DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **21**, Number **6**, August **2016**

doi:10.3934/dcdsb.2016018

pp. **1689–1711**

GLOBAL-IN-TIME GEVREY REGULARITY SOLUTION FOR A CLASS OF BISTABLE GRADIENT FLOWS

Nan Chen

Department of Mathematics and Center for Atmosphere Ocean Science
Courant Institute of Mathematical Sciences
New York University, New York, NY 10012, USA

Cheng Wang*

Mathematics Department, University of Massachusetts
North Dartmouth, MA 02747, USA

Steven Wise

Mathematics Department, University of Tennessee
Knoxville, TN 37996, USA

(Communicated by Shouhong Wang)

Abstract. In this paper, we prove the existence and uniqueness of a Gevrey regularity solution for a class of nonlinear bistable gradient flows, where with the energy may be decomposed into purely convex and concave parts. Example equations include certain epitaxial thin film growth models and phase field crystal models. The energy dissipation law implies a bound in the leading Sobolev norm. The polynomial structure of the nonlinear terms in the chemical potential enables us to derive a local-in-time solution with Gevrey regularity, with the existence time interval length dependent on a certain H^m norm of the initial data. A detailed Sobolev estimate for the gradient equations results in a uniform-in-time-bound of that H^m norm, which in turn establishes the existence of a global-in-time solution with Gevrey regularity.

1. **Introduction.**

Suppose $\ell \in \mathbb{N}+1$, $\wp \in 2\mathbb{N}+4$, and $s \in \{0,1\}$. (We use the notation $\mathbb{N} := \{0,1,2,3,\cdots\}$ and $\mathbb{Z} = \{\cdots,-2,-1,0,1,2,\cdots\}$.) Suppose $\Omega = (0,1)^d$, with $d \in \mathbb{N}+1$. We consider the following bistable energy: for all $\phi : \mathbb{R}^d \to \mathbb{R}$ that are Ω–periodic and sufficiently regular, define

$$E(\phi) := \sum_{j=2}^{\wp/2} \frac{c_{2j}}{2j} \|\nabla\phi\|_{2j}^{2j} + \frac{1}{2}\sum_{j=1}^{\ell} a_j \left\|\nabla^{j-s}\phi\right\|_2^2, \tag{1.1}$$

where $c_\wp = 1$, $a_\ell := \varepsilon^2 > 0$, and otherwise $c_j, a_j \in \mathbb{R}$. We point out that ε is usually a small parameter. But, for the discussion herein, we will not pursue ε dependences in our estimates. Herein, $\|\cdot\|_p$ stands for the L^p norm, with $p \geq 1$. Furthermore,

2010 *Mathematics Subject Classification.* 35G25, 35K30.
Key words and phrases. Bistable gradient flow, Gevrey regularity solution, global-in-time existence.
*Corresponding author.

$\|\nabla\phi\|_p := \| \, |\nabla\phi| \, \|_p \neq |\phi|_{1,p}$, where $|\nabla\phi| := \sqrt{\nabla\phi\cdot\nabla\phi}$ and $|\cdot|_{1,p}$ is the standard semi-norm on $W^{1,p}(\Omega)$. We use the notation

$$\nabla^0\phi := \phi,\ \nabla^2\phi := \Delta\phi,\ \nabla^3\phi := \nabla\Delta\phi,\ \nabla^4\phi := \Delta^2\phi,\ \nabla^5\phi := \nabla\Delta^2\phi,\ \cdots . \tag{1.2}$$

For $k \in 2\mathbb{N}+1$, our notation is

$$\left\|\nabla^k\phi\right\|_2^2 = \int_\Omega \nabla(\Delta^{(k-1)/2}\phi)\cdot\nabla(\Delta^{(k-1)/2}\phi)\,d\mathbf{x} = \int_\Omega \left|\nabla(\Delta^{(k-1)/2}\phi)\right|^2 d\mathbf{x}.$$

The energy E is termed bistable because it can be clearly written as the difference of purely convex energies, according to the signs of the coefficients. Of course, if all of the coefficients are positive, the energy is itself purely convex, with every term being convex. Observe that it is always true that the leading energy terms – consisting of the non-quadratic part, $\frac{c_\wp}{\wp}\|\nabla\phi\|_\wp^\wp$, and the quadratic part, $\frac{a_\ell}{2}\left\|\nabla^{\ell-s}\phi\right\|_2^2$ – are purely positive and convex. This fact will play a key role in our analyses. The variational derivative of the energy may be (formally) calculated as

$$\begin{aligned}\delta_\phi E = & -\nabla\cdot\left(|\nabla\phi|^{\wp-2} + c_{\wp-2}|\nabla\phi|^{\wp-4} + \cdots + c_4|\nabla\phi|^2\right)\nabla\phi \\ & + (-1)^{1-s}a_1\Delta^{1-s}\phi + (-1)^{2-s}a_2\Delta^{2-s}\phi + \cdots \\ & + (-1)^{\ell-1-s}a_{\ell-1}\Delta^{\ell-1-s}\phi + (-1)^{\ell-s}\varepsilon^2\Delta^{\ell-s}\phi,\end{aligned} \tag{1.3}$$

utilizing periodic boundary conditions. Observe that the terms of the form $-\nabla\cdot\left(c_p\,|\nabla\phi|^{p-2}\,\nabla\phi\right)$ are nonlinear p-laplacian operators, where $p\geq 4$ is an even number.

Our principal aim in this paper is to establish the Gevrey regularity of solutions for the following family of nonlinear gradient flow evolution equations:

$$\partial_t\phi + (-\Delta)^s\mu = 0, \quad \mu := \delta_\phi E \quad \text{on } \Omega_T := \Omega\times(0,T), \tag{1.4}$$

where ϕ is Ω–periodic in space, and $s=0$ or $s=1$. Equation (1.4) is the L^2 gradient flow (for $s=0$) and the H^{-1} gradient flow (for $s=1$) with respect to E in (1.1). The rates of energy dissipation along the solution trajectories are

$$d_tE = -\left\|\nabla^s\mu\right\|_2^2, \tag{1.5}$$

and the mass of the solution is a conserved quantity, meaning $d_t\int_\Omega\phi(\mathbf{x},t)d\mathbf{x} = 0$, for all $t\geq 0$. It is often useful to consider the model in the following, less compact form:

$$\begin{aligned}\partial_t\phi = (-\Delta)^s\left\{\nabla\cdot\left(\left\{|\nabla\phi|^{\wp-2} + c_{\wp-2}|\nabla\phi|^{\wp-4} + \cdots + c_4|\nabla\phi|^2\right\}\nabla\phi\right)\right\} \\ + a_1\Delta\phi - a_2\Delta^2\phi + a_3\Delta^3\phi + \cdots + (-1)^\ell a_{\ell-1}\Delta^{\ell-1}\phi + (-1)^{\ell+1}\varepsilon^2\Delta^\ell\phi.\end{aligned} \tag{1.6}$$

The evolution equation is thus a nonlinear "parabolic" equation of order 2ℓ in purely divergence form, and, considering the periodic boundary conditions, the mass conservation is assured.

There are a few special cases of great physical interest that we wish to point out. The first is the epitaxial thin film model with slope selection, also known as the regularized Cross-Newell equation [8, 14]. This equation can be obtained setting $s=0$, $\ell=2$, $\wp=4$, $a_1=-1$:

$$\partial_t\phi = \nabla\cdot\left(|\nabla\phi|^2\nabla\phi\right) - \Delta\phi - \varepsilon^2\Delta^2\phi, \quad E_{\rm ss}(\phi) = \frac{1}{4}\|\nabla\phi\|_4^4 - \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{\varepsilon^2}{2}\|\Delta\phi\|_2^2. \tag{1.7}$$

It has been used as a model for thin film roughening and coarsening [19, 20, 21, 28, 29, 30, 31, 34, 35]. Some numerical works for the equation can be found in more recent articles [6, 7, 39, 42, 44].

The second is the square phase field crystal (SPFC) model, which is obtained by setting $s = 1$, $\ell = 3$, $\wp = 4$, $a_2 = -2\delta^2$:

$$\begin{aligned} \partial_t \phi &= -\Delta(\nabla \cdot (|\nabla \phi|^2 \nabla \phi)) + a_1 \Delta \phi + 2\delta^2 \Delta^2 \phi + \varepsilon^2 \Delta^3 \phi, \\ E_{\mathrm{spfc}}(\phi) &= \frac{1}{4}\|\nabla \phi\|_4^4 + \frac{a_1}{2}\|\phi\|_2^2 - \delta^2 \|\nabla \phi\|_2^2 + \frac{\varepsilon^2}{2}\|\Delta \phi\|_2^2. \end{aligned} \tag{1.8}$$

The SPFC equation is related to another crystal growth model known as the phase field crystal (PFC) equation [10, 11, 37, 41], which is the gradient flow

$$\begin{aligned} \partial_t \phi &= \Delta \left(\delta_\phi E_{\mathrm{pfc}}\right) = \Delta(\phi^3) + a_1 \Delta \phi + 2\delta^2 \Delta^2 \phi + \varepsilon^2 \Delta^3 \phi = 0, \\ E_{\mathrm{pfc}}(\phi) &= \frac{1}{4}\|\phi\|_4^4 + \frac{a_1}{2}\|\phi\|_2^2 - \delta^2 \|\nabla \phi\|_2^2 + \frac{\varepsilon^2}{2}\|\Delta \phi\|_2^2. \end{aligned} \tag{1.9}$$

The PFC model was proposed in [10] for simulating crystal dynamics at the atomic scale in space but on diffusive scales in time, with natural incorporation of elastic and plastic deformations, multiple crystal orientations and defects. The natural lattice for a crystal described by the PFC equation is hexagonal in 2D. The SPFC model, on the other hand, predicts a "square" symmetry crystal lattice in 2D rather than the usual hexagonal structure; see the related references [11, 15, 43]. While the standard PFC model (1.9) is not covered by the following analysis – because the form of the energy is different from and, in fact, somewhat simpler than what is considered in (1.1) – our results can be easily extended for (1.9).

There have been many existing works to establish the existence of Gevrey regularity solutions for time-dependent nonlinear PDEs, such as [3, 13] for 2-D and 3-D incompressible Navier-Stokes equation, [2] for Kuramoto-Sivashinsky equation, [5, 12] for certain nonlinear parabolic equations, [18] for the 3-D Navier-Stokes-Voight equation, [33] for models porous media flow, to mention a few. For gradient flow-type models, Gevrey regularity solutions have been proven by [36] for the Cahn-Hilliard equation with dimension $d = 1$ to $d = 5$. A more recent work [40] gives a further analysis with potentially rough initial data. In addition, a few related works for the Cahn-Hilliard model combined with certain fluid motion equation have also been reported, such as [9] for the convective Cahn-Hilliard equation, and [32] for the Cahn-Hilliard-Hele-Shaw model. Other than the Gevrey regularity solutions, a more general class of analytic solutions for different models of incompressible fluid have been discussed in [4, 16, 22, 23, 24, 25, 26, 27], etc.

A general framework to establish the existence of local-in-time Gevrey regularity solutions for nonlinear parabolic equations

$$\partial_t \phi - \nu \Delta \phi + G(\phi, \nabla \phi) = 0,$$

with periodic boundary conditions in $\mathbb{R}^n$, has been addressed in [5, 12]. The analyses therein apply when the growth of $F(r, \vec{s}) := G(r, \vec{s}) - r$, in either the r or the $\vec{s}$ variable, is bounded by a polynomial, and F is assumed to be real analytic in both variables such that it possesses a majorant. In any case, it is clear that the analyses in [5, 12] will not cover equation (1.1) considered in this article. The reason is that the p laplacian terms of the form $\nabla \cdot (|\nabla \phi|^p \nabla \phi)$, $p \in 2\mathbb{N} + 2$ involve first and second order derivatives combined in a highly nonlinear way, and these terms cannot be recast in the form of $F(\phi, \nabla \phi)$.

While there has been some existing work considering Gevrey regularity of solutions for gradient flows with respect to the Cahn-Hilliard-type energy, no work has been undertaken to study gradient flows with respect to (1.1). For the nonlinear gradient flow considered here, (1.4), which covers a large class of models, the most current result to our knowledge is the proof of a smooth solution for the epitaxial thin film growth model (1.7), as reported by [30]: given any H^m initial data (with $m \geq 2$), there is a unique solution with a uniform-in-time H^m estimate.

In this paper, we provide an analysis of a global-in-time Gevrey regularity solution for the general gradient flow given by (1.4) with respect to the energy (1.1). The paper is organized as follows. In Section 2 we go over some basic notation. In Section 3 we construct an approximate solution to the PDE using the standard Galerkin procedure and give the leading order energy estimate. In Section 4 we prove the existence and uniqueness of a local-in-time Gevrey regularity solution for (1.4), with the existence time interval length dependent upon $\left\| (-\Delta)^{\ell/2} \phi_0 \right\|_2$. Finally, a uniform in time H^ℓ bound $\left\| (-\Delta)^{\ell/2} \phi(t) \right\|_2$ is presented in Section 5, so that a global-in-time Gevrey regularity solution may be established.

