Numerical Simulation of Tear Film in a Blink Cycle using Spectral Collocation Methods

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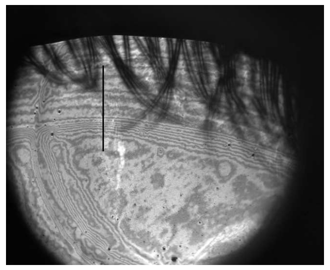
Introduction

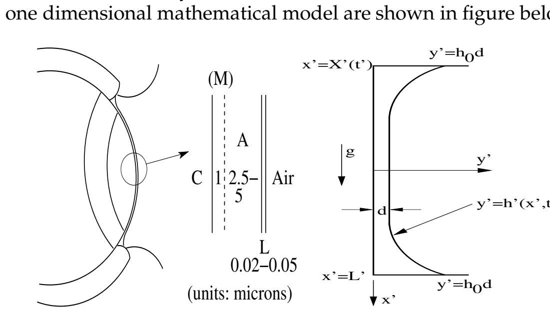
The tear film on the front of the cornea plays important role for eyes to function properly. Although it is studied extensively, the dynamics of the tear film during a blink is still not fully understood quantitatively. An interferogram of the tear film on a contact lens after a half blink is shown.



How do we simulate the dynamics of the tear film ? Although two dimensional model is the final goal, one may get insight from one dimensional case.

Model Formulation

Braun et al 2007 [1] has modeled a single nonlinear partial differential equation (PDE) that governs film thickness over multiple blink cycle from lubrication theory. A sketch of a cross section of human eye and its one dimensional mathematical model are shown in figure below,



with (x, y) and (u, v) are respectively coordinate directions along and velocity components perpendicular to the flat surface representing the corneal surface.

Constants	Description
$L' = 5$ mm	half the width of the palpebral fissure (x direction)
$d = 5\ \mu$m	thickness of the tear film away from ends
$\epsilon = \frac{d}{L'} \approx 10^{-3}$	small parameter for lubrication theory
$U_m = 10$–30c m/s	maximum speed across the film
$L'/U_m = 0.05$ s	time scale for real blink speeds
$\sigma_0 = 45$ mN/m	surface tension
$\mu = 10^{-3}$ Pa·s	viscosity
$\rho = 10^3$ kg/m^3	density

Starting with Navier-Stokes equation, non-dimensionalization results in:

- Viscous incompressible parallel flow inside the film.
- On the impermeable wall at $y = 0$, we have the boundary conditions

$$v = 0, \quad u = \beta u_y;$$

  the first condition is impermeability and the second is the Navier slip condition where $10^{-3} \le \beta \le 10^{-2}$.
- Inertial terms and gravity are neglected.
- Simplified normal stress condition

$$p = -Sh_{xx}, \quad S = \frac{\epsilon^3 \sigma}{\mu U_m}$$

- Kinematic condition.

The free surface evolution is given by

$$h_t + q_x = 0 \text{ on } X(t) \le x \le 1.$$

where

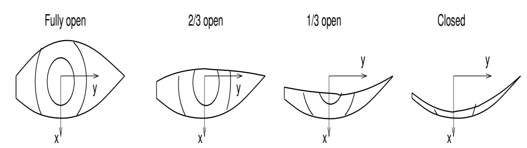
$$q = \int_0^h u(x, y, t)dy$$

is the flux of fluid across any cross-section of the film and $X(t)$ represents movement of the upper eyelid. For a strong insoluble surfactant, we can find one equation limiting case called the uniform stretching limit (USL). In the (USL), we obtain

$$q(x,t) = \frac{h^3}{12}\left(1 + \frac{3\beta}{h+\beta}\right)(Sh_{xxx}) + X_t \frac{1-x}{1-X}\frac{h}{2}\left(1 + \frac{\beta}{h+\beta}\right)$$

Realistic lid motion and Flux BCs

We assume that the bottom lid is fixed and the top lid is moving.



