Introduction
The tear film on the front of the cornea plays important role for eyes to function properly. Although it is studied extensively, the dynamics of the tear film during a blink is still not fully understood quantitatively. An interferogram of the tear film on a contact lens after a half blink is shown.

How do we simulate the dynamics of the tear film? Although two dimensional model is the final goal, one may get insight from one dimensional case.

Model Formulation
Braun et al 2007 [1] has modeled a single nonlinear partial differential equation (PDE) that governs film thickness over multiple blink cycle from lubrication. A sketch of a cross section of human eye and its one dimensional mathematical model are shown in figure below.

with \(x, y\) and \(v, u\) are respectively coordinate directions along and velocity components perpendicular to the flat surface representing the corneal surface.

### Constants
- \(L = 5 \text{ mm}\)
- \(d = 5 \mu \text{m}\)
- \(x = 10^{-5}\)
- \(d_{t} = 10^{3}\)
- \(x_{t} = 10^{-3}\)
- \(y_{t} = 10^{-5}\)
- \(\mu = 10^{-5} \text{ Pa s}\)
- \(\rho = 10^{4} \text{ kg/m}^3\)

Starting with Navier-Stokes equation, non-dimensionalization results in:

- Viscous incompressible parallel flow inside the film.
- On the impermeable wall at \( y = 0 \), we have the boundary conditions
  \[ n = 0, \quad n = n_{0}\] the first condition is impermeability and the second is the Navier slip condition where \( 10^{-1} \leq \beta \leq 10^{-3}\).
- Inertial terms and gravity are neglected.
- Simplified normal stress condition
  \[ p = -\frac{\partial h_{0}}{\partial x} = -

Spectral collocation method
We transform the PDE into fixed domain \([-1, 1]\). The equations become:

\[
\begin{align*}
H_{t} + \frac{x}{2} H_{t} - 2 x H_{x} + \frac{2}{\lambda} S H_{t} + H_{x} = 0, \quad \lambda > 0, \\
H(\pm 1, t) = h_{\pm}, \quad Q(\pm 1, t) = 0, \\
\lambda H(0, t) = h_{0} + \left(h_{0} - h_{\pm}\right) e^{-t}.
\end{align*}
\]

When \( h_{0} = 0 \) and \( h_{\pm} = 0 \), we call \( \lambda = 1 \) as \( \lambda H(0, t) = 0 \). To measure accuracy, conservation of volume during blink cycle with respect to initial volume is used. With spectral collocation methods, the error in volume conservation is around \( 10^{-6} \).

Full Blink Results
We use parameters \( \lambda = 1, \beta = 10^{-5}, S = 2 \times 10^{-3}, h_{\pm} = 0.1, h_{0} = 0.6, \) and initial volume \( V_{c} = 2.576 \). FFLM is used as flux condition. Our simulation is done in MATLAB with ode15s as ODE solver. The number of grid points is 321.

### References


To measure accuracy, conservation of volume during blink cycle with respect to initial volume is used. With spectral collocation methods, the error in volume conservation is around \( 10^{-6} \).

Partial Blink Results
Numerical computations also show a distinct similarity to in vivo observations of the tear film under partial blink conditions. The ULA simulation begins with a 10% open film (\( \lambda = 0.1\)), fully opens, and then repeats opening and closing with \( \lambda = 0.5\). This mimics the upper lid sequence of figure given in the introduction section. We use parameters \( \lambda = 2 \times 10^{-3}, \beta = 10^{-5}, h_{0} = 0.4, \) and initial volume \( V_{c} = 1.576\). The number of grid points is 381. In addition to FFLM, additional term from lacrimal gland supply and punctal drainage is used. Comparison with experimental data for various times is given in the figure below.