2. **Notation and preliminaries.** We use the standard symbols for Lebesgue and Sobolev spaces of complex-valued functions and their norms. To begin, for $u, v \in L^2(\Omega, \mathbb{C}) = L^2(\Omega)$, we set $(u, v) := \int_\Omega u(\vec{x}) v^*(\vec{x})\, d\vec{x}$, where $z^* = a - ib$ is the complex conjugate of $z = a + ib$. Let us also define the following function spaces:

$$\mathring{L}^2(\Omega) := \left\{ u \in L^2(\Omega) \;\middle|\; (u, 1) = 0 \right\},$$

$$C^m_{\text{per}}(\Omega) := \left\{ u \in C^m(\mathbb{R}^d) \;\middle|\; u \text{ is } \Omega\text{-periodic} \right\}, \qquad \mathring{C}^m_{\text{per}}(\Omega) := C^m_{\text{per}}(\Omega) \cap \mathring{L}^2(\Omega),$$

$$W^{m,p}_{\text{per}}(\Omega) := \left\{ u \in W^{m,p}_{\text{loc}}(\mathbb{R}^d) \;\middle|\; u \text{ is } \Omega\text{-periodic} \right\}, \quad \mathring{W}^{m,p}_{\text{per}}(\Omega) := W^{m,p}_{\text{per}}(\Omega) \cap \mathring{L}^2(\Omega),$$

$$H^m_{\text{per}}(\Omega) := W^{m,2}_{\text{per}}(\Omega), \qquad \mathring{H}^m_{\text{per}}(\Omega) := \mathring{W}^{m,2}_{\text{per}}(\Omega),$$

$$H^{-m}_{\text{per}}(\Omega) := \left(H^m_{\text{per}}(\Omega) \right)^*,$$

$$\mathring{H}^{-m}_{\text{per}}(\Omega) := \left\{ v \in H^{-m}_{\text{per}}(\Omega) \,\middle|\, \langle v, 1 \rangle = 0 \right\},$$

where $\langle \cdot\,, \cdot \rangle$ is the duality pairing between H^{-m}_{per} and H^m_{per}. Specifically, for $v \in H^{-m}_{\text{per}}(\Omega)$,

$$\left\langle v, \sum_{k=1}^n \alpha_k u_k \right\rangle := \sum_{k=1}^n \alpha_k^* v(u_k^*) = \sum_{k=1}^n \alpha_k^* \langle v, u_k \rangle.$$

We denote the standard semi-norm and norm on $W^{m,p}(\Omega)$ by $|\cdot|_{m,p,\Omega} = |\cdot|_{m,p}$ and $\|\cdot\|_{m,p,\Omega} = \|\cdot\|_{m,p}$, respectively, dropping the subscript m whenever $m = 0$. Since the domain $\Omega = (0,1)^d$ is understood in our discussion, we usually also drop the subscript Ω in referencing the (semi-)norms.

Define the operator A to be $-\Delta$ paired with Ω–periodic boundary conditions. We define the range of A as $R(A) := \mathring{L}^2(\Omega)$. The domain of A is simply $D(A) = \mathring{H}^2_{\text{per}}(\Omega)$, and $A : D(A) \to R(A)$ is a positive, self-adjoint linear operator that admits a compact inverse. The eigenfunctions of A may be chosen as $\Phi_{\vec{\alpha}}(\vec{x}) = \exp(2\pi i\, \vec{\alpha} \cdot \vec{x}) \in \mathring{C}^\infty_{\text{per}}(\Omega)$, for all $\vec{\alpha} \in \mathbb{Z}^d \setminus \left\{ \vec{0} \right\} =: \mathbb{Z}^d_\star$, in which case the eigenvalues are $\lambda_{\vec{\alpha}} = (2\pi)^2 |\vec{\alpha}|^2 > 0$. Set $\mathring{\mathcal{B}} := \left\{ \Phi_{\vec{\alpha}} \;\middle|\; \vec{\alpha} \in \mathbb{Z}^d_\star \right\}$; this is an orthonormal basis for $\mathring{L}^2(\Omega)$. We can increase $\mathring{\mathcal{B}}$ so the resulting set is an orthonormal basis for all of $L^2(\Omega)$; in particular, $\mathcal{B} := \mathring{\mathcal{B}} \cup \left\{ \Phi_{\vec{0}} \equiv 1 \right\}$ serves this purpose.

Since A is symmetric and positive, we can define the following Hilbert spaces: for any $s \geq 0$, set

$$D(A^s) := \Big\{ u \in \mathring{L}^2(\Omega) \;\Big|\; \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} (2\pi)^{4s} |\vec{\alpha}|^{4s} |\hat{u}_{\vec{\alpha}}|^2 < \infty, \Big\},$$

$$(u,v)_{D(A^s)} := \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} (2\pi)^{4s} |\vec{\alpha}|^{4s} \hat{u}_{\vec{\alpha}}\, \hat{v}^*_{\vec{\alpha}}, \tag{2.1}$$

where $\hat{u}_{\vec{\alpha}} := (u, \Phi_{\vec{\alpha}}) = \int_\Omega u(\vec{x}) \mathrm{e}^{-2\pi i\, \vec{\alpha}\cdot\vec{x}} d\vec{x}$ are the Fourier coefficients. For $u \in D(A^s)$, we define

$$A^s u := \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} (2\pi)^{2s} |\vec{\alpha}|^{2s} \hat{u}_{\vec{\alpha}} \Phi_{\vec{\alpha}}. \tag{2.2}$$

Then, of course, $(u,v)_{D(A^s)} = (A^s u, A^s v)$ and $\|u\|_{D(A^s)} = \|A^s u\|_2$, and it is not difficult to show that, in general, $D(A^s) = \mathring{H}^{2s}_{\mathrm{per}}(\Omega)$. It is possible to define the exponential operator $\exp(\tau A^s) = \mathrm{e}^{\tau A^s}$, for any $\tau, s \geq 0$. To do so we introduce the Hilbert space

$$D\big(\mathrm{e}^{\tau A^s}\big) := \Big\{ u \in \mathring{L}^2(\Omega) \;\Big|\; \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{2\tau (2\pi)^{2s} |\vec{\alpha}|^{2s}} |\hat{u}_{\vec{\alpha}}|^2 < \infty \Big\}. \tag{2.3}$$

For any $u \in D\big(\mathrm{e}^{\tau A^s}\big)$, define

$$\mathrm{e}^{\tau A^s} u := \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{\tau (2\pi)^{2s} |\vec{\alpha}|^{2s}} \hat{u}_{\vec{\alpha}}\, \Phi_{\vec{\alpha}}. \tag{2.4}$$

We introduce the Gevrey space $G_\tau := D\big(\mathrm{e}^{\tau A^{1/2}}\big)$. This is a Hilbert space with the inner product and norm denoted by

$$(u,v)_\tau := \big(\mathrm{e}^{\tau A^{1/2}} u, \mathrm{e}^{\tau A^{1/2}} v\big) = \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} \mathrm{e}^{2\tau 2\pi |\vec{\alpha}|} \hat{u}_{\vec{\alpha}}\, \hat{v}^*_{\vec{\alpha}}, \quad |u|_\tau := \sqrt{(u,u)_\tau}. \tag{2.5}$$

Observe that, for any $u \in G_\tau$,

$$|u|^2_\tau = \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} \sum_{\vec{\alpha}\in\mathbb{Z}^d_\star} (2\pi)^m |\vec{\alpha}|^m |\hat{u}_{\vec{\alpha}}|^2 = \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} \|u\|^2_{D(A^{m/4})}. \tag{2.6}$$

Since $|u|_\tau$ is finite, it follows that every H^k norm of u is also finite.

Set $\mathcal{G}_M := \mathrm{span}\,(\{\Phi_{\vec{\alpha}} \mid |\vec{\alpha}| \leq M\})$. The operator $\mathcal{P}_M : L^2(\Omega) \to \mathcal{G}_M$ is the canonical orthogonal projection:

$$\mathcal{P}_M u := \sum_{|\vec{\alpha}|\leq M} \hat{u}_{\vec{\alpha}}\, \Phi_{\vec{\alpha}}. \tag{2.7}$$

Of course, if $u \in \mathring{L}^2(\Omega)$, then $\hat{u}_{\vec{0}} = 0$. One can extend the domain of definition $\mathcal{P}_M$ to $\mathring{H}^{-r}_{\mathrm{per}}(\Omega)$, for any $r \in (0,\infty)$, as follows: if $u \in \mathring{H}^{-r}_{\mathrm{per}}(\Omega)$, then

$$\mathcal{P}_M u := \sum_{|\vec{\alpha}|\leq M} u\,(\Phi^*_{\vec{\alpha}})\, \Phi_{\vec{\alpha}} = \sum_{|\vec{\alpha}|\leq M} \langle u, \Phi_{\vec{\alpha}}\rangle \Phi_{\vec{\alpha}},$$

which implies that

$$(\mathcal{P}_M u, v) := \langle u, \mathcal{P}_M v\rangle, \quad \forall v \in \mathring{H}^r_{\mathrm{per}}(\Omega).$$

Recall that $\left(\mathring{H}_{\rm per}^{-r}(\Omega), \|\cdot\|_{\mathring{H}_{\rm per}^{-r}}\right)$ is a Hilbert space using the standard operator norm. We have the following basic properties of the orthogonal projection that we state without proof [38]:

Lemma 2.1. *Let $X = \mathring{H}_{\rm per}^{-r}(\Omega)$, or $D(A^s)$, for any $r, s \geq 0$. Then, for any $u \in X$,*

$$\|\mathcal{P}_M u\|_X \leq \|u\|_X\,, \quad \text{and} \quad \|u - \mathcal{P}_M u\|_X \xrightarrow{M\to\infty} 0. \tag{2.8}$$

The results can be modified in a trivial way to accommodate functions that are not of mean zero.

We have the following interpolation inequalities [1]:

Lemma 2.2. *Let $r, k, j \in \mathbb{R}$, with $0 \leq k < j < r$. Then, for any $\psi \in \mathring{H}_{\rm per}^{r}(\Omega) = D(A^{r/2})$,*

$$\left\|A^{j/2}\psi\right\|_2 \leq C\left\|A^{k/2}\psi\right\|_2^{\frac{r-j}{r-k}}\left\|A^{r/2}\psi\right\|_2^{\frac{j-k}{r-k}}. \tag{2.9}$$

For integer values of the indices, we have

$$\left\|\nabla^j\psi\right\|_2 \leq \left\|\nabla^k\psi\right\|_2^{\frac{r-j}{r-k}}\left\|\nabla^r\psi\right\|_2^{\frac{j-k}{r-k}}, \tag{2.10}$$

where a constant of 1 suffices.

Frequent use will be made of following Gagliardo-Nirenberg-type interpolation inequality [1]:

Theorem 2.3. *Let $j, m \in \mathbb{N}$, $q, r, \theta \in \mathbb{R}$. Suppose $1 \leq q, r \leq \infty$, $\frac{j}{m} \leq \theta \leq 1$, and*

$$\frac{1}{p} - \frac{j}{d} = \left(\frac{1}{r} - \frac{m}{d}\right)\theta + \frac{1-\theta}{q}. \tag{2.11}$$

If $\psi \in L^q(\Omega) \cap W_{\rm per}^{m,r}(\Omega)$, then $\psi \in W_{\rm per}^{j,p}(\Omega)$, and there exists a constant $C = C(d, j, m, p, q, r, \Omega) > 0$ such that

$$|\psi|_{j,p} \leq C\left(|\psi|_{m,r}^{\theta}\,\|\psi\|_q^{1-\theta} + \|\psi\|_q\right). \tag{2.12}$$

3. Approximate solutions and uniform energy estimates.

3.1. Lower and upper bounds of the energy.

Proposition 3.1. *Let E be the energy given in (1.1). For any $\phi \in H_{\rm per}^{\ell-s}(\Omega) \cap W_{\rm per}^{1,\wp}(\Omega)$, we have*

$$C_1\|\nabla\phi\|_\wp^\wp + C_2\left\|\nabla^{\ell-s}\phi\right\|_2^2 - C_3 \leq E(\phi) \leq C_4\|\nabla\phi\|_\wp^\wp + C_5\left\|\nabla^{\ell-s}\phi\right\|_2^2, \tag{3.1}$$

where $C_1, \cdots, C_5$ are positive constants that depend only on the model parameters.

