Lecture 8

MTH572/MTH472
Numerical Methods for PDEs

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Main references (quotes):
Trefethen: NumPDE, ATAP, Spectral Methods in MATLAB
Fornberg: PS Guide
Leveque: NumPDE
Driscoll: Learning MATLAB

Stability

Recall the ordinary differential equation (ODE):

$$u_t(t) = f(u(t), t)$$

or for simplicity (by dropping the arguments)

$$u_t = f$$

There are two main stability questions that have proved important over the years:

1. **Stability:** If $t > 0$ is held fixed, do the computed values $v(t)$ remain <u>bounded</u> as the time step $k \to 0$?
2. **Eigenvalue Stability:** If the time step $k > 0$ is held fixed, do the computed values $v(t)$ remain bounded as $t \to \infty$?

Of course, the important question is always: will the numerical solution give the right answer ?

Historical Notes

Stability is also known as "zero-stability" or sometimes "D-stability" for ODEs and "Lax stability" or "Lax-Ritchtmyer stability" for PDEs. Eigenvalue stability is also known as "weak stability" or "absolute stability" for ODEs , and "time-stability", "practical stability", or "P-stability" for PDEs.

First Question

If $t > 0$ is held fixed, do the computed values $v(t)$ remain bounded as the time step $k \to 0$?

Let us recall the characteristic polynomials of the linear multistep method:

$$\rho(z)v^n - k\sigma(z)f^n = 0$$

where $\rho(z) = \sum_{j=0}^{s} \alpha_j z^j$, $\sigma(z) = \sum_{j=0}^{s} \beta_j z^j$.

If we let $k \to 0$, we obtain

$$\rho(z)v^n = 0$$

or more precisely

$$\sum_{j=0}^{s} \alpha_j v^{n+j} = 0 \tag{1}$$

Definition:
A linear multistep formula is stable if all solutions v^n of the recurrence relation (1) are bounded as $n \to \infty$.

Root Condition for Stability

Theorem

A linear multistep method formula is stable if and only if all the roots of $\rho(z)$ satisfy $|z| \le 1$, and any root with $|z| = 1$ is simple.

Proof: See Trefethen's or other NumPDE book.

A "simple" root is a root of multiplicity 1.

Theorem: Stability of standard linear multistep formulas

- The $s-$step Adams-Bashforth, Adams-Moulton are stable for all $s \ge 1$.
- The $s-$step backwards differentiation formulas (BDF) are stable for $1 \le s \le 6$, but unstable for $s \ge 7$.

Theorem: First Dahlquist Stability Barrier (1956)
The order of accuracy p of a stable s-step linear multistep formula satisfies

$$p = \begin{cases} s+2 & \text{if } s \text{ is even,} \\ s+1 & \text{if } s \text{ is odd,} \\ s & \text{if the formula is explicit} \end{cases}$$

Definition: Convergence
A linear multistep formula is <u>convergent</u> if, for all IVPs satisfying

1. The Lipschitz conditions (existence and uniqueness) on an interval $[0,T]$,
2. All starting values $v^0, v^1, \dots, v^{s-1}$ satisfying

$$\lim v^n = u_0, \quad \text{as } k \to 0 \quad \text{for } 0 \le n \le s-1,$$

The solution v^n satisfies

$$\|v(t) - u(t)\| = o(1) \quad \text{as } k \to 0$$

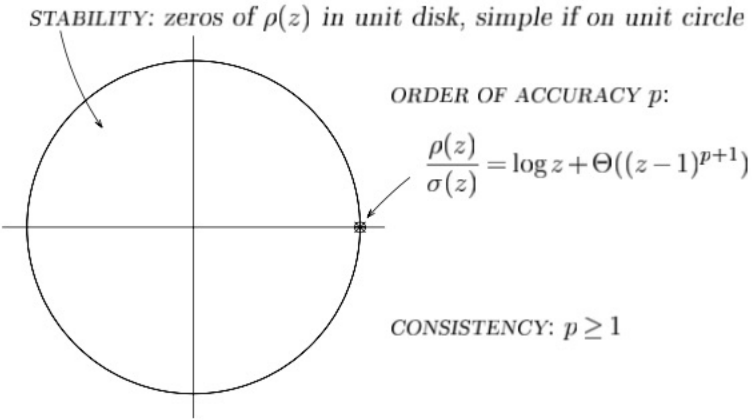
uniformly for all $t \in [0,T]$.

Note:

- The choice of the norm $\|.\|$ does not matter, since all norms on a finite-dimensional space are equivalent. For a scalar ODE, the norm can be replaced by absolute value.
- Uniformly means that $\|v(t) - u(t)\|$ is bounded by a fixed function $\phi(k) = o(1)$ as $k \to 0$, independent of t. A convergent linear multistep formula is one that is guaranteed to get the right answer in the limit $k \to 0$, for each t in a bounded interval $[0, T]$, assuming there are no rounding errors.

The fundamental theorem of linear multistep formulas.

Dahlquist Equivalence Theorem
A linear multistep formula is convergent if and only if it is consistent and stable.



Stability, consistency, and order of accuracy as algebraic conditions on the rational function $\rho(z)/\sigma(z)$

Things to do in class

1. Show that BDF of steps 2,3,4 satisfy root condition for stability. You may use Mathematica or Mupad.

2. Which of the following linear multistep formulas are convergent ? Are the non convergent ones inconsistent, or unstable, or both ?

- $v^{n+2} = \frac{1}{2}v^{n+1} + \frac{1}{2}v^n + 2kf^{n+1}$.
- $v^{n+1} = v^n$.
- $v^{n+4} = v^n + \frac{4}{3}k(f^{n+3} + f^{n+2} + f^{n+1})$.
- $v^{n+3} = v^{n+1} + \frac{1}{3}k(7f^{n+2} - 2f^{n+1} + f^n)$.
- $v^{n+4} = \frac{8}{19}(v^{n+3} - v^{n+1}) + v^n + \frac{6}{19}k(f^{n+4} + 4f^{n+3} + 4f^{n+1} + f^n)$.
- $v^{n+3} = -v^{n+2} + v^{n+1} + v^n + 2k(f^{n+2} + f^{n+1})$.

3. Devise a linear combination of two formulas: trapezoid formula and midpoint formula that has order of accuracy higher than 2. What is the order of accuracy of the new method ?

4. Show that the formula in number 3 is stable.

5. In general, is a convex linear combination of two stable linear multistep formula always stable ? A convex linear combination is

$$\text{new formula} = a \times \text{formula}_1 + (1-a) \times \text{formula}_2, \text{ for } 0 \le a \le 1$$

If not, can you find a counterexample.