C_{19}\nu}}$. Again, C_{20} is a constant independent on time and $\tilde{C}_1$.

In addition, we also have the discrete $\ell^2(0,T;H^2)$ bound for the numerical solution:

$$\nu\Delta t \sum_{k=1}^{N_k} \left\|\Delta_N \omega^k\right\|_2^2 \le C(\tilde{E}_1^0 + C_{18}\, T^*). \tag{99}$$

4.3. Recovery of the a priori $\boldsymbol{H^\delta}$ assumption (43). With the $\ell^\infty(0,T;L^2)$ and $\ell^\infty(0,T;H^1)$ estimate for the numerical vorticity solution, namely, (74) and (98), we are able to recover the H^δ assumption (43):

$$\begin{aligned} \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^\delta} &\le C\left\|\boldsymbol{\omega}^{n+1}\right\|^{1-\delta} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^1}^{\delta} \\ &\le C_\delta \left\|\boldsymbol{\omega}^{n+1}\right\|^{1-\delta} \left\|\nabla\boldsymbol{\omega}^{n+1}\right\|^{\delta} \le C_\delta C_8^{1-\delta} C_{20}^{\delta}. \end{aligned} \tag{100}$$

For simplicity, by taking $\delta = \frac{1}{2}$, we see that (43) is also valid at time step t^{n+1} if we set

$$\tilde{C}_1 = C_\delta \sqrt{C_8 C_{20}}. \tag{101}$$

Note that C_8 and C_{20} are independent of $\tilde{C}_1$ in the derivation. The constant $\tilde{C}_1$ is only used in the time step constraints (66) and (92). Therefore, an induction can be applied so that the a priori H^δ assumption (43) is valid at any time step under a global time step constraint

$$\Delta t \le \frac{\nu}{C_w(C_\delta^2 C_8 C_{20} + 1)} \tag{102}$$

by taking $C_w = \max(4(C_3 + 2C_4 + 3C_5), 4C_{16} + 1, 4C_{17} + 1, 4C_{18} + 1)$.

Therefore, we could choose $M_0 = \tilde{C}_1 C_\delta \sqrt{C_8 C_{20}}$ so that constraint (35) is equivalent to (102). Under this constraint for the time step, the proposed third order fully discrete scheme (24)–(26) is unconditionally stable (in terms of spatial grid size and final time); the asymptotic decay estimates (37), (38), for the L^2 and H^1 norms of the vorticity, can be derived. The first part of Theorem 2 has been proved.

4.4. $\ell^\infty(0,T;H^m)$ estimate for ω. Moreover, the $\ell^\infty(0,T;H^m)$ estimate (42) for the vorticity could be derived in the same manner. For simplicity, we focus our attention on the case $m = 2$; the analysis for higher values of m could be carried out in the same fashion.

For example, by taking a discrete inner product with (24) by $2\Delta_N^2\omega^{n+1}$, we are able to derive the following estimate (with the modified energy E_2^k given by (70)):

$$\begin{aligned}
&\|\Delta_N\omega^{n+1}\|_2^2 \le \tilde{E}_2^{n+1} \le (1 + C_{21}\nu\Delta t)^{-(n+1)}\, \tilde{E}_2^0 + \frac{C_{22}}{C_{21}\nu},\\
&\text{i.e.,}\quad \|\Delta_N\omega^{n+1}\|_2 \le (1 + C_{21}\nu\Delta t)^{-\frac{n+1}{2}}\, (\tilde{E}_2^0)^{1/2} + \sqrt{\frac{C_{22}}{C_{21}\nu}} \le C_{23}\\
&\qquad := (\tilde{E}_2^0)^{1/2} + \sqrt{\frac{C_{22}}{C_{21}\nu}},
\end{aligned} \tag{103}$$

under an additional constraint for the time step (in addition to (66), (92)):

$$\frac{C_{24}C_{20}^2}{\nu}\Delta t \le \frac{1}{4}, \quad \frac{C_{25}C_{20}^2}{\nu}\Delta t \le \frac{1}{4}. \tag{104}$$

The proof of (103) follows similar structures as in sections 4.1 and 4.2; the details are skipped for the sake of brevity. Therefore, the H^2 estimate (42) (with $m = 2$) is available by taking $\gamma_2 = C_{21}$, $Q^{(2)} = \sqrt{\frac{C_{22}}{C_{21}\nu}}$. It is obvious that C_{23} is a time-independent constant.