Proof. First, we decompose the energy (1.1) into non-quadratic and quadratic parts:

$$\begin{aligned} P(\phi) &= \sum_{j=2}^{\wp/2}\frac{c_{2j}}{2j}\|\nabla\phi\|_{2j}^{2j}, \quad c_\wp = 1, \\ Q(\phi) &= \frac{1}{2}\sum_{j=1}^{\ell} a_j\left\|\nabla^{j-s}\phi\right\|_2^2, \quad a_\ell = \varepsilon^2 > 0. \end{aligned} \tag{3.2}$$

We begin with the lower bounds. The non-quadratic energy part obeys the following estimate

$$P(\phi) \geq \frac{1}{\wp}\|\nabla\phi\|_{\wp}^{\wp} - \frac{|c_{\wp-2}|}{\wp-2}\|\nabla\phi\|_{\wp-2}^{\wp-2} - \cdots - \frac{|c_4|}{4}\|\nabla\phi\|_4^4. \tag{3.3}$$

A simple application of Hölder inequality, using $|\Omega| = 1$, shows that $\|\nabla\phi\|_{2j} \leq \|\nabla\phi\|_{\wp}$, for $2 \leq j < \wp/2$. Then, with the help of Young's inequality,

$$a \cdot b \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{3.4}$$

the following general estimate can be derived

$$\frac{|c_{2j}|}{2j}\|\nabla\phi\|_{2j}^{2j} \leq \frac{|c_{2j}|}{2j}\|\nabla\phi\|_{\wp}^{2j} \leq \frac{1}{2\wp^2}\|\nabla\phi\|_{\wp}^{\wp} + A_{2j}, \tag{3.5}$$

where

$$A_{2j} := \frac{\wp-2j}{\wp}\left[2\cdot 2j\cdot\wp\right]^{\frac{2j}{\wp-2j}}\left[\frac{|c_{2j}|}{2j}\right]^{\frac{\wp}{\wp-2j}}, \tag{3.6}$$

upon choosing

$$p = \frac{\wp}{2j}, \quad q = \frac{\wp}{\wp-2j}, \quad a = [2\cdot 2j\cdot\wp]^{-\frac{2j}{\wp}}\|\nabla\phi\|_{\wp}^{2j}, \quad b = [2\cdot 2j\cdot\wp]^{\frac{2j}{\wp}}\frac{|c_{2j}|}{2j}. \tag{3.7}$$

Consequently,

$$\begin{aligned} P(\phi) &\geq \frac{1}{\wp}\left(1 - \frac{1}{2\wp}\cdot\frac{\wp}{2}\right)\|\nabla\phi\|_{\wp}^{\wp} - A_4 - A_6 - \cdots - A_{\wp-2} \\ &= \frac{3}{4\wp}\|\nabla\phi\|_{\wp}^{\wp} - C_6. \end{aligned} \tag{3.8}$$

where the constant $C_6 := A_4 + A_6 + \cdots + A_{\wp-2} > 0$ only depends on the coefficients $c_4, c_6, \cdots, c_{\wp-2}$.

The quadratic part, $Q(\phi)$, is analyzed in two separate cases: $s = 0$ and $s = 1$. If $s = 0$, a direct observation gives

$$Q(\phi) \geq \frac{1}{2}\left(\varepsilon^2\|\nabla^\ell\phi\|^2 - |a_{\ell-1}|\left\|\nabla^{\ell-1}\phi\right\|_2^2 - \cdots - |a_1|\|\nabla\phi\|_2^2\right). \tag{3.9}$$

Meanwhile, an application of the interpolation inequality (2.9), with $r = \ell$, $k = 1$ and $k < j < \ell$, shows that

$$\begin{aligned} |a_j|\cdot\left\|\nabla^j\phi\right\|_2^2 &\leq |a_j|\cdot\|\nabla\phi\|_2^{\frac{2(\ell-j)}{\ell-1}}\cdot\left\|\nabla^\ell\phi\right\|_2^{\frac{2(j-1)}{\ell-1}} \\ &\leq M_j\|\nabla\phi\|_2^2 + \frac{\varepsilon^2}{2(\ell-1)}\left\|\nabla^\ell\phi\right\|_2^2, \end{aligned} \tag{3.10}$$

where Young's inequality was applied in the last step. We remark that the non-negative constants $M_2, M_3, \cdots, M_{\ell-1}$ only depend on $|a_2|, |a_3|, \cdots, |a_{\ell-1}|$ and ε. Substitution of (3.10) into (3.9) yields

$$Q(\phi) \geq \frac{\varepsilon^2}{4}\left\|\nabla^\ell\phi\right\|_2^2 - C_7\|\nabla\phi\|_2^2, \tag{3.11}$$

where $C_7 := \frac{1}{2}(|a_1| + M_2 + M_2 + ... + M_{\ell-1})$. As before, a simple application of Hölder inequality, using $|\Omega| = 1$, shows that $\|\nabla\phi\|_2 \leq \|\nabla\phi\|_{\wp}$. The negative part in (3.11) can be controlled as

$$C_7\|\nabla\phi\|_2^2 \leq C_7\|\nabla\phi\|_{\wp}^2 \leq C_8 + \frac{1}{4\wp}\|\nabla\phi\|_{\wp}^{\wp}, \tag{3.12}$$

with another application of Young's inequality. Again, note that $C_8 > 0$ only depends on $\wp$ and $|a_2|, |a_3|, \cdots, |a_{\ell-1}|$ and ε. Consequently, we arrive at

$$Q(\phi) \geq \frac{\varepsilon^2}{4} \left\|\nabla^\ell \phi\right\|_2^2 - C_7 \left\|\nabla \phi\right\|_2^2 \geq \frac{1}{4}\varepsilon^2 \left\|\nabla^\ell \phi\right\|_2^2 - \frac{1}{4\wp} \left\|\nabla \phi\right\|_\wp^\wp - C_8. \tag{3.13}$$

Finally, a combination of the non-quadratic part (3.8) and the quadratic part (3.13) results in

$$E(\phi) \geq \frac{\varepsilon^2}{4} \left\|\nabla^\ell \phi\right\|_2^2 + \frac{1}{2\wp} \left\|\nabla \phi\right\|_\wp^\wp - C_8. \tag{3.14}$$

Therefore, the energy estimate (3.1) with $s = 0$ is proven with

$$C_1 = \frac{1}{2\wp}, \quad C_2 = \frac{\varepsilon^2}{4}, \quad C_3 = C_8. \tag{3.15}$$

The lower bound for the case with $s = 1$ can be analyzed in a similar manner. We omit the details for the sake of brevity. Likewise, the upper bounds are straightforward, in fact, easier than the lower bounds, and the details are omitted. □

Definition 3.2. Suppose $\ell \in \mathbb{N} + 2$, $s \in \{0, 1\}$, and $d \in \mathbb{N} + 1$. We say that **Condition 1** holds iff $\ell - s \geq 2$ and one of the following cases holds:

1. $2(\ell - s - 1) < d \leq 4(\ell - s - 1)$ and
$$\wp \in \Sigma := \left\{ q \in 2\mathbb{N} + 4 \;\middle|\; q \leq q_\star := \frac{1}{\frac{1}{2} - \frac{\ell - s - 1}{d}} \right\}, \tag{3.16}$$
in which case $H^{\ell-s}_{\mathrm{per}}(\Omega) \hookrightarrow W^{1,\wp}_{\mathrm{per}}(\Omega)$, or
2. $2(\ell - s - 1) = d$ and $\wp \in 2\mathbb{N} + 4$, in which case $H^{\ell-s}_{\mathrm{per}}(\Omega) \hookrightarrow W^{1,\wp}_{\mathrm{per}}(\Omega)$, or
3. $2(\ell - s - 1) > d$ and $\wp \in 2\mathbb{N} + 4$, in which case
$$H^{\ell-s}_{\mathrm{per}}(\Omega) \hookrightarrow C^{\ell-s-1-[d/2]}_{\mathrm{per}}(\Omega) \subseteq C^1_{\mathrm{per}}(\Omega) \subset W^{1,\wp}_{\mathrm{per}}(\Omega).$$

Remark 3.3. Observe that the set Σ, defined in (3.16), is non-empty; in particular, $4 \in \Sigma$. Also note that if Condition 1 holds, then $4\ell \geq d + 4$ is always satisfied. When $s = 1$, the last statement can be strengthened, in particular, $4\ell \geq d + 8$.

If Condition 1 holds, then the quadratic diffusion term has control over the p-laplacian terms, and we have the following:

Corollary 3.4. *If Condition 1 holds and $\phi \in H^{\ell-s}_{\mathrm{per}}(\Omega)$, then the following upper bound holds:*

$$E(\phi) \leq C_6 \left\|\nabla^{\ell-s} \phi\right\|_2^\wp + C_7, \tag{3.17}$$

where $C_6, C_7 > 0$ depend only upon the model parameters.

3.2. **Approximate solutions and uniform energy estimates.** We may write the gradient flow in operator form as

$$\partial_t \phi + \mathcal{N}(\phi) + \mathcal{L}(\phi) + \varepsilon^2 A^\ell \phi = 0, \tag{3.18}$$

denoting the nonlinear term as

$$\mathcal{N}(\phi) := -A^s \left\{ \nabla \cdot \left(\left\{ |\nabla \phi|^{\wp-2} + c_{\wp-2} |\nabla \phi|^{\wp-4} + \cdots + c_4 |\nabla \phi|^2 \right\} \nabla \phi \right) \right\},$$

and the indefinite (unsigned) linear term as

$$\mathcal{L}(\phi) := a_1 A \phi + a_2 A^2 \phi + a_3 A^3 \phi + \cdots + a_{\ell-1} A^{\ell-1} \phi.$$

We refer to the term $\varepsilon^2 A^\ell \phi$ as the "surface diffusion" term, following the physics literature for solid thin film models. We seek the following Galerkin approximation of the original problem: for fixed $M \in \mathbb{N}$, find

$$\phi_M(\vec{x},t) = \sum_{|\vec{\alpha}|\le M} \tilde{\phi}_{\vec{\alpha},M}(t) \mathrm{e}^{2\pi i \vec{\alpha}\cdot\vec{x}}$$

such that

$$\partial_t \phi_M + \mathcal{P}_M\left(\mathcal{N}(\phi_M)\right) + \mathcal{L}(\phi_M) + \varepsilon^2 A^\ell \phi_M = 0, \tag{3.19}$$

with $\phi_M(0) := \phi_M(\,\cdot\,,0) := \mathcal{P}_M(\phi^0)$, where $\phi^0 \in \mathring{L}^2(\Omega)$. Note that we have assumed, for simplicity, that the initial data are mean zero: $|\Omega|^{-1}\int_\Omega \phi^0(\vec{x})\, d\vec{x} = 0$. We will keep this convention for the remainder of the paper.

Lemma 3.5. *Let $\phi^0 \in \mathring{L}^2(\Omega)$. The solution to the Galerkin approximation problem exists for some $T_\star = T_\star(M) > 0$, such that $\tilde{\phi}_{\vec{\alpha},M} \in C^1([0,T_\star])$, for all $|\vec{\alpha}| \le M$, and $\tilde{\phi}_{\vec{0},M}(t) = 0$, for all $t \in [0,T_\star]$. Furthermore, the following energy stability is valid: $E(\phi_M(t)) \le E(\phi_M(0))$, for any $t \in [0,T_\star]$.*

Proof. The approximation problem can be recast as a system of nonlinear ODE's; it has a unique solution up to some finite time $T_\star$, such that $\tilde{\phi}_{\vec{\alpha},M} \in C^\infty([0,T_\star])$, for all $|\vec{\alpha}| \le M$. It is clear that $\tilde{\phi}_{\vec{0},M}(t) = 0$, for all $t \in [0,T_\star]$, since $\int_\Omega \phi_M(\vec{x},t)\, d\vec{x} = 0$ for all $t \in [0,T_\star]$. We define the test function

$$\begin{aligned}\mu_M :=\; & -\mathcal{P}_M\left(\nabla\cdot\left(|\nabla\phi_M|^{\wp-2} + c_{\wp-2}|\nabla\phi_M|^{\wp-4} + \cdots + c_4|\nabla\phi_M|^2\right)\nabla\phi_M\right) \\ & + (-1)^{1-s} a_1 \Delta^{1-s}\phi_M + (-1)^{2-s} a_2 \Delta^{2-s}\phi_M + \cdots \\ & + (-1)^{\ell-1-s} a_{\ell-1}\Delta^{\ell-1-s}\phi_N + (-1)^{\ell-s}\varepsilon^2\Delta^{\ell-s}\phi_N.\end{aligned} \tag{3.20}$$

Observe $\mu_M \in \mathcal{G}_M \cap \mathring{L}^2(\Omega)$. Testing this with the Equation (3.19) and integrating, we arrive (after a standard energy variation calculation) at the result

$$(\partial_t\phi_M, \mu_M) = d_t E(\phi_M(t)) = -\left\|\nabla^s \mu_M(t)\right\|_2^2.$$

Integrating this in time, we have, for any $T \in [0,T_\star]$,

$$E(\phi_M(t)) + \int_0^T \left\|\nabla^s\mu_M(t)\right\|_2^2\, dt = E(\phi_M(0)). \tag{3.21}$$

□

As a consequence of Proposition 3.1, Lemma 3.5, and Corollary 3.4 the following result is valid.

Corollary 3.6. *Suppose that Condition 1 holds and $\phi_0 \in \mathring{H}^{\ell-s}_{\text{per}}(\Omega)$. Then ϕ_M and μ_M, defined as in Lemma 3.5, exist for all time, and, moreover, for any $T > 0$ whatsoever,*

$$\max_{0\le t\le T} \left\|\phi_M(t)\right\|^2_{H^{\ell-s}} + \int_0^T \left\|\nabla^s \mu_M(t)\right\|_2^2\, dt \le C_9, \tag{3.22}$$

where C_9 depends on the initial data and the equation parameters, but is independent of M and T.

