Lecture 6

MTH572/MTH472
Numerical Methods for PDEs
Alfa Heryudono

UMass Dartmouth

Main references (quotes):
Trefethen: NumPDE, ATAP, Spectral Methods in MATLAB
Fornberg: PS Guide
Leveque: NumPDE
Driscoll: Learning MATLAB

Ordinary Differential Equation (ODE):

$$u_t(t) = f(u(t), t)$$

or for simplicity (by dropping the arguments)

$$u_t = f$$

where

- t is the time variable.
- u is a real or complex scalar or vector function of t, i.e $u(t) \in \mathbb{C}^N, N \geq 1$.
- u_t denotes $\frac{du}{dt}$. If $N > 1$, it should be interpreted component wise: $(u^{(1)}, \ldots, u^{(N)})^T_t = (u^{(1)}_t, \ldots, u^{(N)}_t)^T$. u_{tt} denotes $\frac{d^2u}{dt^2}$, and so on.
- f is a function that takes values in $\mathbb{C}^N$.

Historical Notes

Study of ODEs dates back to Newton and Leibniz in the 1670s and Euler in the 18th century. System of ODEs were first considered by Lagrange in the 1750s. Vector notation became standard around 1890.

Classification of the right hand side function f:

$$u_t = f(u,t)$$

Linear:

$$f(u,t) = \alpha(t)u + \beta(t)$$

for some functions $\alpha(t)$ and $\beta(t)$. If $\beta(t) = 0$ it is linear and homogeneous. In the vector case, $\alpha(t)$ is an $N \times N$ matrix and $\beta(t)$ is an $N-$vector.

Otherwise it is nonlinear.

Autonomous:

$f(u,t)$ is independent of t

If $f(u,t)$ is independent of u, the ODE reduces to an indefinite integral.

Initial Value Problems (IVP)

We shall provide initial data $u(0) = u_0$ at $t = 0$ and look for solutions $u(t)$ on some interval $t \in [0, T], T > 0$.

Initial Value Problem

Given f as described in the previous slide, $T > 0$, and $u_0 \in \mathbb{C}^N$, find a differentiable function $u(t)$ defined for $t \in [0, T]$ such that

1. $u(0) = u_0$,
2. $u_t(t) = f(u(t), t)$ for all $t \in [0, T]$

The choice of $t = 0$ as a starting point introduces no loss of generality, since any other t_0 could be treated by the change of variables $t' = t - t_0$.

The ODE above is of first order (contains only a first derivative with respect to t). However, any higher-order ODE can be reduced to an equivalent system of first-order of ODEs.

Existence and Uniqueness for the IVP (Cauchy 1824)

Standard assumptions for f to ensure the existence and uniqueness for the IVP.

Lipschitz Condition

f is continuous with respect to t and satisfies a (uniform) Lipschitz condition with respect to u. This means that there exists a constant $L > 0$ such that for all $u, v \in \mathbb{C}^N$ and $t \in [0, T]$,

$$\|f(u,t) - f(v,t)\| \leq L\|u - v\|$$

where $\|.\|$ denotes some norm on the set of $N-$vectors.

- $N = 1$, $\|.\|$ is usually just the absolute value $|.|$
- $N \geq 2$, the most important examples of norms are $\|.\|_1$, $\|.\|_2$, and $\|.\|_\infty$.

Sufficient Condition: $\frac{\partial f}{\partial u}$ exists and is bounded in the norm by L for all $u \in \mathbb{C}^N$ and $t \in [0, T]$.

Popular Numerical Methods for IVP

1. Linear Multisteps Methods (LMM).
2. Runge-Kutta (RK) Methods.
3. Exponential Integrator Methods.

In this class, we will concentrate on LMM only.

Continuous-Discrete Symbol Conventions:

	Continuous	Discrete
Time Step	Δt	$k, \quad k > 0$
Time	$t = t_0, t_1, \ldots, t_n$	$t = 0, k, \ldots, nk, n \geq 0$
Solution values	$u(t_n)$	v^n
Function values	$f(u(t_n), t_n)$	$f(v^n, t_n) = f^n$

Note: We use k for time step (i.e. Δt) and h for Δx for spatial step. The superscripts in v^n and f^n are not exponents!

Linear Multistep Formulas

$$u(t_{n+s}) - u(t_{n+s-1}) = \int_{t_{n+s-1}}^{t_{n+s}} u_t(t)dt = \int_{t_{n+s-1}}^{t_{n+s}} f(t)dt = \int_{t_{n+s-1}}^{t_{n+s}} q(t)dt$$

$q(t)$ is an interpolating polynomials that interpolates f. A LMM is essentially a formula for calculating each new value v^{n+1} from some of the previous values $v^0, \cdots, v^n$ and $f^0, \ldots, f^n$.

Popular Examples (Recall your MTH362)

s: Number of Steps. p: Order of accuracy.

Name	s	p	Type	Formula
Forward Euler	1	1	Explicit	$v^{n+1} = v^n + kf^n$
Backward Euler	1	1	Implicit	$v^{n+1} = v^n + kf^{n+1}$
Trapezoid	1	2	Implicit	$v^{n+1} = v^n + \frac{k}{2}(f^n + f^{n+1})$
Explicit Midpoint	2	2	Explicit	$v^{n+2} = v^n + 2kf^{n+1}$

In MTH362, utilize Taylor expansions to find p.

