Single-Equation Models for the Tear Film in a Blink Cycle with Realistic Lid Motion

A. Heryudono¹, R.J. Braun¹, T.A. Driscoll¹, K.L. Maki ¹, L.P. Cook¹, and P.E. King-Smith²

¹Mathematical Sciences, U of Delaware
and ²College of Optometry, Ohio State U

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How do we simulate the dynamics of the tear film?

Interference fringes.

(Units: microns)

\[ \epsilon = \frac{d}{L} \text{ is small} \rightarrow \text{Lubrication theory} \]
Inside the film

- Viscous incompressible parallel flow inside the film.
- Inertial terms and gravity are neglected.

At the impermeable wall \( y = 0 \)

- \( \nu = 0, \quad u = \beta u_y; \)

At the free surface \( y = h(x, t) \)

- Simplified stress conditions

\[
p = -Sh_{xx}, \quad S = \frac{\epsilon^3 \sigma}{\mu U_m}, \quad u^{(s)} = X_t \frac{1-x}{1-X}
\]

Kinematic condition

<table>
<thead>
<tr>
<th>Constants</th>
<th>Description</th>
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<tbody>
<tr>
<td>( L' = 5 \text{ mm} )</td>
<td>half the width of the palpebral fissure (( x ) direction)</td>
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<tr>
<td>( d = 5 \text{ ( \mu )m} )</td>
<td>thickness of the tear film away from ends</td>
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<tr>
<td>( \epsilon = \frac{d}{L'} \approx 10^{-3} )</td>
<td>small parameter for lubrication theory</td>
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<td>( U_m = 10-30 \text{ cm/s} )</td>
<td>maximum speed across the film</td>
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<td>( L'/U_m = 0.05 \text{ s} )</td>
<td>time scale for real blink speeds</td>
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<td>( \sigma_0 = 45 \text{ mN/m} )</td>
<td>surface tension</td>
</tr>
<tr>
<td>( \mu = 10^{-3} \text{ Pa\cdot s} )</td>
<td>viscosity</td>
</tr>
<tr>
<td>( \rho = 10^3 \text{ kg/m}^3 )</td>
<td>density</td>
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</tbody>
</table>
$$h_t + q_x = 0 \text{ on } X(t) \leq x \leq 1,$$

where

$$q = \int_0^h u(x, y, t) dy$$

- The uniform stretching limit (USL).

$$q(x, t) = \frac{h^3}{12} \left( 1 + \frac{3\beta}{h + \beta} \right) (Sh_{xxx}) + X_t \frac{1-x}{1-X} \frac{h}{2} \left( 1 + \frac{\beta}{h + \beta} \right)$$

**Boundary conditions**

$$h(X(t), t) = h(1, t) = h_0 \quad q(X(t), t) = X_t h_0 + Q_{top} \quad q(1, t) = -Q_{bot}.$$ 

**Initial condition**

**Polynomial function**
We depart from Berke and Mueller (98) ⇒ Heryudono et al (07)

Flux proportional to lid motion (FPLM) (Jones et al (05))

\[ Q_{top} = -X_t h_e, \quad Q_{bot} = 0 \]
Add in lacrimal gland supply and punctal drainage approximated by Gaussians.
We transform the PDE into a fixed domain $\xi \in [-1, 1]$ via

$$\xi = 1 - 2 \frac{1 - x}{1 - X(t)}.$$ 

The equations become

$$H_t = \frac{1 - \xi}{L - X} X_t H_{\xi} - \left( \frac{2}{L - X} \right) Q_{\xi}$$

$$Q = X_t \frac{1 - \xi}{2} \frac{H}{2} \left( 1 + \frac{\beta}{H + \beta} \right) + \frac{H^3}{12} \left( 1 + \frac{3\beta}{H + \beta} \right) \left[ S \left( \frac{2}{1 - X} \right)^3 H_{\xi\xi\xi} \right]$$

$$H(\pm 1, t) = h_0, \quad Q(-1, t) = X_t h_0 + Q_{\text{top}}, \quad Q(1, t) = -Q_{\text{bot}}, \quad (\text{BCs})$$

$$H(\xi, 0) = h_m + (h_0 - h_m)\xi^m \quad (\text{IC}).$$

$\Rightarrow$ Spectral discretization in space and standard ODE in time.
Use spectral collocation method.

- Map Chebyshev points technique Kosloff & Tal-Ezer (93)
- Use mapping parameter by Don & Solomonoff (97) to reduce roundoff errors near end points.

Imposing boundary conditions.

- Set $Q$

$$Q = X_t \frac{1 - \xi}{2} H_2 \left( 1 + \frac{\beta}{H + \beta} \right) + \frac{H^3}{12} \left( 1 + \frac{3\beta}{H + \beta} \right) \left[ S \left( \frac{2}{1 - X} \right)^3 H_{\xi\xi\xi} \right]$$

- When computing $Q_{\xi}$, overwrite its end values with $Q(-1, t) = X_t h_0 + Q_{top}, \quad Q(1, t) = -Q_{bot}$.

- Solve the initial value problem at inner nodes.

$$H_t = \frac{1 - \xi}{1 - X} X_t H_\xi - \left( \frac{2}{1 - X} \right) Q_\xi$$

with ode solver ode15s.
Parameters $N = 351, \lambda = 0.1, \beta = 10^{-2}, S = 2 \times 10^{-5}, h_0 = 13, h_e = 0.6,$ and initial volume $V_0 = 2.576$. Our simulation is done in MATLAB with ode15s as ODE solver.

Opening phase

fully open (zoom)

The closing phase
Numerical results

Partial blinks results: FPLM + Gaussians

$S = 2 \times 10^{-5}$

$S = 8 \times 10^{-6}$

Experimental data

Conservation of volume

Heryudono et al

Single-Equation models for the Tear Film
SUMMARY

- 1-D simulation of the tear film in a blink cycle.
- Good fit with experimental data for partial blink simulation with FPLM+ type fluxes.
- Use spectral methods for getting higher accuracy solutions.