



Lecture 9

MT4572/MT4472
Numerical Methods for PDEs
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Main references (quotes):

Trefethen: NumPDE, ATAP, Spectral Methods in MATLAB

Fornberg: PS Guide

Leveque: NumPDE

Driscoll: Learning MATLAB

Order Stars (Wanner, Hairer, Norsett (1978))

Order stars can be used to help explaining the global and local properties of the LMM method defined by $R(z) = \rho(z)/\sigma(z)$ by way of geometric arguments. For this purpose,

- ▶ The global property is the root condition stability: The zeros of $\rho(z)$ must lie in the unit disk.
- ▶ The local property is the order of accuracy p .

The key idea is to look at the level lines of the function

$$\phi(z) = \frac{\rho(z)}{\sigma(z)} - \log(z)$$

The **order stars** of the linear multistep formula is defined by

$$\mathcal{A} = \{z \in \mathbb{C} : \operatorname{Re}\phi(z) > 0\}$$

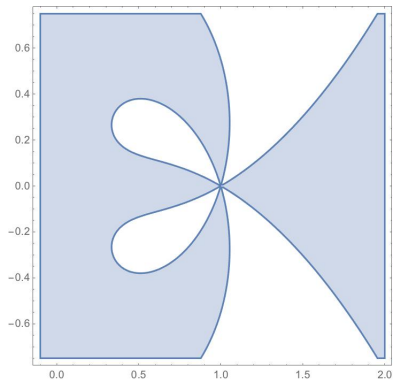
That definition is also sometimes called order stars of the second kind.

Plotting Order Stars

Mathematica or MATLAB can be easily used to plot order-stars. Let's plot the order stars of AB^3 .

$$R[z_] := (z^3 - z^2)/((23/12) z^2 - (16/12) z + (5/12))$$

```
RegionPlot[ Re[R[x + I*y] - Log[x + I*y]] > 0, {x, -1/10, 2},  
           {y, -3/4, 3/4}, PlotPoints -> {100, 100}]
```



- ▶ Accuracy: Near $z = 1$, A must look like a daisy with $p + 1$ evenly spaced petals or "fingers". Fingers can be bounded or unbounded. In our particular case, the number of fingers is 4.
- ▶ For stability: Any bounded finger of A must contain one of the zeros of p . Consequence: Every bounded finger has to intersect the unit disk.

Plotting Order Stars

It is sometimes useful to do a transformation $z = e^{(w-1)}$ and then invert ϕ and look at the level lines of

$$\phi(w) = \frac{\sigma(w-1)}{\rho(w-1)} - \frac{1}{(w-1)}$$

to get a better picture of the bounded fingers. Note that the zeros of ρ is now becoming the poles of the denominators. Plotting the order stars of BDF7 and unit disk on the same figures gives.

```
R[z_] := (z^7 - (980/363) z^6 + (490/121) z^5 - (4900/1089) z^4 + (1225/363) z^3 - (196/121) z^2 + (490/1089) z - (60/1089))/((140/363) z^7)
```

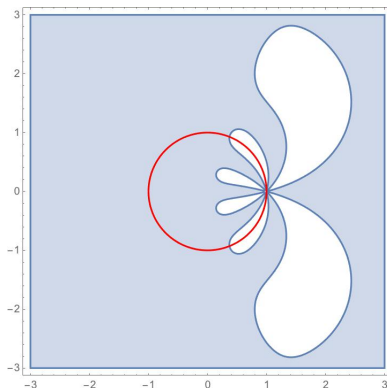
```
g1 := RegionPlot[  
  Re[1/R[Exp[x + I*y-1]] - 1/(x + I*y-1)] > 0, {x, -2, 2}, {y, -2, 2},  
  PlotPoints -> {100, 100}]
```

```
g2 := ParametricPlot[{Cos[t], Sin[t]}, {t, 0, 2*Pi}, PlotStyle -> Red]
```

```
Show[g1, g2]
```

Plotting Order Stars of BDF7

Plot of the order stars of BDF7 where the root condition is violated. See order stars book by Iserles and Norsett.



Little exercise in class. Plot the order stars of the following LMM methods

- ▶ AB4
- ▶ AM4
- ▶ BDF4
- ▶ The hybrid Trapezoid and Midpoint method (the convex linear combination with $\alpha = 4/5$) from the previous lecture.

Second Question

If the time step $k > 0$ is held fixed, do the computed values $v(t)$ remain bounded as $t \rightarrow \infty$?

As a simpler model, let's take a look at a simple linear ODE with constant coefficient, where $f(u(t), t)$ is just $au(t)$ where a is a constant. For simplicity (by dropping the arguments)

$$u_t = au$$

The characteristic polynomials of the linear multistep method simply becomes:

$$\rho(z)v^n - k\sigma(z)av^n = (\rho(z) - (ka)\sigma(z))v^n = (\rho(z) - \bar{k}\sigma(z))v^n = 0$$

where $\rho(z) = \sum_{j=0}^s \alpha_j z^j$, $\sigma(z) = \sum_{j=0}^s \beta_j z^j$ as usual.