Characteristic/Generating Polynomials for LMM

Let $\mathcal{Z}$ denote a time shift operator.

	Continuous	Discrete
Shift u Once	$\mathcal{Z}u(t) = u(t+k)$ $\mathcal{Z}^{-1}u(t) = u(t-k)$	$\mathcal{Z}v^n = v^{n+1}$ $\mathcal{Z}v^n = v^{n-1}$
Shift f Once	$\mathcal{Z}f(u(t),t) = f(u(t+k), t+k)$	$\mathcal{Z}f^n = f^{n+1}$
Shift u Twice	$\mathcal{Z}^2u(t) = u(t+2k)$	$\mathcal{Z}^2v^n = v^{n+2}$

Example: Forward Euler

$$v^{n+1} = v^n + kf^n \text{ becomes } \mathcal{Z}v^n = v^n + kf^n.$$

which can be written as $(\mathcal{Z}-1)v^n = kf^n$ or

$$\rho(\mathcal{Z})v^n - k\sigma(\mathcal{Z})f^n = 0$$

where $\rho(z) = z - 1$ and $\sigma(z) = 1$.

Characteristic/Generating Polynomials for LMM

Several examples of Characteristic Polynomials

s: Number of Steps.
p: Order of accuracy.

Name	s	p	Type	Polynomials	
Forward Euler	1	1	Explicit	$\rho(z) = z - 1$	$\sigma(z) = 1$
Backward Euler	1	1	Implicit	$\rho(z) = z - 1$	$\sigma(z) = z$
Trapezoid	1	2	Implicit	$\rho(z) = z - 1$	$\sigma(z) = \frac{1}{2}(z+1)$
Explicit Midpoint	2	2	Explicit	$\rho(z) = z^2 - 1$	$\sigma(z) = 2z$

Linear multistep formulas are connected with the approximation of the function $\log z$ by the rational function $\frac{\rho(z)}{\sigma(z)}$ at $z = 1$.

Linear Multistep Formulas and Rational Approximation

Theorem

A linear multistep formula with $\sigma(1) \neq 0$ has order of accuracy p if and only if

$$R(z) = \frac{\rho(z)}{\sigma(z)} = \log z + \mathcal{O}((z-1)^{p+1})$$
$$= (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \cdots + \Theta((z-1)^{p+1})$$

as $z \to 1$. It is consistent if and only if

$$\rho(1) = 0 \text{ and } \rho'(1) = \sigma(1)$$

See Trefethen's NumPDE book for the proof. As an example, for trapezoid, $\rho(1) = 0$ and $\rho'(1) = 1 = \sigma(1)$ and

$$R(z) = \frac{z-1}{\frac{1}{2}(z+1)} = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{4}(z-1)^3 - \frac{1}{8}(z-1)^4 + \ldots$$

Comparing it with the series of $\log z$, the coefficient starts to disagree for the term $(z-1)^3$. Hence, $p = 2$.

Mathematica: Mapping a unit disk under a rational function

For example, in Forward Euler case, $R(z) = z - 1$.

- Define $R(z)$

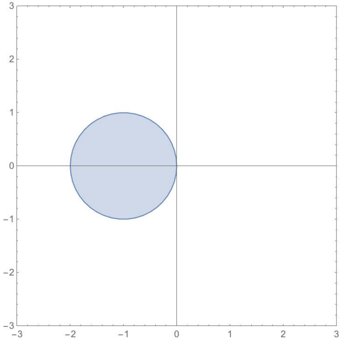
```
R[z_] := z - 1;
```

- Define the unit disk with center at $(0, 0)$

```
disk = Disk[{0, 0}, 1];
```

- Map the unit disk under $R(z)$.

```
ParametricPlot[Block[{w = u + I v}, {Re[R[w]], Im[R[w]]}],
Element[{u, v}, disk], PlotRange -> 3]
```



Things to do in class

1. Check the Lipschitz condition for the initial value problem $u_t = u\cos(t)$, $u(0) = 1$ and find the constant L.
2. Assuming K, m and y^* are constants, rewrite

$$y_{tt} = -\frac{K}{m}(y - y^*), \quad y(0) = a, \quad y_t(0) = b$$

as first order system. Then check the Lipschitz condition and find the constant L.
3. Verify the $\rho(z), \sigma(z)$ and $R(z)$ for Forward, Backward Euler, Trapezoid, and Midpoint as well as their order of accuracy p.
4. Given a unit circle $|z| <= 1$ centered at the origin in the complex plane, find its map under the $R(z)$ for forward, backward Euler, trapezoid, and midpoint. Plot the circle before and after the map.
5. Solve the IVP $u_t = u\cos(t)$, $u(0) = 1$ with Forward, Backward, Trapezoid, and Midpoint with $k = 0.1$ for $t \in [0, 1]$. The exact solution is $u(t) = e^{\sin(t)}$. Compare the error $|u(1) - v(1)|$. Compute the error for $k = 10^{-3}, 5.10^{-3}, 10^{-2}, 5.10^{-2}$. For each method, plot Error vs k in loglog scale. What does the slope tell you ?

Tips: You can use the command `Series[ ]` in Mathematica to generate the series of $R(z) = \frac{\rho(z)}{\sigma(z)}$ around $z = 1$.