Proof. A combination of Proposition 3.1, Lemma 3.5, and Corollary 3.4 indicates that, for any $0 < t \le T_\star$,

$$
\begin{aligned}
C_2 \left\|\nabla^{\ell-s}\phi_M(t)\right\|_2^2 - C_3 + \int_0^t \left\|\nabla^s \mu_M(\tau)\right\|_2^2 d\tau &\le E(\phi_M(t)) + \int_0^t \left\|\nabla^s \mu_M(\tau)\right\|_2^2 d\tau \\
&\le E(\phi_M(0)) \\
&\le C_6 \left\|\nabla^{\ell-s}\phi_M(0)\right\|_2^{\wp} + C_7 \\
&\le C_6 \left\|\nabla^{\ell-s}\phi^0\right\|_2^{\wp} + C_7, \qquad (3.23)
\end{aligned}
$$

where Lemma 2.8 was employed in the last step. By regularity, there is a constant constant, C_{10} such that

$$
C_{10} \left\|\psi\right\|_{H^{\ell-s}}^2 \le \left\|\psi\right\|_{\mathring{H}_{\rm per}^{\ell-s}}^2 = \left\|\nabla^{\ell-s}\psi\right\|_2^2, \qquad (3.24)
$$

for any $\psi \in \mathring{H}_{\rm per}^{\ell-s}(\Omega)$. Therefore, estimate (3.22) is proven for $T = T_\star$. But, since C_9 is independent of the final time, T, the Galerkin approximate solutions do not blow-up and can be extended up to any final time $T > 0$ [38]. ☐

Definition 3.7. Suppose $T > 0$ and $\phi, \mu : \Omega \times [0,T] \to \mathbb{R}$. We say that the pair (ϕ, μ) is a weak solution on the time interval $[0,T]$ iff

$$
\begin{aligned}
\phi &\in L^\infty\left(0,T;\mathring{H}_{\rm per}^{\ell-s}(\Omega) \cap \mathring{W}_{\rm per}^{1,\wp}(\Omega)\right) \cap C^0\left(0,T;\mathring{L}^2(\Omega)\right), \\
\mu &\in L^2\left(0,T;\mathring{H}_{\rm per}^{s}(\Omega)\right), \\
\partial_t\phi &\in L^2\left(0,T;\mathring{H}_{\rm per}^{-s}(\Omega)\right), \qquad (3.25)
\end{aligned}
$$

and, for almost all $t \in [0,T]$,

$$
\langle \partial_t\phi, \nu\rangle + \left(\nabla^s\mu, \nabla^s\nu\right) = 0, \ \forall\ \nu \in H_{\rm per}^s(\Omega), \qquad (3.26)
$$

$$
\sum_{j=1}^{\ell} a_j \left(\nabla^{j-s}\phi, \nabla^{j-s}\psi\right) + \sum_{j=2}^{\wp/2} c_{2j}\left(\left|\nabla\phi\right|^{2j-2}\nabla\phi, \nabla\psi\right) - (\mu,\psi) = 0, \ \forall\ \psi \in H_{\rm per}^{\ell-s}(\Omega), \qquad (3.27)
$$

with $\phi(0) = \phi^0 \in \mathring{L}^2(\Omega)$, where $a_\ell = \varepsilon^2$ and $c_\wp = 1$, as usual.

Theorem 3.8. *Suppose that Condition 1 holds and $\phi^0 \in \mathring{H}_{\rm per}^{\ell-s}(\Omega)$. Then a weak solution exists on any time interval $[0,T]$, however large the final time T may be.*

Proof. Since the bound (3.22) is uniform in M, there exist subsequences ϕ_{M_m} and μ_{M_m} and limit points $\phi \in L^\infty\left(0,T;\mathring{H}_{\rm per}^{\ell-s}(\Omega)\right)$ and $\mu \in L^2\left(0,T;\mathring{H}_{\rm per}^{s}(\Omega)\right)$, such that ϕ_{M_m} converges weakly to ϕ, μ_{M_m} converges weakly to μ, and

$$
\left\|\phi\right\|_{L^\infty\left(0,T;H_{\rm per}^{\ell-s}(\Omega)\right)} + \left\|\mu\right\|_{L^2\left(0,T;H_{\rm per}^{s}(\Omega)\right)} \le C_{11}, \qquad (3.28)
$$

where $C_{11} > 0$ is independent of T. Passing to limits, one can prove that the pair (ϕ,μ) is a weak solution to the gradient equation (1.4). The details are standard and are skipped for brevity. ☐

4. **A local-in-time solution with Gevrey regularity.** In this section, we establish a crucial technical estimate that will be used in the Gevrey analysis of the solution for (1.4). In a standard way, we need to analyze the Galerkin approximate solution (3.19) and pass to the limit to obtain the desired results.

4.1. **A preliminary estimate of the nonlinear terms.** We define the following nonlinear terms: for ϕ sufficiently regular and Ω-periodic, set

$$\mathcal{N}_p^0(\phi) := -\nabla\cdot\left(|\nabla\phi|^p\nabla\phi\right), \quad \mathcal{N}_p^1(\phi) := -\Delta\mathcal{N}_p^0(\phi), \quad p\in\{2,4,\cdots,\wp-2\}\,.$$

Then, using the formula

$$\nabla\left(|\nabla\phi|^q\right) = q|\nabla\phi|^{q-2}\mathbf{H}(\phi)\nabla\phi, \quad \forall\, q\in 2\mathbb{N}, \tag{4.1}$$

we find

$$\mathcal{N}_p^0(\phi) = -|\nabla\phi|^p\Delta\phi - p|\nabla\phi|^{p-2}\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi, \tag{4.2}$$

where $\mathbf{H}(\phi)$ is the $d\times d$ symmetric Hessian matrix of ϕ with components $[\mathbf{H}(\phi)]_{i,j} = \partial_i\partial_j\phi$. Furthermore, using

$$\Delta(fg) = g\Delta f + 2\nabla f\cdot\nabla g + f\Delta g, \tag{4.3}$$

we find

$$\begin{aligned}\mathcal{N}_p^1(\phi) = {}& \Delta\left(|\nabla\phi|^p\right)\Delta\phi + 2\nabla\left(|\nabla\phi|^p\right)\cdot\nabla\Delta\phi + |\nabla\phi|^p\Delta^2\phi \\ &+ p\Delta\left(|\nabla\phi|^{p-2}\right)\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi + 2p\nabla\left(|\nabla\phi|^{p-2}\right)\cdot\nabla\left(\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi\right) \\ &+ p|\nabla\phi|^{p-2}\Delta\left(\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi\right).\end{aligned} \tag{4.4}$$

This expression becomes quite a bit more complicated upon further expansion. For instance,

$$\nabla\left(\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi\right) = 2\mathbf{H}^2(\phi)\nabla\phi + \left[\mathbf{C}(\phi)\nabla\phi\right]\nabla\phi, \tag{4.5}$$

where $\mathbf{C}(\phi)$ is the symmetric 3-tensor with the components $[\mathbf{C}(\phi)]_{i,j,k} = \partial_i\partial_j\partial_k\phi$, and

$$\begin{aligned}\Delta\left(\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi\right) = {}& 2\left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\Delta\phi + 4\left[\mathbf{C}(\phi)\nabla\phi\right]:\mathbf{H}(\phi) \\ &+ 2\mathbf{H}^2(\phi):\mathbf{H}(\phi) + \left[\mathbf{H}(\phi)\nabla\phi\right]\cdot\nabla\phi.\end{aligned} \tag{4.6}$$

where $\mathbf{A}:\mathbf{B} = \sum_{j,k=1}^d \mathbf{A}_{j,k}\mathbf{B}_{j,k}$, for two-tensors (matrices) $\mathbf{A}$ and $\mathbf{B}$.

In the next lemma, we examine a single representative term of $\mathcal{N}_p^s$, $s=0,1$.

Definition 4.1. Let $p\in 2\mathbb{N}+2$, i.e., $p=2r$, $r\in\mathbb{N}+1$. Define, for $u^{(1)},\cdots u^{(p)},v:\Omega\to\mathbb{R}$ sufficiently regular,

$$\mathcal{N}_{p,\star}^s(u^{(1)},\cdots,u^{(p)},v) := \left[\prod_{j=1}^r \nabla u^{(2j-1)}\cdot\nabla u^{(2j)}\right]\Delta^{1+s}v, \tag{4.7}$$

Observe that

$$\mathcal{N}_{p,\star}^s(\phi,\cdots,\phi,\phi) = \left(\nabla\phi\cdot\nabla\phi\right)^{p/2}\Delta^{1+s}\phi = |\nabla\phi|^p\Delta^{1+s}\phi, \tag{4.8}$$

which is the first term of $\mathcal{N}_p^0(\phi)$ ($s=0$) in (4.2) and third term of $\mathcal{N}_p^1(\phi)$ ($s=1$) in (4.4), modulo the appropriate signs. Then, we have the following result.

Lemma 4.2. *Suppose* $p\in 2\mathbb{N}+2$, $s\in\{0,1\}$, $u^{(1)},u^{(2)},\cdots,u^{(p)},v,w\in D(A^\ell e^{\tau A^{1/2}})$, $\tau>0$, *where* $A=-\Delta$. *Then, if Condition 1 is satisfied, the following estimate* *holds:*

$$\left|\left(e^{\tau A^{1/2}}\mathcal{N}^s_{p,\star}(u^{(1)},u^{(2)},\cdots,u^{(p)},v), e^{\tau A^{1/2}}A^\ell w\right)\right|$$
$$\leq C_{12}\begin{cases} \left[\prod_{j=1}^p \left|A^{\frac{\ell}{2}}u^{(j)}\right|_\tau^{\frac{4\ell-2-d}{2\ell}} \left|A^\ell u^{(j)}\right|_\tau^{\frac{d-2\ell+2}{2\ell}}\right] \left|A^{\frac{\ell}{2}}v\right|_\tau^{\frac{2(\ell-s-1)}{\ell}} \left|A^\ell v\right|_\tau^{\frac{2s+2-\ell}{\ell}} \left|A^\ell w\right|_\tau \\ \qquad\qquad \text{for } {}^{d>2\ell-2}_{\frac{\ell}{2}<1+s\leq\ell} \\ \left[\prod_{j=1}^p \left|A^{\frac{\ell}{2}}u^{(j)}\right|_\tau^{\frac{4\ell-2-d}{2\ell}} \left|A^\ell u^{(j)}\right|_\tau^{\frac{d-2\ell+2}{2\ell}}\right] \left|A^{\frac{\ell}{2}}v\right|_\tau \left|A^\ell w\right|_\tau \qquad \text{for } {}^{d>2\ell-2}_{1+s\leq\frac{\ell}{2}} \\ \left[\prod_{j=1}^p \left|A^{\frac{\ell}{2}}u^{(j)}\right|_\tau\right] \left|A^{\frac{\ell}{2}}v\right|_\tau^{\frac{2(\ell-s-1)}{\ell}} \left|A^\ell v\right|_\tau^{\frac{2s+2-\ell}{\ell}} \left|A^\ell w\right|_\tau \qquad \text{for } {}^{d\leq 2\ell-2}_{\frac{\ell}{2}<1+s\leq\ell} \\ \left[\prod_{j=1}^p \left|A^{\frac{\ell}{2}}u^{(j)}\right|_\tau\right] \left|A^{\frac{\ell}{2}}v\right|_\tau \left|A^\ell w\right|_\tau \qquad \text{for } {}^{d\leq 2\ell-2}_{1+s\leq\frac{\ell}{2}} \end{cases}. \tag{4.9}$$

for some $C_{12} > 0$.

Proof. Suppose $p = 2r$. Let us set

$$v(\vec{x}) = \sum_{\vec{\alpha}\in\mathbb{Z}^d} \hat{v}_{\vec{\alpha}} e^{2\pi i\vec{\alpha}\cdot\vec{x}}, \qquad e^{\tau A^{1/2}}v(\vec{x}) = \tilde{v}(\vec{x}) = \sum_{\vec{\alpha}\in\mathbb{Z}^d} \hat{\tilde{v}}_{\vec{\alpha}} e^{2\pi i\vec{\alpha}\cdot\vec{x}}, \qquad \hat{\tilde{v}}_{\vec{\alpha}} := e^{\tau 2\pi|\vec{\alpha}|}\hat{v}_{\vec{\alpha}},$$

with similar notation for $u^{(1)}, u^{(2)}, \cdots, u^{(p)}$, w. Observe that $\hat{v}_{\vec{0}} = 0$, et cetera. We have

$$\begin{aligned} &\left(e^{\tau A^{1/2}}\mathcal{N}^s_{p,\star}(u^{(1)},u^{(2)},\cdots,u^{(p)},v), e^{\tau A^{1/2}}A^\ell w\right) \\ &= (-1)^r(2\pi)^{p+2(1+s)+2\ell} \sum_{\substack{\vec{\beta}\in\mathbb{Z}^d \\ \sum_{k=1}^p \vec{\alpha}^{(k)}+\vec{\alpha}=\vec{\beta}}} \left[\prod_{j=1}^r \hat{u}^{(2j-1)}_{\vec{\alpha}^{(2j-1)}} \hat{u}^{(2j)}_{\vec{\alpha}^{(2j)}} \vec{\alpha}^{(2j-1)}\cdot\vec{\alpha}^{(2j)}\right] \\ &\qquad \times \hat{v}_{\vec{\alpha}}|\vec{\alpha}|^{2(1+s)} \hat{w}^*_{\vec{\beta}}|\vec{\beta}|^{2\ell} e^{2\tau 2\pi|\vec{\beta}|} \\ &= (-1)^r(2\pi)^{p+4+2\ell} \sum_{\substack{\vec{\beta}\in\mathbb{Z}^d \\ \sum_{k=1}^p \vec{\alpha}^{(k)}+\vec{\alpha}=\vec{\beta}}} \left[\prod_{j=1}^r \hat{\tilde{u}}^{(2j-1)}_{\vec{\alpha}^{(2j-1)}} \hat{\tilde{u}}^{(2j)}_{\vec{\alpha}^{(2j)}} \vec{\alpha}^{(2j-1)}\cdot\vec{\alpha}^{(2j)}\right] \\ &\qquad \times \hat{\tilde{v}}_{\vec{\alpha}}|\vec{\alpha}|^{2(1+s)} \hat{\tilde{w}}^*_{\vec{\beta}}|\vec{\beta}|^{2\ell} e^{\tau 2\pi\left(|\vec{\beta}|-\sum_{k=1}^p|\vec{\alpha}^{(k)}|-|\vec{\alpha}|\right)}. \end{aligned}$$

