Project 1
Finite Difference Differentiation Matrices and 1D Boundary Value Problems.
Deadline and submission information: please log in to mycourses. All problems are of equal weights.

Name : ________________________________

1. In class, you have practiced to compute 3-point stencil finite difference weights to form **first derivative** differentiation matrix **D**.
   (a) By using equally-spaced points on the interval $[-1, 1]$, redo what we have done in class (you may choose Lagrange or Taylor series technique) for 5-point stencil finite difference weights to form the matrix **D**. You can Computer Algebra System software such as MATLAB-Mupad, Mathematica, or Maple.
   (b) By using $u(x) = e^{-x^2}$ with $u'(x) = -2xe^{-x^2}$ as a test case, use MATLAB to test the algebraic error convergence trend (error $\approx h^p$) by plotting h vs $\|.\|_\infty$ in **loglog** style plot. What is the order p ?
   (c) Redo part (c) for the periodic case with $u(x) = \sin(\pi x)$ with $u'(x) = \pi\cos(\pi x)$ as a test case.

2. In his paper (published in 1998), Fornberg presents a one-line Mathematica formula for finding finite difference weights:

```
CoefficientList[Normal[Series[x^s Log[x]^m, {x, 1, n}]/h^m], x]
```

   where m is the order of the derivative, n is the number of grid intervals enclosed in the stencil, and s is the number of grid intervals between the point at which the derivative is approximated and the leftmost edge of the stencil. As an example, for 3 point stencils,

```
m = 1;
n = 2;
s = 1;
CoefficientList[Normal[Series[x^s Log[x]^m, {x, 1, n}]/h^m], x]
```

   will result in weights $\left\{-\frac{1}{2h}, 0, \frac{1}{2h}\right\}$. Additionally,

```
m = 1;
n = 2;
s = 0;
CoefficientList[Normal[Series[x^s Log[x]^m, {x, 1, n}]/h^m], x]
```

   will result in "left one sided" weights $\left\{-\frac{3}{2h}, \frac{2}{h}, -\frac{1}{2h}\right\}$.

(a) Use the MATHEMATICA code above to check your answers for the 5 point stencil weights in problem 1 (a).

(b) Use the MATHEMATICA code to find 7 point stencil weights and then form the differentiation matrix **D** with MATLAB and redo part 1 (b) and 1 (c).

3. Write a MATLAB function

```
function [x, h, D] = FDDiffMat(domain,n,stencilsize,option)
```

that accepts domain $[a,b]$, number of equally-spaced points n, stencil size, and option as inputs and spits out a column vector x of size $n \times 1$, which is the discrete representation of $[a,b]$, spacing size h, and differentiation matrix $\mathbf{D}$ of size $n \times n \times 2$. Users can choose stencil sizes $2,3,5,7$ and an option either 'non-periodic' or 'periodic'.

For example, if a user calls

```
[x, h, D] = FDDiffMat([-1 1], 20, 5, 'non-periodic')
```

he/she will obtain x with entries $x_j = -1 + jh,\ j = 0,\dots,n-1,\ h = 2/(n-1)$, and the non-periodic 5-point stencil case differentiation matrix $\mathbf{D}$, where $\mathbf{D}(:,:,1)$ stores the first derivative differentiation matrix and $\mathbf{D}(:,:,2)$ stores the second derivative differentiation matrix.

4. By using 3-point stencil differentiation matrices from number 3, solve the boundary value problem (BVP)

$$u'' + u' + 2u = x, \quad x \in (-1,1)$$

(a) with boundary conditions $u(-1) = u(1) = 0$.

(b) with boundary conditions $u(-1) = u'(1) = 0$.

Plot the solution on $[-1,1]$. You are free to choose the number of n. For part (a) and (b), find the exact solutions with Mathematica or MATLAB or wolframalpha, and check the error convergence (i.e. h vs $\|.\|_\infty$ in **loglog** scale as usual).

5. By setting $u'(x) = v(x)$, redo problem 4 (b), by solving the system

$$\begin{aligned} v' + v + 2u &= x \\ u' - v &= 0 \end{aligned}$$

with boundary conditions $u(-1) = v(1) = 0$. The system will be twice as large compared to 4 (b). However the solution u computed this way should give you the same solution as in 4 (b).

6. By using 5-point stencil differentiation matrices from number 3, solve the Airy equation boundary value problem

$$u'' - xu = 0, \quad x \in (-30, 30)$$

with boundary conditions $u(-30) = 1$ and $u(30) = 0$. Plot the solution on $[-30, 30]$. You are free to choose the number of n.

7. By using 3-point stencil differentiation matrices, solve the Poisson equation

$$-u'' = e^{-40x^2}, \quad x \in (-1, 1),$$
$$u(-1) = 0, u(1) = 0.$$

Since the forcing function e^{-40x^2} on the right hand side shows its peak between $x = -\frac{1}{2}$ and $x = \frac{1}{2}$, you can try to discretize the interval $[-1, 1]$ with non-uniform nodes distributions. For example, the subinterval $[-\frac{1}{2}, \frac{1}{2}]$ can have spacing h twice smaller than other parts of the domain. Plot the solution on $[-1, 1]$.

8. Solve 1D nonlinear pendulum equation

$$\frac{d^2u}{ds^2} = -\sin(u)$$

on $s \in [0, 6]$ with boundary conditions $u(0) = -\pi/2$ and $u(6) = \pi/2$. You are free to try with different stencil sizes and the number of n. Use iteration technique to solve the nonlinear system. If we define a variable "height" $H = -\cos(u)$, plot H vs s.