It is obvious that we could set $C_w = 4\max(C_{24}, C_{25})$, and $M_0^{(2)} = C_{20}$ in (40), and $M_1^{(2)} = C_{28}$ in (41). The proof of Theorem 2 is completed.

Remark 3. It is observed that the stability condition (23), as reported in [21], plays a crucial role in the long time stability analysis, since it ensures that the diffusion coefficient at time step t^{n+1} dominates the rest.

Remark 4. With a uniform in time H^m bound derived for the numerical solution (24)–(26), a statistical convergence is expected to be available, using similar techniques presented in [20, 46]. The details are left to future works.

5. Numerical results. We present a numerical test to verify the theoretical analysis in this article, including both the local in time convergence and the long time stability for the third and fourth order numerical solutions.

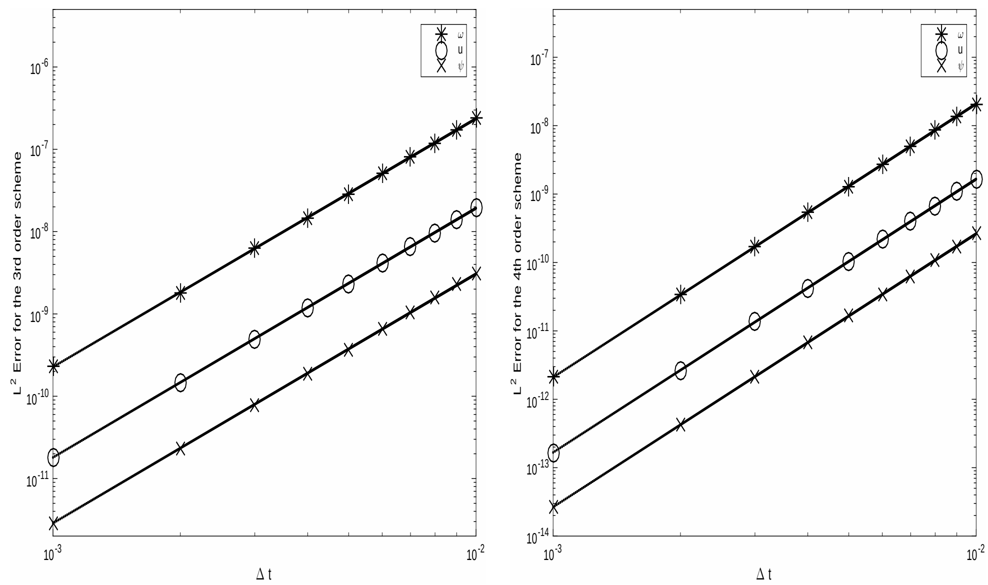

FIG. 1. *Discrete $\|\cdot\|_2$ numerical errors at $T = 6$ plotted versus Δt, for the vorticity, velocity, and stream function variables. The kinematic viscosity parameter is taken to be $\nu = 0.5$. Left: Log-log error plot for the third order numerical scheme (24)–(26); the data lie roughly on curves $C\Delta t^{a_3}$, with $a_3 = 3.0282$, for appropriate choices of C, confirming the third order accuracy. Right: The error plot for the fourth order numerical scheme (27)–(29); the data lie roughly on curves $C\Delta t^{a_4}$, with $a_4 = 3.9956$, which confirms the fourth order accuracy.*

5.1. Local in time convergence. In this subsection we perform a numerical accuracy check for the proposed multistep numerical schemes, including the third order one (24)–(26) and the fourth order one (27)–(29). The 2D computational domain is set to be $\Omega = (0, 1)^2$, and the exact profile for the fluid flow is given by

(105)

$$\psi_e(x, y, t) = \frac{1}{2\pi^2} \sin(2\pi x) \sin(2\pi y) \cos t, \quad \omega_e(x, y, t) = -4 \sin(2\pi x) \sin(2\pi y) \cos t,$$
$$u_e(x, y, t) = -\frac{1}{\pi} \sin(2\pi x) \cos(2\pi y) \cos t, \quad v_e(x, y, t) = \frac{1}{\pi} \cos(2\pi x) \sin(2\pi y) \cos t.$$

