Project 2
Linear and Nonlinear Boundary Value Problems (1D and 2D)
Deadline and submission information are available in mycourses
All problems are of equal weights

Name : ______________________________

1. Use second-order accurate (non-periodic) finite difference method to solve the following two dimensional boundary value problem

$$-u_{xx} - \frac{3}{2}u_{yy} - \frac{1}{4}u = f(x,y) \qquad (x,y) \in \Omega,$$
$$u = g(x,y) \qquad (x,y) \in \partial\Omega,$$

where $\overline{\Omega} = [-1,1] \times [-1,1]$ and $g(x,y) = \sin(\pi x)\cos(\pi y)$. The right hand side function $f(x,y)$ is derived by "plugging" in the exact solution $u_{\text{ex}}(x,y) = \sin(\pi x)\cos(\pi y)$ to the PDE operator. Please review the "Things to Do in Class" from Lecture 4.

   (a) Following Lecture 4, solve the numerical solution $\underline{u}$ for $n = m = 30$, plot the solution, and compute the $\|\underline{u} - \underline{u}_{\text{ex}}\|_\infty$.

   (b) Do the convergence test and plot the trend in **loglog** scale. Do you get second-order convergence ? (Check the slope)

   (c) Redo part (a) and (b) with fourth-order FDM.

2. Use second-order accurate (non-periodic) finite difference method to solve the following two dimensional boundary value problem

$$(1 - \frac{1}{4}x)u_{xx} + (1 + \frac{1}{2}y)u_{yy} = f(x,y) \qquad (x,y) \in \Omega,$$
$$u = g(x,y) \qquad (x,y) \in \partial\Omega,$$

where $\overline{\Omega} = [-1,1] \times [-1,1]$ and $g(x,y) = \sin(\pi x)\cos(\pi y)$. The right hand side function $f(x,y)$ is derived by "plugging" in the exact solution $u_{\text{ex}}(x,y) = \sin(\pi x)\cos(\pi y)$ to the PDE operator. Please review the "Things to Do in Class" from Lecture 4.

   (a) Following Lecture 4, solve the numerical solution $\underline{u}$ for $n = m = 30$, plot the solution, and compute the $\|\underline{u} - \underline{u}_{\text{ex}}\|_\infty$.

   (b) Do the convergence test and plot the trend in **loglog** scale. Do you get second-order convergence ? (Check the slope)

   (c) Redo part (a) and (b) with fourth-order FDM.

3. Use second-order accurate (non-periodic) finite difference method to solve the following two dimensional eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \qquad (x,y) \in \Omega, \\ u &= 0 \qquad (x,y) \in \partial\Omega, \end{aligned}$$

where $\overline{\Omega} = [-1,1] \times [-1,1]$.

   (a) Find three largest (in magnitude) eigenvalues.

   (b) Plot the eigenvectors that correspond to eigenvalues from part (a).

   (c) Redo part (a) and (b) with fourth-order FDM.

4. Use second-order accurate (non-periodic) finite difference method to solve the following two dimensional boundary value problem

$$\begin{aligned} -\Delta u &= f(x,y) \qquad (x,y) \in \Omega, \\ u &= g(x,y) \qquad (x,y) \in \partial\Omega, \end{aligned}$$

where $\overline{\Omega}$ is an L shape domain defined by $\overline{\Omega} = \{[-1,1] \times [-1,1]\} \backslash \{(0,1] \times (0,1]\}$ and $g(x,y) = \sin(\pi x)\cos(\pi y)$. The right hand side function $f(x,y)$ is derived by "plugging" in the exact solution $u_{\text{ex}}(x,y) = \sin(\pi x)\cos(\pi y)$ to the PDE operator. Please review the "Things to Do in Class" from Lecture 4.

   (a) Following Lecture 4, solve the numerical solution $\underline{u}$, plot the solution, and compute the $\|\underline{u} - \underline{u}_{\text{ex}}\|_\infty$.

   (b) Do the convergence test and plot the trend in **loglog** scale. Do you get second-order convergence ? (Check the slope).