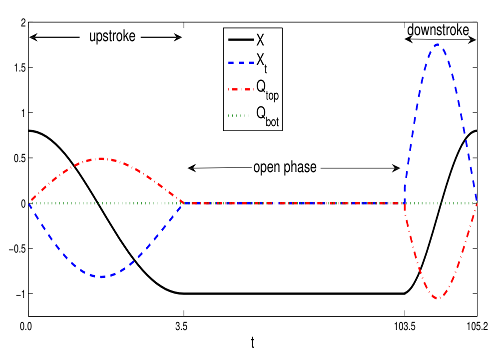
Referring to Doane[2] and to Berke and Mueller[3], the time-varying domain that mimics realistic movement of upper eyelid is modeled by

$$X(t) = \begin{cases} 1 - 2\lambda - 2(1-\lambda)\left(\frac{t}{\Delta t_{co}}\right)^2 \exp\left[1 - \left(\frac{t}{\Delta t_{co}}\right)^2\right], & t \in I_u \\ -1, & t \in I_o \\ -1 + 2(1-\lambda)\left(\frac{t-\Delta t_{co}-\Delta t_o}{\Delta t_{oc}}\right)^2 \exp\left[1 - \left(\frac{t-\Delta t_{co}-\Delta t_o}{\Delta t_{oc}}\right)^2\right], & t \in I_d \end{cases}$$

where $\Delta t_{bc} = \Delta t_{co} + \Delta t_o + \Delta t_{oc}$ is the nondimensional period of the complete blink cycle, $\Delta t_{co} = 3.52$, $\Delta t_{oc} = 1.64$ and $\Delta t_o = 100$.
We also use two flux functions:

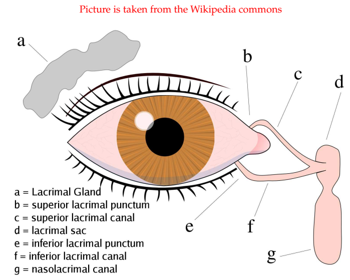
- Flux from exposing layer under lids of thickness h_e by Jones et al[4]

$$Q_{top} = -X_t h_e, \quad Q_{bot} = 0$$



  We call Q_{top} to be flux proportional to lid motion (FPLM).
- Add in lacrimal gland supply and punctal drainage approximated by Gaussians.

Picture is taken from the Wikipedia commons



Spectral collocation method

We transform the PDE into fix domain $[-1, 1]$. The equations become

$$H_t = \frac{1-\xi}{L-X} X_t H_\xi - \left(\frac{2}{L-X}\right) Q_\xi$$
$$Q = S\left(\frac{2}{L-X}\right)^3 \left(\frac{H^3}{3} + \beta H^2\right) H_{\xi\xi\xi}$$
$$H(\pm 1, t) = h_0,\ Q(1,t) = 0,\ Q(-1,t) = X_t h_0$$
$$H(\xi, 0) = h_m + (h_0 - h_m)\xi^m.$$

The interval $\xi \in [-1, 1]$ is discretized using stretched Chebyshev points found by using Kosloff and Tal-Ezer mapping[5] and solutions H and Q are interpolated with Nth-order polynomials there. In discretized forms, H and Q are each $N + 1$ dimensional vectors

$$H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{N-1} \\ H_N \end{bmatrix} \qquad Q = \begin{bmatrix} Q_0 \\ Q_1 \\ \vdots \\ Q_{N-1} \\ Q_N \end{bmatrix}. \tag{0.1}$$

