Lecture 5

MTH572/MTH472
Numerical Methods for PDEs

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Main references (quotes):
Trefethen: NumPDE, ATAP, Spectral Methods in MATLAB
Fornberg: PS Guide
Leveque: NumPDE
Driscoll: Learning MATLAB

Dealing with Mutiple Boundary Conditions in 1D

A simple BVP in 1-D

$$\frac{d^3u}{dx^3} = f(x) \text{ for } x \in (a, b)$$
$$u(a) = g_l$$
$$u(b) = g_n \quad u'(b) = w_n$$

with one boundary condition at $x = a$ and two boundary conditions at $x = b$. Question: How do we impose two boundary conditions at one point ?

- Technique 1: Row Addition (leads to rectangular (non square) system). Easy! But is this a good practice ?
- Technique 2: Ghost (Fictitious) Point + Row Replacement (square system).
- Technique 3: Ghost (Fictitious) Point + Strip Row Move Over Column (reduced square system)

Take an easy example where $n = 7$ with 2^{nd} order accuracy.

Technique 1: Row Addition

For our particular case right now, the Row Addition technique will lead to a rectangular (augmented) system of size $(n+1) \times n$.

Step 1: Collocate the PDE all the way to the boundaries

$$\frac{1}{h^3}\begin{bmatrix} -\frac{5}{2} & 9 & -12 & 7 & -\frac{3}{2} & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\ 0 & 0 & \frac{3}{2} & -7 & 12 & -9 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

In order to keep the accuracy of 2nd order, watch out for the "non-centered" stencil weights.

Technique 1: Row Addition

Step 2: Replace/modify the corresponding rows of boundary conditions: ($u_1 = g_1$) at $x = x_1 = a$ and ($u_n = g_n$) at $x = x_n = b$.

$$\frac{1}{h^3}\begin{bmatrix} h^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & h^3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} g_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ g_7 \end{bmatrix}$$

Note: Since we factor out $\frac{1}{h^3}$, the Dirichlet boundary condition $u_1 = g_1$ and $u_n = g_n$ can be written as

$$\frac{1}{h^3}(h^3 u_1) = g_1 \quad \frac{1}{h^3}(h^3 u_n) = g_n$$

Technique 1: Row Addition

Step 3: Add an extra row to impose the additional (Neumann) boundary condition ($u_n' = w_n$) at $x = b$.

$$\frac{1}{h^3}\begin{bmatrix} h^3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & h^3 \\ 0 & 0 & 0 & 0 & \frac{1}{2}h^2 & -2h^2 & \frac{3}{2}h^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} g_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ g_7 \\ w_7 \end{bmatrix}$$

Note: Since we factor out $\frac{1}{h^3}$, the "right-sided" Neumann boundary condition $u_n' = w_n$ can be written as

$$\frac{1}{h^3}\left(\frac{1}{2}h^2 u_{n-2} - 2h^2 u_{n-1} + \frac{3}{2}h^2 u_n\right) = w_n$$

Technique 1: Row Addition

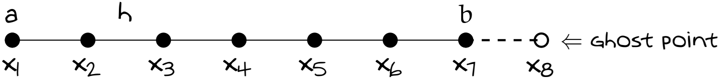
Step 4: Solve the resulting rectangular system of size $(n+1)\times n$ in least-squares sense.

> To avoid having entries with varying scale due to h, you may want to scale the system by multiplying all rows with h^3 except for the last row where we multiply by h.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 \\
-\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \frac{1}{2} & -2 & \frac{3}{2}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix}
=
\begin{bmatrix} g_1 \\ h^3 f_2 \\ h^3 f_3 \\ h^3 f_4 \\ h^3 f_5 \\ h^3 f_6 \\ g_7 \\ h w_7 \end{bmatrix}
$$

Technique 2: Ghost (Fictitious) Point + Row Replacement

A ghost or fictitious point can be added outside of the domain as an extra degree of freedom.



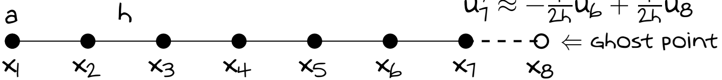
Step 1: Collocate the PDE all the way to the fictitious point.

$$\frac{1}{h^3}\begin{bmatrix} -\frac{5}{2} & 9 & -12 & 7 & -\frac{3}{2} & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & -7 & 12 & -9 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}$$

Watch out for the "non-centered" stencil weights.

Technique 2: Ghost (Fictitious) Point + Row Replacement

$$u_7' \approx -\frac{1}{2h}u_6 + \frac{1}{2h}u_8$$

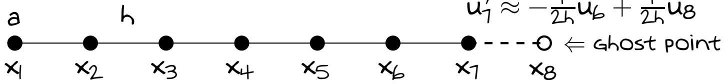


Step 2: Modify row 1, 7, and 8 to impose boundary conditions $u_1 = g_1$, $u_7 = g_7$, and $u_7' = w_7$.

$$\frac{1}{h^3}\begin{bmatrix} h^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & h^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}h^2 & 0 & \frac{1}{2}h^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} g_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ g_7 \\ w_7 \end{bmatrix}$$

