Lecture 4

MTH572/MTH472
Numerical Methods for PDEs
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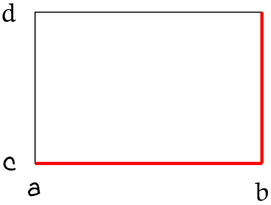
Main references (quotes):
Trefethen: NumPDE, ATAP, Spectral Methods in MATLAB
Fornberg: PS Guide
Leveque: NumPDE
Driscoll: Learning MATLAB

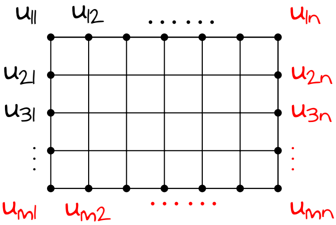
2D Silly Boundary Value Problem (BVP) with Dirichlet BCs

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = f(x,y) \quad (x,y) \in \Omega \cup \partial\Omega_{north} \cup \Omega_{west}$$

$$u(x,y) = g(x,y) \quad (x,y) \in \partial\Omega_{south} \cup \partial\Omega_{east}$$

Ex: 2-D rectangular domain $\overline{\Omega} = \{a \le x \le b, c \le y \le d\}$.

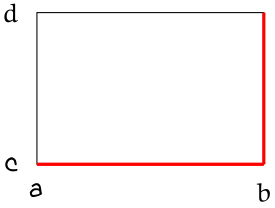


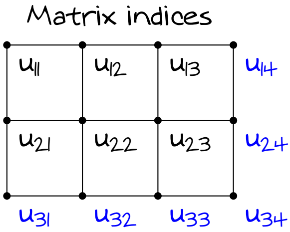


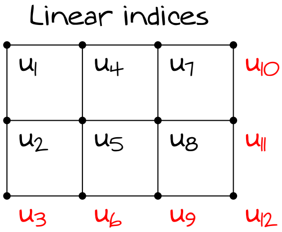
- The BVP is collocated at interior, north, and west nodes.
- Boundary conditions are imposed at south and east nodes.

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

A Simple Example



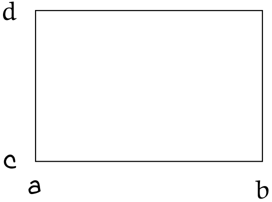


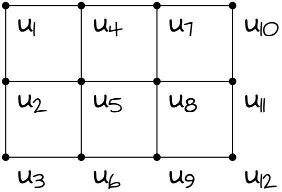


- Discretize x into $n = 4$ equally-spaced points with equal spacing $h_x = \frac{b-a}{n-1}$.
- Discretize y into $m = 3$ equally-spaced points with equal spacing h_y with equal spacing $h_y = \frac{d-c}{m-1}$.
- Entries of $\underline{u} = \mathrm{vec}(\mathbf{U})$ is ordered using "lexicographic" ordering (linear index). Hence $\underline{u} = [u_1, \cdots, u_{12}]^T$.
- In MATLAB, conversion from linear index to "matrix indices" and vice versa can be done with the commands `ind2sub` and `sub2ind` respectively.

Differentiation Matrices

Differentiate x and y directions independently.





Derivatives with respect to y

Ex: 3-point stencil, 2-nd order accurate. Note that the prime $'$ here means partial derivative with respect to y.

$$\begin{bmatrix} u_1' & u_4' & u_7' & u_{10}' \\ u_2' & u_5' & u_8' & u_{11}' \\ u_3' & u_6' & u_9' & u_{12}' \end{bmatrix} \approx \frac{1}{2h_y} \begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} u_1 & u_4 & u_7 & u_{10} \\ u_2 & u_5 & u_8 & u_{11} \\ u_3 & u_6 & u_9 & u_{12} \end{bmatrix}$$

$$\mathbf{U}_y \approx D_y \mathbf{U}$$

$(\mathbf{D}_y)$: Differentiation Matrix with respect to y

In lexicographic ordering, the relation of $\underline{u}_y = \mathbf{D}_y \underline{u}$, where $\underline{u}_y = \mathrm{vec}(\mathbf{U}_y)$ and $\underline{u} = \mathrm{vec}(\mathbf{U})$, can be written as