Root Condition for Absolute Stability

Let us define $\pi_{\bar{k}}(z) = \rho(z) - \bar{k}\sigma(z)$

Theorem

A linear multistep method formula is absolutely stable for a particular value $\bar{k} = k_a$ if and only if all the roots of $\pi_{\bar{k}}$ satisfy $|z| \leq 1$, and any root with $|z| = 1$ is simple.

A "simple" root is a root of multiplicity 1.

Stability Region

The stability region S of a linear multistep formula is the set of all $\bar{k} \in \mathbb{C}$ (complexplane) for which the formula is absolutely stable.

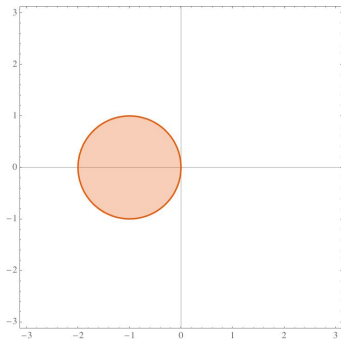
Example

For Forward Euler's formula, $\rho(z) = z - 1$ and $\sigma(z) = 1$. Hence

$$\pi_{\bar{k}}(z) = (z - 1) - \bar{k} = z - (1 + \bar{k})$$

The root is $z = (1 + \bar{k})$. Therefore, the stability region S is the set $\bar{k} \in \mathbb{C}$ with $|1 + \bar{k}| \leq 1$, that is the disk in the complex plane centered at $(-1, 0)$ with radius 1. If the root can be written as a function of \bar{k} , then plotting the stability region S with Mathematica is straightforward.

```
kbar[x_, y_] := x + I*y  
RegionPlot[Abs[kbar[x, y] + 1] <= 1, {x, -3, 3}, {y, -3, 3},  
PlotTheme -> "Scientific"]
```



Moral of the story.

If you are solving

$$u_t = au$$

with forward Euler, then make sure that you choose your time-step k such that

$$\bar{k} = ka$$

lies in the stability region S on the left.

An Alternative Way of Plotting S

In most cases, finding the roots of $\pi_{\bar{k}}(z)$ as a function of \bar{k} can be quite difficult. As an alternative, we can write \bar{k}

$$\bar{k} = \frac{\rho(z)}{\sigma(z)}$$

Since root condition of the absolute stability requires $|z| \leq 1$, then plotting S is the same as mapping a unit disk under the rational function

$R(z) = \rho(z)/\sigma(z)$ that we discussed in the past lecture.

Note: In some cases, it is sometimes easier to plot the unstable region by mapping the outer of unit disk.

Example: BDF3 case.

- ▶ Define the Rational Function for BDF3.

$$R[z_] := (z^3 - (18/11) z^2 + (9/11) z - (2/11))/((6/11) z^3)$$

- ▶ Define Complex number in polar coordinates

$$zp[r_ , t_] := r*Exp[I*t]$$

- ▶ Map the outer of unit disk (say map an annulus with inner and outer radius 1 and 100 (since we can't take the outer radius to infinity) respectively.

```
g1 := ParametricPlot[{Re[R[zp[r, t]]], Im[R[zp[r, t]]]},  
  {r, 1, 100}, {t, 0, 2*Pi}, BoundaryStyle -> None,  
  PlotRange -> 10, PlotTheme -> "Scientific", PlotStyle -> Red]
```

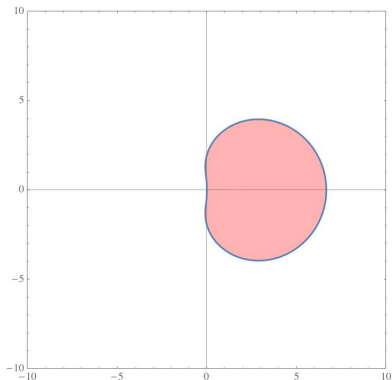
Stability region of BDF3

- ▶ Plot the boundary of the stability region, i.e map of the boundary of the unit disk.

```
g2 := ParametricPlot[{Re[R[zp[1, t]]], Im[R[zp[1, t]]]}, {t, 0, 2*Pi}]
```

- ▶ Show both g1 and g2

```
Show[g1, g2]
```



The white area is the stability region of the BDF3. The pinkish one is the "unstable" region.

Things to do in class

1. Derive the stability region of Midpoint by hand. With Mathematica, plot the stability regions of Backward Euler, Trapezoid, AB, AM, and BDF.

2. Consider a 2×2 system of linear initial value problem

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \begin{bmatrix} -1000 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \text{ with initial condition at } t = 0 \text{ to be } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

With MATLAB, simulate the problem for $t \in [0, 1]$ with Forward Euler, Backward Euler, Trapezoid, and Midpoint. What is the time-stepping restriction to stably march the solution in time? Note that the system matrix is a diagonal matrix with Eigenvalues -1000 , and -1 . Can you find out what is the relation of the Eigenvalues and time-step k ? What happens to the time-stepping restriction if you change the matrix to -100000 for the left top corner entry?

3. What is the maximum time step k for which the third-order Adams-Bashforth formula is absolutely stable when applied to

1. $u_t = -u$
2. $u_t = iu$

4. Plot the stability region of the hybrid Trapezoid and Midpoint method (the convex linear combination with $\alpha = 4/5$) from the previous lecture.