Project 3
Stability-Regions and Time-Dependent Problems (1D and 2D)
Deadline and submission information are available in mycourses
All problems are of equal weights

Name : ______________________________

1. **[Stability Regions and Order Stars]** You may use Mathematica to plot the order stars and stability regions for the following problems.
   (a) Derive the stability region of Midpoint by hand and plot it.
   (b) Plot the order stars of AB1-2-3-4, AM1-2-3-4-5, BDF2-3-4. For each method, confirm that your plots show the correct number of "bounded fingers". Moreover, for each method check whether the unit disk intersects those fingers.
   (c) Plot the stability regions of AB2-3-4-5, AM1-2-3-4-5, and BDF2-3-4.
   (d) Plot the stability region and order stars of the hybrid Trapezoid and Midpoint method (the convex linear combination with $\alpha = 4/5$) from the lecture.

2. **[Multiple boundary condition problems]** With MATLAB and MOL (with ode solver of your own choice), simulate solutions $u(x,t)$ starting from $t=0$ until $t=1$ of

$$u_t = u_{xxx} + e^{-x}((\pi^3 - 4\pi)\cos(\pi(x-t)) + (1-3\pi^2)\sin(\pi(x-t)),$$

   for $x \in [-1, \frac{1}{2})$ with boundary conditions at $x=-1$ and $x=\frac{1}{2}$ given by

$$\begin{aligned} u(-1,t) &= e\sin(\pi t) \\ u(\tfrac{1}{2},t) &= e^{-\frac{1}{2}}\cos(\pi t), \quad u_x(\tfrac{1}{2},t) = e^{-\frac{1}{2}}(\pi\sin(\pi t) - \cos(\pi t)). \end{aligned}$$

   and initial condition at $t=0$

$$u(x,0) = e^{-x}\sin(\pi(x))$$

   Simulate the numerical solution vs the exact solution $u(x) = e^{-x}\sin(\pi(x-t))$ on the same figure. Plot $\|\cdot\|_\infty$ vs time t to measure the error of numerical solution with respect to the exact solution.

3. **[1D advection equation with variable speed]** With MATLAB and MOL (with ode solver of your own choice), simulate solutions $u(x,t)$ starting from $t=0$ until $t=1$ of the following PDE

$$u_t = a(x,t)u_x$$

   all defined in the interval $x \in [-1,1]$, with right hand boundary condition $u(1,t) = 1+t$. The initial condition is $u(x,0) = x$ and $a(x,t) = e^{(1+t)(1+\cos 3x)}$. Is it true that $u(x,t)$ is constant along **characteristic curves** with slope $-\frac{1}{a(x,t)}$?

4. **[2D Space-Time BVP]** Suppose you are getting tired of MOL and you want to join space-time fans club whose members treat time t as another space variable. Solve problem 2 with 2D space time boundary value problem in a space time domain $[-1,1] \times [0,1]$ using 2D finite-difference that you use in project 2. Can this thing be done ? Does the space-time solution look like the solution in number 3 ? Do you see any instability ? Plot the surface of $u(x,t)$ on the space-time domain $[-1,1] \times [0,1]$.

5. **[1D Burgers' Equation]** With MATLAB and MOL (with ode solver of your own choice), simulate solutions $u(x,t)$ starting from $t=0$ until $t=0.9$ of the 1D Burgers' equation

$$u_t = \varepsilon u_{xx} - u u_x$$

all defined in the interval $x \in [0,1]$, with boundary conditions $u(0,t)=0$, $u(1,t)=0$, with initial condition $u(x,0) = \sin(2\pi x) + \frac{1}{2}\sin(\pi x)$ and $\varepsilon = 10^{-1}$. Devise whatever strategies you can think of to handle the nonlinearities successfully. What will happen if you choose smaller ε say 10^{-2} or 10^{-3} ?

6. **[2D Wave Equation]** With the knowledge that you learned from the program **wave2dfd** and by reading create a program to solve the 2D wave equation

$$u_{tt} = \Delta u, \quad u = 0 \quad \text{on the boundary}$$

with initial data at $t=0$ to be $u(x,y,0) = e^{-40((x-0.75)^2+y^2)}$, $u_t(x,y,0)=0$ on **an annulus** with inner radius $r_{\text{in}} = 0.25$ and outer radius $r_{\text{out}} = 1.25$. **Hint:** Write the Laplacian in terms of polar coordinates. Use non-periodic FD in r and periodic FD in θ.