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \\ u_6' \\ u_7' \\ u_8' \\ u_9' \\ u_{10}' \\ u_{11}' \\ u_{12}' \end{bmatrix} = \frac{1}{2h_y} \left[\begin{array}{ccc|ccc|ccc|ccc} -3 & 4 & -1 & & & & & & & & & \\ -1 & 0 & 1 & & & & & & & & & \\ 1 & -4 & 3 & & & & & & & & & \\ \hline & & & -3 & 4 & -1 & & & & & & \\ & & & -1 & 0 & 1 & & & & & & \\ & & & 1 & -4 & 3 & & & & & & \\ \hline & & & & & & -3 & 4 & -1 & & & \\ & & & & & & -1 & 0 & 1 & & & \\ & & & & & & 1 & -4 & 3 & & & \\ \hline & & & & & & & & & -3 & 4 & -1 \\ & & & & & & & & & -1 & 0 & 1 \\ & & & & & & & & & 1 & -4 & 3 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$$

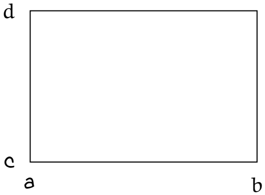
The differentiation matrix with respect to y can be concisely written using Kronecker product symbol as

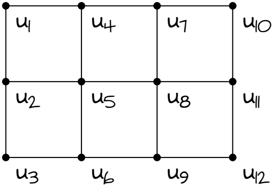
$$\mathbf{D}_y = I_n \otimes D_y = I_4 \otimes D_y$$

where,

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad D_y = \frac{1}{2h_y} \begin{bmatrix} -3 & 4 & -1 \\ -1 & 0 & 1 \\ 1 & -4 & 3 \end{bmatrix}$$

Derivatives with respect to x





An example using 3-point stencil (2-nd order accurate, see slides from the previous lecture). Note that the prime $'$ here means partial derivative with respect to x.

$$\begin{bmatrix} u_1' & u_2' & u_3' \\ u_4' & u_5' & u_6' \\ u_7' & u_8' & u_9' \\ u_{10}' & u_{11}' & u_{12}' \end{bmatrix} \approx \frac{1}{2h_x} \begin{bmatrix} -3 & 4 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \\ u_{10} & u_{11} & u_{12} \end{bmatrix}$$

Transposing the matrix gives

$$\begin{bmatrix} u_1' & u_4' & u_7' & u_{10}' \\ u_2' & u_5' & u_8' & u_{11}' \\ u_3' & u_6' & u_9' & u_{12}' \end{bmatrix} \approx \begin{bmatrix} u_1 & u_4 & u_7 & u_{10} \\ u_2 & u_5 & u_8 & u_{11} \\ u_3 & u_6 & u_9 & u_{12} \end{bmatrix} \begin{bmatrix} & & \\ & D_x & \\ & & \end{bmatrix}^T$$

$$U_x \approx U D_x^T$$

$(\mathbf{D}_x)$: Differentiation Matrix with respect to x

In lexicographic ordering, the relation of $\underline{u}_x = \mathbf{D}_x \underline{u}$, where $\underline{u}_x = \texttt{vec}(\mathbf{U}_x)$ and $\underline{u} = \texttt{vec}(\mathbf{U})$, can be written as

$$
\begin{bmatrix} u_1' \\ u_2' \\ u_3' \\ u_4' \\ u_5' \\ u_6' \\ u_7' \\ u_8' \\ u_9' \\ u_{10}' \\ u_{11}' \\ u_{12}' \end{bmatrix}
= \frac{1}{2h_x}
\left[\begin{array}{ccc|ccc|ccc|ccc}
-3 & & & 4 & & & -1 & & & 0 & & \\
 & -3 & & & 4 & & & -1 & & & 0 & \\
 & & -3 & & & 4 & & & -1 & & & 0 \\
\hline
-1 & & & 0 & & & 1 & & & 0 & & \\
 & -1 & & & 0 & & & 1 & & & 0 & \\
 & & -1 & & & 0 & & & 1 & & & 0 \\
\hline
0 & & & -1 & & & 0 & & & 1 & & \\
 & 0 & & & -1 & & & 0 & & & 1 & \\
 & & 0 & & & -1 & & & 0 & & & 1 \\
\hline
0 & & & 1 & & & -4 & & & 3 & & \\
 & 0 & & & 1 & & & -4 & & & 3 & \\
 & & 0 & & & 1 & & & -4 & & & 3
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
$$

The differentiation matrix with respect to x can be concisely written using Kronecker product symbol as

$$\mathbf{D}_x = D_x \otimes I_m = D_x \otimes I_3$$

where,

$$
I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\qquad
D_x = \frac{1}{2h_x} \begin{bmatrix} -3 & 4 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -4 & 3 \end{bmatrix}
$$