Technique 2: Ghost (Fictitious) Point + Row Replacement

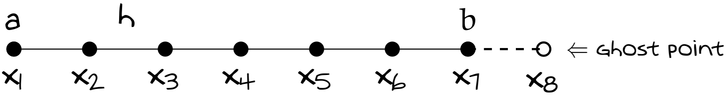
$$u_7' \approx -\frac{1}{2h}u_6 + \frac{1}{2h}u_8$$



Step 3: Solve the $(n+1)\times(n+1)$ square system but you only care about values of $u_1, \cdots, u_n$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} g_1 \\ h^3 f_2 \\ h^3 f_3 \\ h^3 f_4 \\ h^3 f_5 \\ h^3 f_6 \\ g_7 \\ h w_7 \end{bmatrix}$$

Technique 3: Ghost (Fictitious) Point, SR + MOC



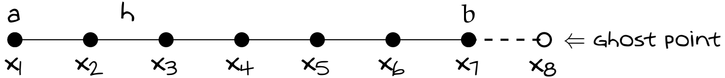
Step 1: By using the extra Neumann condition $u_7' = w_7$, solve the ghost value u_8 in terms of values at non ghost points. Particularly, for our example we have

$$u_7' = w_7 = -\frac{1}{2h}u_6 + \frac{1}{2h}u_8.$$

Hence,

$$u_8 = 2hw_7 + u_6$$

Technique 3: Ghost (Fictitious) Point, SR + MOC

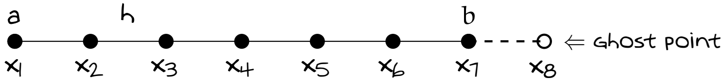


Step 1: Collocate the PDE all the way to the fictitious point.

$$\frac{1}{h^3}\begin{bmatrix} -\frac{5}{2} & 9 & -12 & 7 & -\frac{3}{2} & 0 & 0 & 0 \\ -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & -3 & 6 & -5 & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & -7 & 12 & -9 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix}$$

Watch out for the "non-centered" stencil weights.

Technique 3: Ghost (Fictitious) Point, SR + MOC



Step 3: Strip the rows of 1, 7, and 8 since we don't impose PDE there. After multiplying both sides by h^3, the stripped down system is given as the following:

$$\begin{bmatrix} -\frac{3}{2} & 5 & -6 & 3 & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} h^3 f_2 \\ h^3 f_3 \\ h^3 f_4 \\ h^3 f_5 \\ h^3 f_6 \end{bmatrix}$$

Technique 3: Ghost (Fictitious) Point, SR + MOC

Step 5: Split columns 1, 7, and 8 from the main matrix

$$\begin{bmatrix} 5 & -6 & 3 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \\ u_8 \end{bmatrix} = \begin{bmatrix} h^3 f_2 \\ h^3 f_3 \\ h^3 f_4 \\ h^3 f_5 \\ h^3 f_6 \end{bmatrix}$$

Note that since $u_1 = g_1$, $u_7 = g_7$ and $u_8 = 2hw_7 + u_6$, the additional columns become

$$\begin{bmatrix} -\frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_7 \\ u_8 \end{bmatrix} = g_1 \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + g_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -1 \end{bmatrix} + 2hw_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} + u_6 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Move columns with known values g_1, g_7 and w_7 to the RHS
Integrate column that contains u_6 with the system matrix.

Technique 3: Ghost (Fictitious) Point, SR + MOC

Step 6: Move over and integrate columns.

$$\begin{bmatrix} 5 & -6 & 3 & -\frac{1}{2} & 0 \\ 1 & 0 & -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & 0 & -1 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 1 & 0 & -1 \\ 0 & 0 & -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} h^3 f_2 \\ h^3 f_3 \\ h^3 f_4 \\ h^3 f_5 \\ h^3 f_6 \end{bmatrix} - g_1 \begin{bmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} - g_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2} \\ -1 \end{bmatrix} - 2hw_7 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Step 7: Solve the $(n-2) \times (n-2)$ system.

> Technique 1 (Row Addition) and Technique 2 (Ghost Point + RR) are suitable for boundary value (time-independent) problems with multiple BCs. Technique 3 (Ghost Point + SR-MOC) can be quite handy for the time-dependent problems (or analyzing eigenvalue stability of the DMs) with multiple BCs.

Things to do in class

Use the techniques 1, 2, and 3 for the following problems

1. Write the MATLAB code (with $n = 20$) to solve

$$\frac{d^3u}{dx^3} = e^{-x}\left(\left(3\pi^2 - 1\right)\sin(\pi x) - \pi\left(\pi^2 - 3\right)\cos(\pi x)\right)$$

   for $x \in [-1, \frac{1}{2})$ with multiple boundary conditions at $x = \frac{1}{2}$ given by

$$u(-1) = 0$$
$$u(\tfrac{1}{2}) = e^{-\frac{1}{2}} \quad u'(\tfrac{1}{2}) = -e^{-\frac{1}{2}}.$$

   Plot the numerical solution vs the exact solution $u(x) = e^{-x}\sin(\pi x)$ on the same figure. Compute $\|\cdot\|_\infty$ to measure the error of numerical solution with respect the exact solution.

2. Using different values of $n = 10, 100, 1000, \cdots$, redo problem 1 and plot the $\|\cdot\|_\infty$ vs h in log scale. Do you observe 2^{nd}-order convergence (i.e. $\mathcal{O}(h^2)$ or $\mathcal{O}(n^{-2})$) ?