Mathematics Department, University of Massachusetts Dartmouth

Discrete Mathemtics II

MTH182 – Section 03 – Spring 2015

Problem set 5

Bijective Functions and cardinality

Reading:	Discrete Mathematics, first edition, section Sections 5.4, 5.5
	Section 5.4, 1, 3, 5, 7, 9, 11, 13, 19, 21, 29, 31, 35, 37
	Section 5.5, 1, 3, 9

Section 5.4

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5\}$. Given an example of
- (**a**) a function $f : A \to B$ that is not one-to-one but where $f(1) \neq f(2)$
- (**b**) a one-to-one function $g : A \to B$ where $g(1) \neq 3 \neq g(2)$
- (**c**) a one-to-one function $h : A \to B$ such that $h(n) > n$ for every $n \in A$

3. Let $A = \{a, b, c, d\}$ and $B = \{a, b, c\}$. Give an example of
- (**a**) a function $f : A \to B$ that is not onto but where $f(a) \neq f(b)$
- (**b**) a function $g : A \to B$ that is onto, but where $g(a) = g(b)$
- (**c**) a function $h : A \to B$ that is onto but $h(x) \neq x$ for each $x \in B$

5. Determine, with explanation, whether the following functions are onto.
- (**a**) $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 4n + 1$ for $n \in \mathbb{Z}$
- (**b**) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $f(x) = x^2$ for $x \in \mathbb{R}^+$
- (**c**) $f : \mathbb{R} \to \mathbb{Z}$ is defined by $f(x) = \lceil x \rceil$ for $x \in \mathbb{R}$
- (**d**) $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2 + 2x + 1$ for $x \in \mathbb{R}$.

7. Let $A = \{a, b, c, d\}$ and $B = \{q, r, s, t\}$. Give an example, if such an example exists, of a function $f : A \to B$ that is
- (**a**) one-to-one but not onto
- (**b**) onto but not one-to-one
- (**c**) one-to-one and onto
- (**d**) neither one-to-one nor onto

9. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Does there exist a function from A to B that is onto?

11. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 2n$. Determine with explanation whether f is (a) one-to-one, (b) onto.

13. A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 5n + 1$. Determine with explanation whether f is (a) one-to-one, (b) onto.

19. Each of the following is a function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. Which of these are one-to-one?

(**a**) $f(a, b) = 2a + b$

(**b**) $f(a, b) = b$

(**c**) $f(a, b) = a^2 + b^2$

(**d**) $f(a, b) = a^b$

(**e**) $f(a, b) = 2^a 3^b$

21. Let $f : A \to B$ be a function for nonempty sets A and B. Prove or disprove the following.

(**a**) If, for every two nonempty subsets A_1 and A_2 of A, $f(A_1) = f(A_2)$ implies that $A_1 = A + 2$, then f is one-to-one.

(**b**) If, for every two nonempty subset B_1 of B, there exists a subset A_1 of A such that $f(A_1) = B_1$, then f is onto.

29. For $A = \{a, b, c\}$ and $B = \{w, x, y, z\}$, let $f : A \to B$ be the function defined by $f(a) = y$, $f(b) = z$, and $f(c) = w$. Determine, with explanation

(**a**) whether f is one-to-one

(**b**) whether f has an inverse function from B to A

31. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x - 7$ is bijective and determine $f^{(-1)}(x)$ for $x \in \mathbb{R}$.

35. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and $C = \{x, y, z\}$. The functions $f : A \to B$ and $g : B \to C$ defined by $f = \{(1, c), (2, a), (3, b)\}$ and $g = \{(a, y), (b, z), (c, x)\}$ are bijective.

(**a**) Determine $g \circ f : A \to C$ and $(g \circ f)^{-1} : C \to A$

(**b**) Determine $f^{-1} : B \to A$, $g^{-1} : C \to B$, and $f^{-1} \circ g^{-1} : C \to A$

37. Give an example of a finite set S and a bijection $f : S \to S$ such that all of the elements $f(a)$, $(f \circ f)(a)$, $f^{-1}(a)$ are distinct for each $a \in S$.

Section 5.5

1. Let E denote the set of even integers and O denote the set of odd integers. Show that $|E| = |O|$.

3. Let A denote a denumerable set and let B be a nonempty finite set. Show that if A and B are disjoint, then $A \cup B$ is a denumerable set.

9. Prove or disprove: If A is a nonempty subset of a denumerable set, then A is denumerable.