Now, observe that

$$|\vec{\beta}| - \sum_{\ell=1}^p |\vec{\alpha}^{(\ell)}| - |\vec{\alpha}| = \left|\sum_{\ell=1}^p \vec{\alpha}^{(\ell)} + \vec{\alpha}\right| - \sum_{\ell=1}^p |\vec{\alpha}^{(\ell)}| - |\vec{\alpha}| \leq 0,$$

and, consequently,

$$\begin{aligned} &\left|\left(e^{\tau A^{1/2}}\mathcal{N}^s_{p,\star}(u^{(1)},u^{(2)},\cdots,u^{(m)},v), e^{\tau A^{1/2}}A^\ell w\right)\right| \\ &\leq (2\pi)^{p+2(1+s)+2\ell} \sum_{\substack{\vec{\beta}\in\mathbb{Z}^d \\ \sum_{\ell=1}^p \vec{\alpha}^{(\ell)}+\vec{\alpha}=\vec{\beta}}} \left[\prod_{j=1}^p \left|\hat{\tilde{u}}^{(j)}_{\vec{\alpha}^{(j)}}\right| \cdot \left|\vec{\alpha}^{(j)}\right|\right] \left|\hat{\tilde{v}}_{\vec{\alpha}}\right| \cdot |\vec{\alpha}|^{2(1+s)} \cdot \left|\hat{\tilde{w}}_{\vec{\beta}}\right| \cdot \left|\vec{\beta}\right|^{2\ell} \end{aligned}$$

$$=: \left(\psi \prod_{j=1}^{p} \xi_j, \theta\right), \tag{4.10}$$

where, for all $\vec{x} \in \Omega$,

$$\xi_j(\vec{x}) := 2\pi \sum_{\vec{\alpha}\in\mathbb{Z}^d} \left|\hat{\hat{u}}^{(j)}_{\vec{\alpha}}\right| |\vec{\alpha}| \mathrm{e}^{2\pi i\vec{\alpha}\cdot\vec{x}}, \quad \psi(\vec{x}) := (2\pi)^{2(1+s)} \sum_{\vec{\alpha}\in\mathbb{Z}^d} \left|\hat{\hat{v}}_{\vec{\alpha}}\right| |\vec{\alpha}|^{2(1+s)} \mathrm{e}^{2\pi i\vec{\alpha}\cdot\vec{x}},$$
$$\theta(\vec{x}) := (2\pi)^{2\ell} \sum_{\vec{\alpha}\in\mathbb{Z}^d} \left|\hat{\hat{w}}_{\vec{\alpha}}\right| \cdot |\vec{\alpha}|^{2\ell} \mathrm{e}^{2\pi i\vec{\alpha}\cdot\vec{x}}.$$

According to the Nirenberg inequality (2.12),

$$\|\xi_j\|_\infty \le C \left\|A^{\frac{d}{4}}\xi_j\right\|_2 = C \left\|A^{\frac{2+d}{4}} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2, \qquad \forall\, j \in \{1,2,\cdots,p\}, \tag{4.11}$$

furthermore,

$$\|\psi\|_2^2 = \left\|A^{1+s}\mathrm{e}^{\tau A^{1/2}} v\right\|_2^2, \quad \|\theta\|_2^2 = \left\|A^{\ell}\mathrm{e}^{\tau A^{1/2}} w\right\|_2^2. \tag{4.12}$$

Since Condition 1 holds, it follows (see Remark 3.3) that $4\ell \ge d+2$, and, we will need to consider four cases in the analysis. If $d > 2\ell - 2$, there are two sub-cases, Cases 1 and 2:

Case 1. If

$$\frac{\ell}{2} < \frac{2+d}{4} \le \ell \qquad \text{and} \qquad \frac{\ell}{2} < 1+s \le \ell,$$

we have, using Lemma 2.2,

$$\left\|A^{\frac{2+d}{4}} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2 \le C \left\|A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2^{\frac{4\ell-2-d}{2\ell}} \left\|A^{\ell} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2^{\frac{d-2\ell+2}{2\ell}},$$
$$\left\|A^{1+s} \mathrm{e}^{\tau A^{1/2}} v\right\|_2 \le C \left\|A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} v\right\|_2^{\frac{2(\ell-s-1)}{\ell}} \left\|A^{\ell} \mathrm{e}^{\tau A^{1/2}} v\right\|_2^{\frac{2s+2-\ell}{\ell}}.$$

Therefore,

$$\begin{aligned}&\left|\left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}^s_{p,\star}(u^{(1)}, u^{(2)}, \cdots, u^{(p)}, v), \mathrm{e}^{\tau A^{1/2}} A^{\ell} w\right)\right| \\ &\quad \le C \prod_{j=1}^{p} \left|A^{\frac{\ell}{2}} u^{(j)}\right|_\tau^{\frac{4\ell-2-d}{2\ell}} \left|A^{\ell} u^{(j)}\right|_\tau^{\frac{d-2\ell+2}{2\ell}} \left|A^{\frac{\ell}{2}} v\right|_\tau^{\frac{2(\ell-s-1)}{\ell}} \left|A^{\ell} v\right|_\tau^{\frac{2s+2-\ell}{\ell}} \left|A^{\ell} w\right|_\tau.\end{aligned} \tag{4.13}$$

Case 2. If

$$\frac{\ell}{2} < \frac{2+d}{4} \le \ell \qquad \text{and} \qquad 1 \le 1+s \le \frac{\ell}{2},$$

we have, appealing to Lemma 2.2 and the Sobolev embedding $D(A^{\ell/2}) \hookrightarrow D(A^{1+s})$,

$$\left\|A^{\frac{2+d}{4}} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2 \le C \left\|A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2^{\frac{4\ell-2-d}{2\ell}} \left\|A^{\ell} \mathrm{e}^{\tau A^{1/2}} u^{(j)}\right\|_2^{\frac{d-2\ell+2}{2\ell}},$$
$$\left\|A^{1+s} \mathrm{e}^{\tau A^{1/2}} v\right\|_2 \le C \left\|A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} v\right\|_2.$$

Therefore,

$$\begin{aligned}&\left|\left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}^s_{p,\star}(u^{(1)}, u^{(2)}, \cdots, u^{(p)}, v), \mathrm{e}^{\tau A^{1/2}} A^{\ell} w\right)\right| \\ &\qquad\qquad \le C \prod_{j=1}^{p} \left|A^{\frac{\ell}{2}} u^{(j)}\right|_\tau^{\frac{4\ell-2-d}{2\ell}} \left|A^{\ell} u^{(j)}\right|_\tau^{\frac{d-2\ell+2}{2\ell}} \left|A^{\frac{\ell}{2}} v\right|_\tau \left|A^{\ell} w\right|_\tau.\end{aligned} \tag{4.14}$$

If $d < 2\ell - 2$, two more sub-cases must be examined, Cases 3 and 4:

Case 3. If
$$\frac{2+d}{4} \le \frac{\ell}{2} \qquad \text{and} \qquad \frac{\ell}{2} < 1 + s \le \ell,$$
we have
$$\left\| A^{\frac{2+d}{4}} \mathrm{e}^{\tau A^{1/2}} u^{(j)} \right\|_2 \le C \left\| A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} u^{(j)} \right\|_2,$$
$$\left\| A^{1+s} \mathrm{e}^{\tau A^{1/2}} v \right\|_2 \le C \left\| A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} v \right\|_2^{\frac{2(\ell-s-1)}{\ell}} \left\| A^{\ell} \mathrm{e}^{\tau A^{1/2}} v \right\|_2^{\frac{2s+2-\ell}{\ell}},$$
and therefore
$$\left| \left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}^s_{p,\star}(u^{(1)}, u^{(2)}, \cdots, u^{(p)}, v), \mathrm{e}^{\tau A^{1/2}} A^{\ell} w \right) \right| \le C \prod_{j=1}^{p} \left| A^{\frac{\ell}{2}} u^{(j)} \right|_\tau \left| A^{\frac{\ell}{2}} v \right|_\tau^{\frac{2(\ell-s-1)}{\ell}} \left| A^{\ell} v \right|_\tau^{\frac{2s+2-\ell}{\ell}} \left| A^{\ell} w \right|_\tau. \tag{4.15}$$

Case 4. If
$$\frac{2+d}{4} \le \frac{\ell}{2} \qquad \text{and} \qquad 1 \le 1 + s \le \frac{\ell}{2},$$
we obtain
$$\left\| A^{\frac{2+d}{4}} \mathrm{e}^{\tau A^{1/2}} u^{(j)} \right\|_2 \le C \left\| A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} u^{(j)} \right\|_2,$$
$$\left\| A^{1+s} \mathrm{e}^{\tau A^{1/2}} v \right\|_2 \le C \left\| A^{\frac{\ell}{2}} \mathrm{e}^{\tau A^{1/2}} v \right\|_2,$$
and thus
$$\left| \left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}^s_{p,\star}(u^{(1)}, u^{(2)}, \cdots, u^{(p)}, v), \mathrm{e}^{\tau A^{1/2}} A^{\ell} w \right) \right| \le C \prod_{j=1}^{p} \left| A^{\frac{\ell}{2}} u^{(j)} \right|_\tau \left| A^{\frac{\ell}{2}} v \right|_\tau \left| A^{\ell} w \right|_\tau. \tag{4.16}$$

The proof of Lemma 4.2 is finished. □

We summarize the analysis of the nonlinear terms in the following result:

Corollary 4.3. *Suppose that Condition 1 holds, $\phi \in D(A^{\ell} \mathrm{e}^{\tau A^{1/2}})$, $\ell \in \mathbb{N} + 2$, $\wp \in 2\mathbb{N} + 4$, $s \in \{0, 1\}$, $A = -\Delta$. Let $\mathcal{N}$ be as defined in (3.2). Then*
$$\left| \left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}(\phi), \mathrm{e}^{\tau A^{1/2}} A^{\ell} \phi \right) \right| \le C_{13} \left| A^{\frac{\ell}{2}} \phi \right|_\tau^{\sigma_1} \left| A^{\ell} \phi \right|_\tau^{\sigma_2}, \tag{4.17}$$
for some constant $C_{13} > 0$ that depends upon the parameters $\wp$ and ℓ, where
$$\sigma_1 = \begin{cases} \frac{(4\ell-2-d)(\wp-2)+4(\ell-1-s)}{2\ell} & \text{if } \left\{ \begin{matrix} d>2\ell-2 \\ \frac{\ell}{2}<1+s\le\ell \end{matrix} \right\} \quad \textit{(Case 1)} \\ \frac{(4\ell-2-d)(\wp-2)+2\ell}{2\ell} & \text{if } \left\{ \begin{matrix} d>2\ell-2 \\ 1+s\le\frac{\ell}{2} \end{matrix} \right\} \quad \textit{(Case 2)} \\ \frac{\ell(\wp-2)+2(\ell-1-s)}{\ell} & \text{if } \left\{ \begin{matrix} d\le2\ell-2 \\ \frac{\ell}{2}<1+s\le\ell \end{matrix} \right\} \quad \textit{(Case 3)} \\ \wp - 1 & \text{if } \left\{ \begin{matrix} d\le2\ell-2 \\ 1+s\le\frac{\ell}{2} \end{matrix} \right\} \quad \textit{(Case 4)} \end{cases},$$
$$\sigma_2 = \begin{cases} \frac{(d-2\ell+2)(\wp-2)+4(s+1)}{2\ell} & \text{if } \left\{ \begin{matrix} d>2\ell-2 \\ \frac{\ell}{2}<1+s\le\ell \end{matrix} \right\} \quad \textit{(Case 1)} \\ \frac{(d-2\ell+2)(\wp-2)+2\ell}{2\ell} & \text{if } \left\{ \begin{matrix} d>2\ell-2 \\ 1+s\le\frac{\ell}{2} \end{matrix} \right\} \quad \textit{(Case 2)} \\ \frac{2s+2}{\ell} & \text{if } \left\{ \begin{matrix} d\le2\ell-2 \\ \frac{\ell}{2}<1+s\le\ell \end{matrix} \right\} \quad \textit{(Case 3)} \\ 1 & \text{if } \left\{ \begin{matrix} d\le2\ell-2 \\ 1+s\le\frac{\ell}{2} \end{matrix} \right\} \quad \textit{(Case 4)} \end{cases}. \tag{4.18}$$

Remark 4.4. Observe that we always have $\sigma_1 + \sigma_2 = \wp$, regardless of the case. Because of Condition 1, all of the exponents in the last corollary are non-negative.