Steps for solving the BVP

Collocate the PDE all the way to the boundaries:

$$\mathcal{L}\underline{u} = (D_x \otimes I_m + I_n \otimes D_y)\underline{u} = (D_x \otimes I_3 + I_4 \otimes D_y)\underline{u} = \underline{f}$$

$$\mathcal{L}\underline{u} = \left[\begin{array}{ccc|ccc|ccc|ccc}
\bullet & \bullet & \bullet & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & & \bullet & & & \bullet & & & & \\
{\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & & & {\color{red}\bullet} & & & {\color{red}\bullet} & & & \\
\hline
\bullet & & & \bullet & \bullet & \bullet & \bullet & & & & & \\
 & \bullet & & \bullet & \bullet & \bullet & & \bullet & & & & \\
 & & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & & & {\color{red}\bullet} & & & \\
\hline
 & & & \bullet & & & \bullet & \bullet & \bullet & \bullet & & \\
 & & & & \bullet & & \bullet & \bullet & \bullet & & \bullet & \\
 & & & & & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & & & {\color{red}\bullet} \\
\hline
 & & & {\color{red}\bullet} & & & {\color{red}\bullet} & & & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} \\
 & & & & {\color{red}\bullet} & & & {\color{red}\bullet} & & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} \\
 & & & & & {\color{red}\bullet} & & & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet} & {\color{red}\bullet}
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ {\color{red}u_3} \\ u_4 \\ u_5 \\ {\color{red}u_6} \\ u_7 \\ u_8 \\ {\color{red}u_9} \\ {\color{red}u_{10}} \\ {\color{red}u_{11}} \\ {\color{red}u_{12}} \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ {\color{red}f_3} \\ f_4 \\ f_5 \\ {\color{red}f_6} \\ f_7 \\ f_8 \\ {\color{red}f_9} \\ {\color{red}f_{10}} \\ {\color{red}f_{11}} \\ {\color{red}f_{12}} \end{bmatrix}$$

Apply Row Replacement Method

Modify rows that correspond to the boundary conditions (row replacement method) and then solve the resulting system of linear equations to obtain $\underline{u}$.

$$\tilde{\mathcal{L}}\underline{u} = \left[\begin{array}{ccc|ccc|ccc|ccc}
\bullet & \bullet & \bullet & \bullet & & & \bullet & & & & & \\
\bullet & \bullet & \bullet & & \bullet & & & \bullet & & & & \\
 & & 1 & & & & & & & & & \\
\hline
\bullet & & & \bullet & \bullet & \bullet & \bullet & & & & & \\
 & \bullet & & \bullet & \bullet & \bullet & & \bullet & & & & \\
 & & & & & 1 & & & & & & \\
\hline
 & & & \bullet & & & \bullet & \bullet & \bullet & \bullet & & \\
 & & & & \bullet & & \bullet & \bullet & \bullet & & \bullet & \\
 & & & & & & & & 1 & & & \\
\hline
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & 1 & \\
 & & & & & & & & & & & 1
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}
=
\begin{bmatrix} f_1 \\ f_2 \\ g_3 \\ f_4 \\ f_5 \\ g_6 \\ f_7 \\ f_8 \\ g_9 \\ g_{10} \\ g_{11} \\ g_{12} \end{bmatrix}$$

Note: If the size of the system is too large, you may want to consider using an iterative solver to solve $\underline{u}$.

Things to do in class

1. By using $n = m = 10$, $h_x = h_y = h = 2/(n-1)$, solve the following 2D Poisson equation

$$-\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -(4x^2 + y^2 - 3)e^{-(x^2+\frac{1}{2}y^2)} \quad (x,y) \in \Omega$$

$$u(x,y) = e^{-(x^2+\frac{1}{2}y^2)} \quad (x,y) \in \partial\Omega$$

   on the domain $\overline{\Omega} = [-1,1] \times [-1,1]$. Use $\|\cdot\|_\infty$ to measure the error of the numerical solution with respect the exact solution $u(x,y) = e^{-(x^2+\frac{1}{2}y^2)}$.

2. Redo problem 1 by applying the Cuthill-McKee permutation to the matrix operator $\widetilde{\mathcal{L}}$.

3. Using different values of $n = 10, 60, 110, 160, \ldots, 310$, redo problems 1 and 2 and plot the $\|\cdot\|_\infty$ vs h in log scale. Do you observe 2^{nd}-order convergence (i.e. $\mathcal{O}(h^2)$) ?

Tips: In MATLAB, use the command `symrcm` to perform sparse reverse Cuthill-McKee permutation .