To make $(\omega_e, \boldsymbol{u}_e, \psi_e)$ satisfy the original PDE (1)–(3), we have to add an artificial, time-dependent force term, which could be viewed as $\boldsymbol{f}$. For either the third or fourth order multistep scheme, only two FFT-based Poisson solvers are needed at each time step. We fix the spatial resolution as $N = 256$ and compute solutions with a sequence of time step sizes, from $\Delta t = 0.001$ to $\Delta t = 0.01$, with an increment 0.001, and the numerical errors are reported at the final time $T = 6$. The kinematic viscosity parameter is taken to be $\nu = 0.5$. Figure 1 shows the discrete $\|\cdot\|_2$ norms of the errors between the numerical and exact solutions for the vorticity, velocity, and stream function variables. With a choice of $N = 256$, the spatial error, which comes from the Fourier pseudospectral approximation, is negligible, so that the numerical errors are dominated by the temporal discretization. Clear third and fourth order accuracy is observed in all cases, and a strong verification of Theorem 1 is given by these results.

5.2. Long time numerical stability. In this subsection we present long time numerical simulation results. For the 2D flow with the same force term $\boldsymbol{f}$ in section 5.1

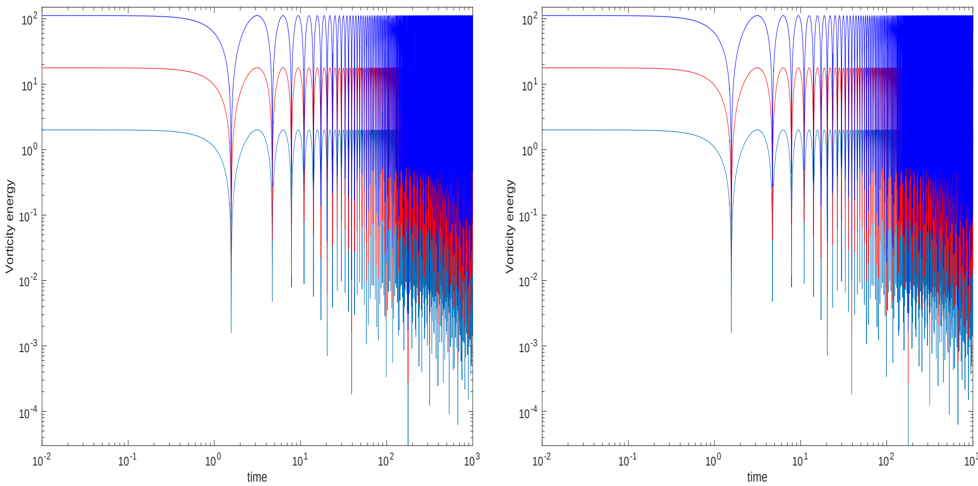

FIG. 2. *Time evolution of the vorticity profile up to* $T = 1000$. *The gray, red, and blue lines represent the plot for* $\|\omega(t)\|_2$, $\|\nabla_N \omega(t)\|_2$, *and* $\|\Delta_N \omega(t)\|_2$, *respectively. Left: Log-log time evolution for the third order numerical scheme* (24)–(26). *Right: The time evolution for the fourth order numerical scheme* (27)–(29).

(determined by the fluid profiles (105)), we perform the computation up to $T = 1000$. For such a time scale, the local in time convergence result (34) could hardly provide us any theoretical insight, since the convergence constant grows exponentially in terms of the final time T. Instead, we examine the time evolution of the vorticity profile by recording its discrete L^2, H^1, and H^2 norms, namely, $\|\omega(t)\|_2$, $\|\nabla_N \omega(t)\|_2$, and $\|\Delta_N \omega(t)\|_2$, respectively. These time evolution plots, for both the third order scheme (24)–(26) and the fourth order one (27)–(29), are presented in Figure 2. These two log-log plots are almost identical. A clear observation shows that, for both high order multistep schemes, all three energy norms are globally in time bounded. Also, these global in time bounds are expected to be valid for even a larger time scale. This numerical result provides strong evidence of the long time numerical stability analysis given by Theorem 2.

6. Conclusions. In this paper, we propose a few multistep numerical schemes for the 2D incompressible Navier–Stokes equations, up to the fourth order temporal accuracy. In addition, we provide a long time numerical stability analysis for the proposed schemes, combined with Fourier pseudospectral spatial approximation; uniform in time bounds for these high order schemes, in both L^2 and H^m (for $m \geq 1$) norms, are derived. The numerical experiments have also verified such a long time stability.

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