4.2. **A local-in-time solution with Gevrey regularity.**

Definition 4.5. We say that **Condition 2** holds iff $\sigma_2 = \wp - \sigma_1 < 2$.

The following theorem is the main result of this section.

Theorem 4.6. *Suppose that Conditions 1 and 2 hold, and assume that $\phi^0 \in D(A^{\ell/2})$. Then there exists T_* that depends upon $\left\|A^{\ell/2}\phi_0\right\|_2$, such that the weak solution is regular and unique on $(0, T_\star)$, and $t \to \mathrm{e}^{tA^{1/2}}\phi(t)$ is analytic on $(0, T_\star)$.*

Proof. Considering the Galerkin approximation, $\phi_M(\tau)$, constructed earlier, we take the scalar inner product of (3.19) with $A^\ell \phi_M(\tau)$ in the space $D(\mathrm{e}^{\tau A^{1/2}})$:

$$0 = \left(\mathrm{e}^{\tau A^{1/2}}\partial_t \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) + \left(\mathrm{e}^{\tau A^{1/2}}\mathcal{N}(\phi_M), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right)$$
$$+ \sum_{j=1}^{\ell} a_j \left(\mathrm{e}^{\tau A^{1/2}} A^j \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) + \varepsilon^2 \left(\mathrm{e}^{\tau A^{1/2}} A^\ell \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right). \tag{4.19}$$

The terms above are evaluated as follows. For the time-derivative term, using Caucy's inequality, and Lemma 2.2, we have

$$\begin{aligned}
&\left(\mathrm{e}^{\tau A^{1/2}}\partial_t \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\
&= \left(A^{\frac{\ell}{2}}\partial_t \left[\mathrm{e}^{tA^{1/2}}\phi_M(t)\right]_{t=\tau} - A^{\frac{\ell+1}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau), \mathrm{e}^{\tau A^{1/2}} A^{\frac{\ell}{2}}\phi_M(\tau)\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left\|A^{\frac{\ell}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right\|_2^2 - \left(A^{\frac{\ell+1}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau), A^{\frac{\ell}{2}}\mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \\
&= \frac{1}{2}\frac{d}{d\tau}\left|A^{\frac{\ell}{2}}\phi_M\right|_\tau^2 - \left(A^{\frac{\ell+1}{2}}\phi_M(\tau), A^{\frac{\ell}{2}}\phi_M(\tau)\right)_\tau \\
&\geq \frac{1}{2}\frac{d}{d\tau}\left|A^{\frac{\ell}{2}}\phi\right|_\tau^2 - \frac{\varepsilon^2}{2}\left|A^{\frac{\ell+1}{2}}\phi_M(\tau)\right|_\tau^2 - \frac{1}{2\varepsilon^2}\left|A^{\frac{\ell}{2}}\phi_M(\tau)\right|_\tau^2 \\
&\geq \frac{1}{2}\frac{d}{d\tau}\left|A^{\frac{\ell}{2}}\phi_M\right|_\tau^2 - \frac{\varepsilon^2}{2}\left|A^{\frac{\ell}{2}}\phi_M(\tau)\right|_\tau^{\frac{2(\ell-1)}{\ell}}\left|A^\ell\phi_M(\tau)\right|_\tau^{\frac{2}{\ell}} - \frac{1}{2\varepsilon^2}|A^{\frac{\ell}{2}}\phi_M(\tau)|_\tau^2 \\
&\geq \frac{1}{2}\frac{d}{d\tau}\left|A^{\frac{\ell}{2}}\phi_M\right|_\tau^2 - \left(\frac{\varepsilon^2}{2}\frac{\ell-1}{\ell} + \frac{1}{2\varepsilon^2}\right)\left|A^{\frac{\ell}{2}}\phi_M(\tau)\right|_\tau^2 - \left(\frac{\varepsilon^2}{2}\frac{1}{\ell}\right)\left|A^\ell\phi_M(\tau)\right|_\tau^2.
\end{aligned} \tag{4.20}$$

For the surface diffusion term, we have

$$\varepsilon^2 \left(\mathrm{e}^{\tau A^{1/2}} A^\ell \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) = \varepsilon^2 \left|A^\ell \phi_M(\tau)\right|_\tau^2. \tag{4.21}$$

For the linear terms, we have

$$a_j \left(\mathrm{e}^{\tau A^{1/2}} A^j \phi_M(\tau), A^\ell \mathrm{e}^{\tau A^{1/2}}\phi_M(\tau)\right) \leq |a_j| \left|A^j \phi_M\right|_\tau \left|A^\ell \phi_M\right|_\tau, \tag{4.22}$$

for $1 \leq j \leq \ell - 1$. Now, if $j \leq \frac{\ell}{2}$, we get

$$\begin{aligned}
|a_j| \cdot \left|A^j \phi_M\right|_\tau \left|A^\ell \phi_M\right|_\tau &\leq C|a_j| \cdot \left|A^{\frac{\ell}{2}}\phi_M\right|_\tau \left|A^\ell \phi_M\right|_\tau \\
&\leq \frac{|a_j|}{4} \cdot \frac{8\ell}{\varepsilon^2}\left|A^{\frac{\ell}{2}}\phi_M\right|_\tau^2 + \frac{\varepsilon^2}{8\ell}\left|A^\ell \phi_M\right|_\tau^2.
\end{aligned} \tag{4.23}$$

If $\ell > j > \frac{\ell}{2}$, using Lemma 2.2, we have

$$\begin{aligned} |a_j| \cdot \left|A^j \phi_M\right|_\tau \left|A^\ell \phi_M\right|_\tau &\le C|a_j| \cdot \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^{\frac{2(\ell-j)}{\ell}} \left|A^\ell \phi_M\right|_\tau^{\frac{2j}{\ell}} \\ &\le C_{14} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^2 + \frac{\varepsilon^2}{8\ell} \left|A^\ell \phi_M\right|_\tau^2 . \end{aligned} \tag{4.24}$$

We now use the nonlinear estimate given by (4.17) in Corollary 4.3. Considering Condition 2, we have using Young's inequality,

$$\begin{aligned} \left(\mathrm{e}^{\tau A^{1/2}} \mathcal{N}(\phi_M), A^\ell \mathrm{e}^{\tau A^{1/2}} \phi_M(\tau)\right) &\le C_{13} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^{\sigma_1} \left|A^\ell \phi_M\right|_\tau^{\sigma_2} \\ &\le C_{15} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^{\frac{2\sigma_1}{2-\sigma_2}} + \frac{\varepsilon^2}{8} \left|A^\ell \phi_M\right|_\tau^2 . \end{aligned} \tag{4.25}$$

Putting together estimates (4.20), (4.21), (4.23), (4.24), and (4.25) we have

$$\frac{1}{2}\frac{d}{d\tau} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^2 + \left(\varepsilon^2 - \frac{\varepsilon^2}{4} - \frac{\varepsilon^2}{8} - \frac{\varepsilon^2}{8}\right) \left|A^\ell \phi_M\right|_\tau^2 \le C_{16} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^{\frac{2\sigma_1}{2-\sigma_2}} + C_{17} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^2 . \tag{4.26}$$

This in turn gives

$$\frac{1}{2}\frac{d}{d\tau} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^2 \le C_{18} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^{\frac{2\sigma_1}{2-\sigma_2}} + C_{17} \left|A^{\frac{\ell}{2}} \phi_M\right|_\tau^2 . \tag{4.27}$$

Set $\gamma_1 := \frac{\sigma_1}{2-\sigma_2}$, and $\gamma_2 := \frac{2}{2-\sigma_2}$. Observe that, since $0 < \sigma_2 < 2$ (Condition 2) and $\wp \ge 4$ (Condition 1),

$$\gamma_1 = \frac{\wp - \sigma_2}{2-\sigma_2} = \frac{\wp - 2}{2-\sigma_2} + 1 \ge \frac{2}{2-\sigma_2} + 1 = \gamma_2 + 1 \ge 2. \tag{4.28}$$

In particular, $\gamma_1 - 1 \ge \gamma_2 \ge 1$. Setting $y(\tau) := 1 + \left|A^{\frac{\ell}{2}} \phi_M(\tau)\right|_\tau^2$, it follows that

$$y' \le C_{18} y^{\gamma_1},$$

for some $C_{18} > 0$. Then we have the estimates

$$y(\tau) \le \left(\frac{1}{1-(\gamma_1-1)C_{18}\tau y^{\gamma_1-1}(0)}\right)^{\frac{1}{\gamma_1-1}} y(0) \le \left(\frac{1}{1-\gamma_2 C_{18}\tau y^{\gamma_2(0)}}\right)^{\frac{1}{\gamma_2}} y(0), \tag{4.29}$$

valid for $0 \le \tau < T_1$, where

$$T_1 := \frac{1}{(\gamma_1-1)C_{18} y^{\gamma_1-1}(0)} \le \frac{1}{\gamma_2 C_{18} y^{\gamma_2}(0)}.$$

Using the stability of the L^2 projection, this result implies the uniform (in M) estimate

$$y(\tau) = 1 + \left|A^{\frac{\ell}{2}} \phi_M(\tau)\right|_\tau^2 \le 2y(0) = 2 + 2\left\|A^{\frac{\ell}{2}} \phi_M(0)\right\|_2^2 \le 2 + 2\left\|A^{\frac{\ell}{2}} \phi(0)\right\|_2^2,$$

for $0 \le \tau \le T_2$, where,

$$\begin{aligned} T_2 :&= \frac{2^{\gamma_1-1}-1}{2^{\gamma_1-1}(\gamma_1-1)C_{18}} \left(1 + \left\|A^{\frac{\ell}{2}} \phi_0\right\|_2^2\right)^{1-\gamma_1} \\ &\le \frac{2^{\gamma_1-1}-1}{2^{\gamma_1-1}(\gamma_1-1)C_{18}} \left(1 + \left\|A^{\frac{\ell}{2}} \phi_M(0)\right\|_2^2\right)^{1-\gamma_1} . \end{aligned} \tag{4.30}$$

Thus $\phi_M(\tau) \in D\left(A^{\frac{\ell}{2}} e^{\tau A^{1/2}}\right)$, for all $\tau \in [0, T_2]$, uniformly in M, provided $\phi_0 \in D\left(A^{\frac{\ell}{2}}\right)$.

We can now extract a further subsequence of ϕ_M and pass to limits to obtain our estimates for the limit point ϕ, which is observed to be Gevrey regular on the time interval $(0, T_2)$. The uniqueness analysis of the Gevrey regularity solution is straightforward, due to the high order regularity. The details are left to interested readers. The theorem is proven with $T_\star = T_2$. □

5. **Global-in-time existence of a Gevrey regularity solution.**

Note that the existence time interval length $T_\star$ in Theorem 4.6 for the Gevrey regularity solution depends on the initial data, specifically $\left\| A^{\frac{\ell}{2}} \phi_0 \right\|_2$. To obtain a global-in-time solution with Gevrey regularity, we have to establish a uniform-in-time bound for $\left\| A^{\frac{\ell}{2}} \phi(t) \right\|_2$, so that the constructed solution can be extended to any time. For the case $s = 0$, this follows from Theorem 3.8 immediately, and we have the following:

Theorem 5.1. *Suppose that Conditions 1 and 2 hold, and assume that $\phi^0 \in D(A^{\ell/2})$. If $s = 0$, then there exists a unique global-in-time Gevrey regular solution to (1.4).*

To establish a uniform-in-time bound for $\left\| A^{\frac{\ell}{2}} \phi(t) \right\|_2$ for the case $s = 1$, we will need another condition, namely

Definition 5.2. We say that **Condition 3** holds iff when $s = 1$,

$$\begin{aligned} 2 + 1/2 < \ell \quad &\text{if} \quad d = 1, \\ (\wp - 2)(d/2 - 1) + 4 < 2\ell \quad &\text{if} \quad d \geq 2. \end{aligned} \tag{5.1}$$

Theorem 5.3. *Suppose that Conditions 1 – 3 hold, $s = 1$, and $\phi^0 \in \mathring{H}^\ell_{\text{per}}(\Omega) = D\left(A^{\ell/2}\right)$. Then the weak solution ϕ has the additional regularity $\phi \in L^\infty\left(0, T; \mathring{H}^\ell_{\text{per}}(\Omega)\right)$, however large the final time T may be. Furthermore, we have the uniform-in-time bound*

$$\left\| A^{\frac{\ell}{2}} \phi(t) \right\|_2 \leq C_{19}, \tag{5.2}$$

for all $t \geq 0$, where C_{19} is independent of t.