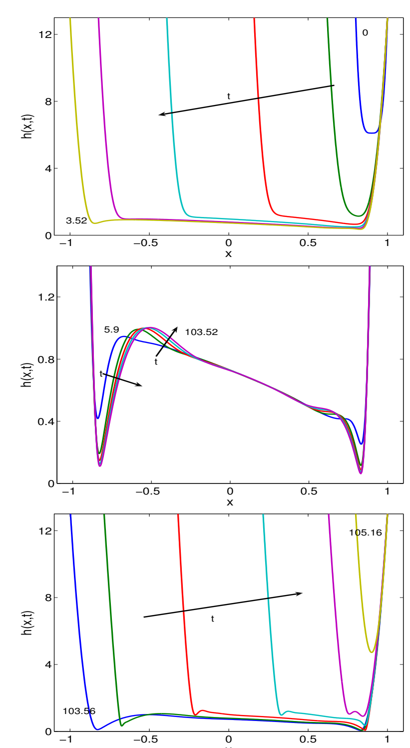
Approximation of kth partial derivatives of $H(\xi, t)$ and $Q(\xi, t)$ with respect to ξ can be written as vectors

$$H^{(k)} = D^{(k)} H \text{ and } Q^{(k)} = D^{(k)} Q \tag{0.2}$$

respectively and $D^{(k)}$ is the kth-order spectral differentiation matrix. H_0, H_N, Q_0 and Q_N are known from boundary conditions and hence only values of H at inner nodes ξ_i, where $i = 1, .., N-1$, need to be found.

Full Blink Results

We use parameters $\lambda = 0.1$, $\beta = 10^{-2}$, $S = 2 \times 10^{-5}$, $h_0 = 13$, $h_e = 0.6$, and initial volume $V_0 = 2.576$. FPLM is used as flux condition. Our simulation is done in MATLAB with ode15s as ODE solver. The number of grid points is 321.

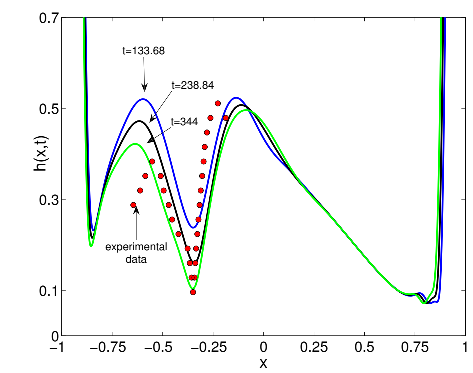


Top: Opening phase of the blink cycle. **Middle:** Zoom in the vertical direction for various times while the domain is fully open. **Bottom:** The closing phase of a blink cycle.

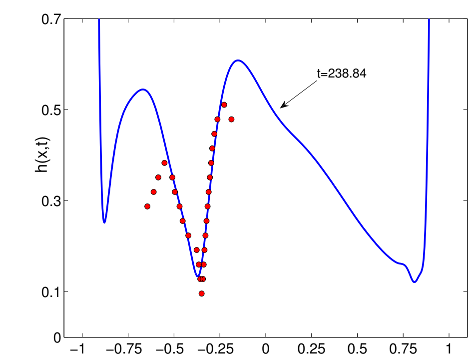
To measure accuracy, conservation of volume during blink cycle with respect to initial volume is used. With spectral collocation methods, the error in volume conservation is around 10^{-5}.

Partial Blink Results

Numerical computations also show a distinct similarity to in vivo observations of the tear film under partial blink conditions. The USL simulation begins with a 10% open film ($\lambda = 0.1$), fully opens, and then repeats opening and closing with $\lambda = 0.5$. This mimics the upper lid sequence of figure given in the introduction section. We use parameters $S = 2 \times 10^{-5}$, $\beta = 10^{-3}$, $h_e = 0.4$, and initial volume $V_0 = 1.576$. The number of grid points is 381. In addition to FPLM, additional term from lacrimal gland supply and punctal drainage is used. Comparison with experimental data for various times is given in the figure below.



$S = 2 \times 10^{-5}$ is too large for eyes but is an intermediate value for the computational range. Recently, we have been able to compute results for the more realistic value of 2×10^{-7}. Figure below is result with the same parameters as previous figure except with smaller $S = 8 \times 10^{-6}$.



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