Proof. For simplicity, we only focus on the case of odd ℓ. The case with an even ℓ can be handled in a similar way. Taking the inner product of (3.19) with $-\Delta^\ell \phi_M$ gives

$$\begin{aligned} &\left(\partial_t \phi_M, -\Delta^\ell \phi_M\right) + \varepsilon^2 \left\| \Delta^\ell \phi_M \right\|_2^2 \\ &= \sum_{j=1}^{\ell-1} a_j (-1)^j \left(\Delta^j \phi_M, \Delta^\ell \phi_M\right) - \left(\Delta^\ell \phi_M, \sum_{j=2}^{\wp/2} c_{2j} \mathcal{N}^1_{2j}(\phi_M)\right). \end{aligned} \tag{5.3}$$

For the temporal derivative term, since ℓ is odd, we have

$$\left(\partial_t \phi_M, -\Delta^\ell \phi_M\right) = \left(\partial_t \nabla \Delta^{\frac{\ell-1}{2}} \phi_M, \nabla \Delta^{\frac{\ell-1}{2}} \phi_M\right) = \frac{1}{2} d_t \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi_M \right\|_2^2. \tag{5.4}$$

For the lower-order linear terms, we start from an application of Cauchy's inequality:

$$\begin{aligned}\left|a_j\left(\Delta^j\phi_M,\Delta^\ell\phi_M\right)\right| \le& |a_j|\cdot\left\|\Delta^j\phi_M\right\|_2\left\|\Delta^\ell\phi_M\right\|_2\\ \le&\frac{(\ell-1)|a_j|}{\varepsilon^2}\left\|\Delta^j\phi_M\right\|_2^2+\frac{\varepsilon^2}{4(\ell-1)}\left\|\Delta^\ell\phi_M\right\|_2^2,\end{aligned} \tag{5.5}$$

for all $1\le j\le\ell-1$. Using (3.22), we have

$$\left\|\Delta^j\phi_M\right\|_2^2\le C_9,\quad\text{for}\quad 1\le j\le\frac{\ell-1}{2}. \tag{5.6}$$

Using Lemma 2.2, Young's inequality, and (3.22), for $\frac{\ell-1}{2}<j\le\ell-1$, we observe that

$$\left\|\Delta^j\phi_M\right\|_2^2\le C\left\|\Delta^{\frac{\ell-1}{2}}\phi_M\right\|_2^{\frac{4(\ell-j)}{\ell+1}}\cdot\left\|\Delta^\ell\phi_M\right\|_2^{\frac{4j-2(\ell-1)}{\ell+1}}\le C(\gamma_j)C_9+\gamma_j\|\Delta^\ell\phi\|^2, \tag{5.7}$$

for any $\gamma_j>0$, for some $C=C(\gamma_j)>0$, where we have used $p=\frac{\ell+1}{2(\ell-j)}$ and $q=\frac{\ell+1}{2j-(\ell-1)}$ in Young's inequality. Thus

$$\frac{(\ell-1)|a_j|}{\varepsilon^2}\left\|\Delta^j\phi_M\right\|_2^2\le\frac{(\ell-1)|a_j|C_9}{\varepsilon^2},\quad\text{for}\quad 1\le l\le\frac{\ell-1}{2}, \tag{5.8}$$

$$\begin{aligned}\frac{(\ell-1)|a_j|}{\varepsilon^2}\left\|\Delta^j\phi_M\right\|_2^2\le&\frac{(\ell-1)|a_J|C(\gamma)C_9}{\varepsilon^2}+\frac{\varepsilon^2}{2(\ell-1)}\left\|\Delta^\ell\phi_M\right\|_2^2,\\ &\text{for}\quad\frac{\ell+1}{2}<j\le\ell-1.\end{aligned} \tag{5.9}$$

Here we have taken

$$\gamma_j=\frac{\varepsilon^4}{2(\ell-1)^2|a_j|}. \tag{5.10}$$

Putting (5.8) – (5.9) into (5.5) yields

$$\begin{aligned}&\left|\sum_{j=1}^{\ell-1}a_j(-1)^j\left(\Delta^j\phi_M,\Delta^\ell\phi_M\right)\right|\\ \le&\sum_{j=1}^{\frac{\ell-1}{2}}\frac{(\ell-1)|a_j|}{\varepsilon^2}\left\|\Delta^j\phi_M\right\|_2^2+\sum_{j=\frac{\ell+1}{2}}^{\ell-1}\frac{(\ell-1)|a_j|}{\varepsilon^2}\left\|\Delta^j\phi_M\right\|_2^2+\frac{\varepsilon^2}{4}\left\|\Delta^\ell\phi\right\|_2^2\\ \le& S_1+\frac{\varepsilon^2}{4}\left\|\Delta^\ell\phi_M\right\|_2^2+\frac{\varepsilon^2}{4}\left\|\Delta^\ell\phi_M\right\|_2^2=S_1+\frac{\varepsilon^2}{2}\left\|\Delta^\ell\phi_M\right\|_2^2,\end{aligned} \tag{5.11}$$

where

$$S_1:=\frac{(\ell-1)C_9}{\varepsilon^2}\left(\sum_{j=1}^{\frac{\ell-1}{2}}|a_j|+\sum_{j=\frac{\ell+1}{2}}^{\ell-1}|a_j|C(\gamma_j)\right).$$

Now, turning our attention to the nonlinear terms, we see that, after some tedious computations – see, in particular, (4.4) – and the application of some Sobolev inequalities, for even number p, $2\le p\le\wp-2$,

$$\begin{aligned}&\left\|\Delta\left(\nabla\cdot\left(|\nabla\phi_M|^p\nabla\phi_M\right)\right)\right\|_2\\ \le& C\Big(\|\phi_M\|_{W^{1,\infty}}^p\cdot\|\phi_M\|_{H^4}+\|\phi_M\|_{W^{1,\infty}}^{p-1}\cdot\|\phi_M\|_{W^{2,\infty}}\cdot\|\phi_M\|_{H^3}\\ &+\|\phi_M\|_{W^{1,\infty}}^{p-2}\cdot\|\phi_M\|_{W^{2,\infty}}^2\cdot\|\phi_M\|_{H^2}\Big).\end{aligned} \tag{5.12}$$

Recall the Sobolev embedding inequalities in $\mathbb{R}^d$: for any $\psi \in H_{\rm per}^{k+d/2+\delta}(\Omega)$, $\delta > 0$, $\psi \in W_{\rm per}^{k,\infty}(\Omega)$ and, for some $C = C(\delta) > 0$,

$$\|\psi\|_{W^{k,\infty}} \leq C(\delta) \|\psi\|_{H^{k+d/2+\delta}} . \tag{5.13}$$

By Condition 1, $\ell - 1 \geq 2$, and we have, for all $t \geq 0$,

$$\|\phi_M(t)\|_{H^2}^2 \leq C \|\phi_M(t)\|_{H^{\ell-1}}^2 \leq CC_9 := C_{20}. \tag{5.14}$$

In addition, since $s = 1$, Condition 1 ensures that $2 + d/2 < 4 + d/2 \leq 2\ell$. Thus, there is a $\delta > 0$ such that $2 + d/2 + \delta \leq 2\ell$. In particular any $0 < \delta < 2$ works, though we will see that we need to make δ as small as possible for the results to be most meaningful. The following interpolation inequality can be derived

$$\|\phi_M\|_{H^{2+d/2+\delta}} \leq \ C \|\phi_M\|_{H^2}^{\frac{2\ell-2-d/2-\delta}{2(\ell-1)}} \cdot \|\phi_M\|_{H^{2\ell}}^{\frac{d/2+\delta}{2(\ell-1)}} \leq CC_{20}^{\frac{2\ell-2-d/2-\delta}{4(\ell-1)}} \|\phi_M\|_{H^{2\ell}}^{\frac{d/2+\delta}{2(\ell-1)}} . \tag{5.15}$$

To estimate $\|\phi_M\|_{H^{1+d/2+\delta}}$ we will need two cases. Case 1: $d = 1$. In this case, for small δ,

$$\|\phi_M\|_{H^{1+d/2+\delta}} \leq C \|\phi_M\|_{H^2} \leq CC_9. \tag{5.16}$$

Case 2: $d \geq 2$. In this case, for any $0 < \delta < 1$, $2 < 1 + d/2 + \delta \leq 2\ell$, and we have the interpolation inequality

$$\|\phi_M\|_{H^{1+d/2+\delta}} \leq C \|\phi_M\|_{H^2}^{\frac{2\ell-1-d/2-\delta}{2(\ell-1)}} \cdot \|\phi_M\|_{H^{2\ell}}^{\frac{d/2+\delta-1}{2(\ell-1)}} \leq CC_{20}^{\frac{2\ell-1-d/2-\delta}{4(\ell-1)}} \|\phi_M\|_{H^{2\ell}}^{\frac{d/2+\delta-1}{2(\ell-1)}} , \tag{5.17}$$

Since $2 < 3, 4 \leq 2\ell$, we also have

$$\|\phi_M\|_{H^3} \leq C \|\phi_M\|_{H^2}^{\frac{2(\ell-2)+1}{2(\ell-1)}} \cdot \|\phi_M\|_{H^{2\ell}}^{\frac{1}{2(\ell-1)}} \leq CC_{20}^{\frac{2(\ell-2)+1}{4(\ell-1)}} \|\phi_M\|_{H^{2\ell}}^{\frac{1}{2(\ell-1)}} , \tag{5.18}$$

$$\|\phi_M\|_{H^4} \leq C \|\phi_M\|_{H^2}^{\frac{2(\ell-2)}{2(\ell-1)}} \cdot \|\phi_M\|_{H^{2\ell}}^{\frac{2}{2(\ell-1)}} \leq CC_{20}^{\frac{2(\ell-2)}{4(\ell-1)}} \|\phi_M\|_{H^{2\ell}}^{\frac{2}{2(\ell-1)}} . \tag{5.19}$$

Substitution of these estimates into (5.12) leads to

$$\left\|\mathcal{N}_p^1(\phi)\right\|_2 = \left\|\Delta\left(\nabla\cdot\left(|\nabla\phi_M|^p\,\nabla\phi_M\right)\right)\right\|_2 \leq C \begin{cases} \|\phi_M\|_{H^{2\ell}}^{\frac{d/2+\delta+1}{\ell-1}} & \text{for} \quad d = 1 \\ \|\phi_M\|_{H^{2\ell}}^{\frac{p(d/2+\delta-1)+2}{2(\ell-1)}} & \text{for} \quad d \geq 2 \end{cases} \tag{5.20}$$

for all even integers p satisfying $2 \leq p \leq \wp - 2$. Therefore, for some $C_{21} > 0$, and for any $0 < \delta \leq 1$,

$$\|\mathcal{N}(\phi_M)\|_2 \leq C_{21} \left\|\Delta^\ell \phi\right\|_2^\beta , \tag{5.21}$$

where

$$\beta = \begin{cases} \frac{d/2+\delta+1}{\ell-1} & \text{for} \quad d = 1 \\ \frac{(\wp-2)(d/2+\delta-1)+2}{2(\ell-1)} & \text{for} \quad d \geq 2 \end{cases} , \tag{5.22}$$

where the following norm equivalence was applied in the last step:

$$\|\psi\|_{H^{2\ell}} \leq C \left\|\Delta^\ell \psi\right\|_2 = C \|\psi\|_{D(A^\ell)} , \tag{5.23}$$

for all $\psi \in D(A^\ell) = \mathring{H}^{2\ell}(\Omega)$. Finally, for some $C_{22} > 0$, we have

$$\left(\Delta^\ell \phi_M, \mathcal{N}(\phi_M)\right) \leq \left\|\Delta^\ell \phi_M\right\|_2 \cdot C_{21} \left\|\Delta^\ell \phi_M\right\|_2^\beta \leq C_{22} + \frac{\varepsilon^2}{4} \left\|\Delta^\ell \phi_M\right\|_2^2 , \tag{5.24}$$

provided $\beta < 1$. Observe that, if Condition 3 holds, there is always some $0 < \delta < 1$ that ensures that $\beta < 1$.

Finally, a combination of (5.3), (5.4), (5.11) and (5.24) results in

$$d_t \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi_M \right\|_2^2 + \frac{1}{2}\varepsilon^2 \left\| \Delta^\ell \phi_M \right\|_2^2 \le C_{23}. \tag{5.25}$$

Setting $y(t) := \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi_M \right\|_2^2$ and making use of the elliptic regularity

$$\left\| \nabla \Delta^{\frac{\ell-1}{2}} \psi_M \right\|^2 \le C \left\| \Delta^\ell \psi \right\|^2, \tag{5.26}$$

for every $\psi \in \mathring{H}^{2\ell}(\Omega)$, we arrive at

$$d_t y(t) + y(t) \le C_{24}, \tag{5.27}$$

where $C_{24} > 0$ is independent of $t \ge 0$. Integrating in time yields

$$y(t) = \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi_M(t) \right\|_2^2 \le \mathrm{e}^{-t} \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi_M(0) \right\|_2^2 + C_{24} \le \mathrm{e}^{-t} \left\| \nabla \Delta^{\frac{\ell-1}{2}} \phi(0) \right\|_2^2 + C_{24}. \tag{5.28}$$

Therefore, a global-in-time, uniform in M bound of $\left\| A^{\frac{\ell}{2}} \phi_M(t) \right\|$ is available. We can now extract a further subsequence of the Galerkin approximation and pass to limits to establish the bound for the weak solution, ϕ. The proof is complete. □

As a consequence of Theorem 4.6 and Theorem 5.3, we arrive at the following theorem, the main result of this paper.

Theorem 5.4. *Suppose that Conditions 1 – 3, $s = 1$, and $\phi^0 \in \mathring{H}^\ell_{\mathrm{per}}(\Omega) = D\left(A^{\ell/2}\right)$. Then there exists a unique global-in-time Gevrey regular solution to* (1.4).

Remark 5.5. For the local-in-time solution, the mapping $t \to e^{tA^{1/2}}\phi(t)$ is analytic within the time interval $(0, T_\star)$. Meanwhile, let us denote by $T_{\star\star}$ the Gevrey regularity solution existence time interval length, determined by Theorem 4.6, with $\left\| A^{\ell/2}\phi_0 \right\|_2 \le C_{19}$, where $C_{19} > 0$ is given in Theorem 5.3. After time $T_\star$, we can only ensure that the norm of $e^{T_{\star\star}A^{1/2}}\phi(t)$ is bounded; we cannot ensure, by the present theory, that the norm of $e^{tA^{1/2}}\phi(t)$ is bounded for large time.

Remark 5.6. Before we conclude, let us check that Conditions 1 – 3 are not so stringent as to exclude all of the interesting PDE's introduced earlier.

- For the Slope Selection (SS) epitaxial thin film growth model (1.7), we have the parameters, $s = 0$, $\ell = 2$ and $\wp = 4$ in $d = 2$. Condition 1 is easily satisfied. For the calculation of exponents, we are in Case 4, and $\sigma_2 = 1$. Thus Condition 2 is satisfied. Condition 3 is not applicable.
- One can envision an SS epitaxial growth model with $s = 0$, $\ell = 2$ and $\wp = 6$ in $d = 2$ [43]. Thus the highest nonlinear term is a 6-laplacian. Again, Conditions 1 is easily satisfied. For the calculation of exponents, we are again in Case 4, and $\sigma_2 = 1$. Conditions 1 and 2 are satisfied. Condition 3 is not applicable.
- For the regularized Cross-Newell (RCN) equation, the parameters are the same as the SS equation (in fact the equation is the same), but the dimension $d = 3$ may be appropriate. In this instance, Conditions 1 is satisfied. For the RCN equation with $d = 3$, $\sigma_2 = 3/2$ and Condition 2 is also satisfied. Here exponents are calculated according to Case 2. Condition 3 is not applicable.

- We can imagine an RCN-type equation with the following parameters: $s = 0$, $\ell = 2$, $\wp = 6$, and $d = 3$. The exponents are covered by Case 1, and $\sigma_2 = 2$. Unfortunately, our analysis does not cover this model, since Condition 2 fails to hold. Condition 3 is not applicable.
- For the SPFC model (1.8), we have the parameters $d = 3$, $s = 1$, $\ell = 3$ and $\wp = 4$. Condition 1 is satisfied. The exponents are covered by Case 3, and $\sigma_2 = 4/3$, showing that Condition 2 is once again satisfied. Condition 3 is also satisfied, since

$$(\wp - 2)(d/2 - 1) + 4 = 5 < 6 = 2\ell.$$

Remark 5.7. For a gradient flow with the Allen-Cahn/Cahn-Hilliard type energy

$$E(\phi) = \frac{1}{4}\|\phi\|_4^4 - \frac{1}{2}\|\phi\|_2^2 + \frac{\varepsilon^2}{2}\|\nabla\phi\|_2^2, \tag{5.29}$$

we see that the global-in-time Gevrey regularity solution could be derived in the same manner, based on the fact that the degree of nonlinearity associated with $\|\phi\|_4^4$ is much lower than that of $\|\nabla\phi\|_4^4$.

On the other hand, the Gevrey regularity for the Cahn-Hilliard equation has already been proved in an existing work [36].

Remark 5.8. For the sake of comparison with the Slope Selection (SS) model (1.7), the energy for the No-Slope-Selection (NSS) epitaxial thin film growth model is given by

$$E(\phi) = \int_\Omega \left(-\frac{1}{2}\ln(1 + |\nabla\phi|^2) + \frac{\varepsilon^2}{2}(\Delta\phi)^2 \right) d\mathbf{x}, \tag{5.30}$$

which includes a logarithmic term. The L^2 gradient flow with respect to this energy is

$$\partial_t\phi = -\mu = -\delta_\phi E = -\nabla \cdot \left(\frac{\nabla\phi}{1 + |\nabla\phi|^2} \right) - \varepsilon^2\Delta^2\phi. \tag{5.31}$$

For the NSS model (5.31), the existence of a global-in-time smooth solution has been established in [30]. However, the framework to establish the Gevrey regularity solution, as presented in this article, can not be directly applied to this problem. The primary difficulty derives from the fact that the preliminary estimate Lemma 4.2 is not available for this gradient flow, since the nonlinear term in (5.31) is not in a polynomial pattern; instead, the nonlinear denominator makes a Fourier-type-analysis not feasible any more.

The analysis of the analytic solution for the NSS model (5.31) will be explored in a future work. The techniques related to the analyticity radius for nonlinear parabolic equations in a bounded domain, as reported by [4, 17, 22, 23, 24], are expected to be useful for this work.

Acknowledgments. The authors greatly appreciate many helpful discussions with Xiaoming Wang and Edriss Titi, in particular for their insightful suggestions and comments. This work is supported in part by the NSF DMS-1418689 (C. Wang), NSFC 11271281 (C. Wang), NSF-DMS 1418692 (S. Wise). C. Wang also thanks Shanghai Key Laboratory for Contemporary Applied Mathematics, Fudan University, for support during his visit.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Biswas and D. Swanson, Existence and generalized Gevrey regularity of solutions to the Kuramoto-Sivashinsky equation in R^n, *J. Differential Equations*, **240** (2007), 145–163.

[3] A. Biswas and D. Swanson, Gevrey regularity of solutions to the 3-D Navier-Stokes equations with weighted ℓ_p initial data, *Indiana Univ. Math. J.*, **56** (2007), 1157–1188.

[4] Z. Bradshaw, Z. Grujic and I. Kukavica, Local analyticity radii of solutions to the 3d Navier-Stokes equations with locally analytic forcing, *J. Differential Equations*, **259** (2015), 3955–3975.

[5] C. Cao, M. Rammaha and E. Titi, Gevrey regularity for nonlinear analytic parabolic equations on the sphere, *J. Dynam. Differential Equations*, **12** (2000), 411–433.

[6] W. Chen, S. Conde, C. Wang, X. Wang and S. M. Wise, A linear energy stable scheme for a thin film model without slope selection, *J. Sci. Comput.*, **52** (2012), 546–562.

[7] W. Chen, C. Wang, X. Wang and S. M. Wise, A linear iteration algorithm for a second-order energy stable scheme for a thin film model without slope selection, *J. Sci. Comput.*, **59** (2014), 574–601.

[8] M. C. Cross and A. C. Newell, Convection patterns in large aspect ratio systems, *Physica D*, **10** (1984), 299–328.

[9] A. Eden and V. Kalantarov, The convective Cahn-Hilliard equation, *Appl. Math. Lett.*, **20** (2007), 455–461.

[10] K. R. Elder, M. Katakowski, M. Haataja and M. Grant, Modeling elasticity in crystal growth, *Phys. Rev. Lett.*, **88** (2002), 245701.

[11] K. R. Elder, M. Katakowski, M. Haataja and M. Grant, Modeling elastic and plastic deformations in nonequilibrium processing using phase field crystals, *Phys. Rev. E*, **70** (2004), 051605.

[12] A. Ferrari and E. Titi, Gevrey regularity for nonlinear analytic parabolic equations, *Comm. Partial Differential Equations*, **23** (1998), 1–16.

[13] C. Foias and R. Temam, Gevrey class regularity for the solution of the Navier-Stokes equations, *J. Funct. Anal.*, **87** (1989), 359–369.

[14] K. B. Glasner, Grain boundary motion arising from the gradient flow of the Aviles-Giga functional, *Physica D*, **215** (2006), 80–98.

[15] A. A. Golovin and A. A. Nepomnyashchy, Disclinations in square and hexagonal patterns, *Phys. Rev. E*, **67** (2003), 056202, 7pp.

[16] Z. Grujic and I. Kukavica, Space analyticity for the Navier-Stokes and related equations with initial data in L^p, *J. Funct. Anal.*, **152** (1998), 447–466.

[17] Z. Grujic and I. Kukavica, Space analyticity for the nonlinear heat equation in a bounded domain, *J. Differential Equations*, **154** (1999), 42–54.

[18] V. Kalantarov, B. Levant and E. Titi, Gevrey regularity for the attractor of the 3D Navier-Stokes-Voight equations, *J. Nonlinear Sci.*, **19** (2009), 133–152.

[19] R. V. Kohn and F. Otto, Upper bound on coarsening rate, *Commun. Math. Phys.*, **229** (2002), 375–395.

[20] R. V. Kohn and X. Yan, Upper bound on the coarsening rate for an epitaxial growth model, *Comm. Pure Appl. Math.*, **56** (2003), 1549–1564.

[21] J. Krug, Four lectures on the physics of crystal growth, *Physica A*, **313** (2002), 47–82.

[22] I. Kukavica, R. Temam, V. Vlad, and M. Ziane, On the time analyticity radius of the solutions of the two-dimensional Navier-Stokes equations, *J. Dynam. Differential Equations*, **3** (1991), 611–618.

[23] I. Kukavica, R. Temam, V. Vlad and M. Ziane, Existence and uniqueness of solutions for the hydrostatic Euler equations on a bounded domain with analytic data, *C. R. Math. Acad. Sci. Paris*, **348** (2010), 639–645.

[24] I. Kukavica and V. Vlad, On the radius of analyticity of solutions to the three-dimensional Euler equations, *Proc. Amer. Math. Soc.*, **137** (2009), 669–677.

[25] I. Kukavica and V. Vlad, The domain of analyticity of solutions to the three-dimensional Euler equations in a half space, *Discrete Contin. Dyn. Syst.*, **29** (2011), 285–303.

[26] I. Kukavica and V. Vlad, On the analyticity and Gevrey-class regularity up to the boundary for the Euler equations, *Nonlinearity*, **24** (2011), 765–796.

[27] I. Kukavica and V. Vlad, On the local existence of analytic solutions to the Prandtl boundary layer equations, *Commun. Math. Sci.*, **11** (2013), 269–292.

[28] A. Levandovsky and L. Golubovic, Epitaxial growth and erosion on (001) crystal surfaces: Far-from-equilibrium transitions, *Phys. Rev. B*, **65** (2004), 241402.

[29] A. Levandovsky, L. Golubovic and D. Moldovan, Interfacial states and far-from-equilibrium transitions in the epitaxial growth and erosion on (110) crystal surfaces, *Phys. Rev. E*, **74** (2006), 061601.

[30] B. Li and J. G. Liu, Thin film epitaxy with or without slope selection, *Euro. J. Appl. Math.*, **14** (2003), 713–743.

[31] B. Li and J. G. Liu, Epitaxial growth without slope selection: Energetics, coarsening, and dynamic scaling, *J. Nonlinear Sci.*, **14** (2004), 429–451.

[32] J. S. Lowengrub, E. Titi and K. Zhao, Analysis of a mixture model of tumor growth, *Euro. J. Appl. Math.*, **24** (2013), 691–734.

[33] H. Ly and E. Titi, Global Gevrey regularity for the bénard convection in a porous medium with zero Darcy-Prandtl number, *J. Nonlinear Sci.*, **9** (1999), 333–362.

[34] D. Moldovan and L. Golubovic, Interfacial coarsening dynamics in epitaxial growth with slope selection, *Phys. Rev. E*, **61** (2000), 6190.

[35] M. Ortiz, E. A. Repetto and H. Si, A continuum model of kinetic roughening and coarsening in thin films, *J. Mech. Phys. Solids*, **47** (1999), 697–730.

[36] K. Promislow, Time analyticity and Gevrey regularity for solutions of a class of dissipative partial differential equations, *Nonlinear Anal.*, **16** (1991), 959–980.

[37] N. Provatas, J. A. Dantzig, B. Athreya, P. Chan, P. Stefanovic, N. Goldenfeld and K. R. Elder, Using the phase-field crystal method in the multiscale modeling of microstructure evolution, *JOM*, **59** (2007), p83.

[38] J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, Cambridge, UK, 2001.

[39] J. Shen, C. Wang, X. Wang and S. M. Wise, Second-order convex splitting schemes for gradient flows with Ehrlich-Schwoebel type energy: Application to thin film epitaxy, *SIAM J. Numer. Anal.*, **50** (2012), 105–125.

[40] D. Swanson, Gevrey regularity of certain solutions to the Cahn-Hilliard equation with rough initial data, *Methods Appl. Anal.*, **18** (2011), 417–426.

[41] J. Swift and P. C. Hohenberg, Hydrodynamic fluctuations at the convective instability, *Phys. Rev. A*, **15** (1977), p319.

[42] C. Wang, X. Wang and S. M. Wise, Unconditionally stable schemes for equations of thin film epitaxy, *Discrete Contin. Dyn. Sys. A*, **28** (2010), 405–423.

[43] K. A. Wu, M. Plapp and P. W. Voorhees, Controlling crystal symmetries in phase-field crystal models, *J. Phys.: Condensed Matter*, **22** (2010), 364102.

[44] C. Xu and T. Tang, Stability analysis of large time-stepping methods for epitaxial growth models, *SIAM J. Numer. Anal.*, **44** (2006), 1759–1779.

Received August 2015; revised March 2016.

E-mail address: chennan@cims.nyu.edu
E-mail address: cwang1@umassd.edu
E-mail address: swise1@utk.edu