Chapter 12. Introduction to Graphs

Section 12.1. Fundamental Concepts of Graph Theory

1. Yes. For example, the company representative can take the route: \( c_2, c_5, c_3, c_6 \).
2. See Figure 58.

![Graphs](image1)

Figure 58: The graphs in Exercises 2

3-6. See Figure 59. For Exercise 6, the vertex \( \{i, j\} \) is expressed as \( ij \).

![Graphs](image2)

Figure 59: The graphs in Exercises 3-6

7. The vertex set of \( G \) is \( S \) and the edge set of \( G \) is
\[
E = \{\{4, 6\}, \{4, 10\}, \{6, 9\}, \{6, 10\}, \{6, 15\}, \{9, 12\}, \{9, 15\}, \{10, 12\}, \{10, 15\}, \{12, 15\}\}.
\]

8. The edge set of \( G \) is \( \{\{1, 2\}, \{1, 5\}, \{1, 9\}, \{2, 4\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{6, 9\}, \{7, 8\}\} \).

9. The graph \( G \) is shown in Figure 60.

![Graph](image3)

Figure 60: The graph in Exercise 9
10. Since no vertex can have degree 2 (because no two vertices have a common neighbor), the maximum number of edges is 3.

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
\end{bmatrix}
\]

11. By the First Theorem of Graph Theory, \(1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 5 + (n-6) \cdot 4 = 2 \cdot 17\) and so \(18 + 4n - 24 = 34\). Solving for \(n\), we obtain \(n = 10\).

12. See Figure 61.

Figure 61: The graph in Exercise 12

13. The statement is false. The number of 1’s in row \(i\) is the degree of \(v_i\). The total number of 1’s in the adjacency matrix is the sum of the degrees of the vertices of the graph, which is twice the size of the graph.

14. See Figure 62.

Figure 62: The graph in Exercise 14

15. Since \(G\) has one vertex of degree 1, two vertices of degree 2, one vertex of degree 3 and two vertices of degree 5, there are \(n - 6\) vertices of degree 4. By the First Theorem of Graph Theory, \(1 \cdot 1 + 2 \cdot 2 + 1 \cdot 3 + 2 \cdot 5 + (n-6) \cdot 4 = 2 \cdot 17\) and so \(18 + 4n - 24 = 34\). Solving for \(n\), we obtain \(n = 10\).

16. Let \(x\) be the number of vertices of degree 3. Then there are \(12 - x\) vertices of degree 5. By the First Theorem of Graph Theory, \(x \cdot 3 + 2 \cdot 4 + (12 - x) \cdot 5 = 2 \cdot 62\) and so \(3x + 8 + 60 - 5x + 66 = 124\). Solving for \(x\), we obtain \(x = 5\).

17. There are \(21 - 3 - 6 - 4 = 8\) vertices of degree \(r\). By the First Theorem of Graph Theory, \(3 \cdot 2 + 6 \cdot 5 + 4 \cdot 8 + 8 \cdot r = 2 \cdot 50 = 100\). Solving for \(r\), we obtain \(r = 4\).

18. There are \(25 - 5 - 8 - 6 = 6\) vertices of degree \(r\) or \(2r\). Furthermore, there are \(\frac{2}{3} \cdot 6 = 4\) vertices of degree \(r\) and \(\frac{1}{3} \cdot 6 = 2\) vertices of degree \(2r\). By the First Theorem of Graph Theory, \(5 \cdot 4 + 8 \cdot 5 + 6 \cdot 7 + 4r + 2(2r) = 2 \cdot 63\) and so \(20 + 40 + 42 + 8r = 126\). Solving for \(r\), we obtain \(r = 3\).

19. See Figure 63.

Figure 63: The graph in Exercise 19
20. See Figure 64.

21. **Proof.** Suppose that $G$ contains $a$ vertices of degree $2k$, $b$ vertices of degree $2k + 1$ and $c$ vertices of degree $2k + 2$. Assume, to the contrary, that $a \leq 2k$, $b \leq 2k + 1$ and $c \leq 2k$. Thus $a + b + c \leq 2k + (2k + 1) + 2k = 6k + 1$. Since $G$ has $6k + 1$ vertices, $a = 2k$, $b = 2k + 1$ and $c = 2k$. However then, $G$ contains an odd number $b$ of odd vertices, which is impossible.

22. Construct a graph $G$ with vertex set $\{v_1, v_2, \ldots, v_8\}$ (that is, the set consisting of the eight students) and $v_i$ and $v_j$ are adjacent if $v_i$ and $v_j$ know each other. Since one student knows all other students, $\deg v_i \geq 1$ for $1 \leq i \leq 8$. Furthermore, there are seven vertices with different degrees and exactly two vertices have the same degree. We may assume that $\deg v_i = i$ for $1 \leq i \leq 7$. Thus $v_1$ is adjacent to all other vertices, $v_6$ is adjacent to every vertex except $v_1$, $v_5$ is adjacent to all vertices except $v_1$ and $v_2$ and $v_4$ is adjacent to all vertices except $v_1$, $v_2$ and $v_3$. Then $\deg v_8 = \deg v_4 = 4$ and $v_8$ and $v_4$ are adjacent and so these two students know each other.

23. (a) $P_4$
(b) $C_5$
(c) Suppose that there is a self-complementary graph $G$ of order 6. Then $G \cong \overline{G}$. Necessarily, $G$ and $\overline{G}$ have the same size $m$. Thus $2m = \binom{6}{2} = 15$, which says that $m = 15/2$. This is impossible.

24. Since $10r = 2 \cdot 30$, it follows that $r = 6$.

25. The statement is false. There is no 5-regular graph of order $2n + 1$.

26. **Proof.** Assume, to the contrary, that there exists a graph $G$ of order $n \geq 2$, whose vertices have distinct degrees. These degrees must be among the $n$ integers $0, 1, 2, \ldots, n - 1$. So $G$ must have one vertex of each such degree. Let $u$ and $v$ be the vertices of $G$ such that $\deg u = 0$ and $\deg v = n - 1$. Since $\deg u = 0$, it follows that $u$ is adjacent to no vertex of $G$, including $v$. On the other hand, since $\deg v = n - 1$, it follows that $v$ is adjacent to all other vertices of $G$, including $u$. This is impossible.

27. The graph $H$ is not a subgraph of $G$ since $vy$ is not an edge of $G$.

28. (a) The graph $H_1$ is not a subgraph of $G$ since $vy$ is not an edge of $G$. Although $H_2$ is a subgraph of $G$, it is not an induced subgraph of $G$ since $uw$ is not an edge of $H_2$ but $uw$ is an edge of $G$.
(b) $S = \{u, y, v, z\}$
(c) The induced subgraphs $G[S_1], G[S_2], G[S_3]$ are shown in Figure 65; while $G[S_4] = G$.

29. The statement is false. The graph $G$ in Figure 66 contains two subgraphs isomorphic to $K_4$; while $\overline{G}$ contains no such subgraph.
30. **Proof.** If $G$ is an $r$-regular graph of order $n$, then $\overline{G}$ is an $(n - 1 - r)$-regular graph. Similarly, if $\overline{G}$ is an $r$-regular of order $n$, then $\overline{\overline{G}} = G$ is an $(n - 1 - r)$-regular graph. □

31. **Proof.** Let $v \in V(G)$. Then $\deg_G v = \deg_{\overline{G}} v = r$. Since $\deg_G v + \deg_{\overline{G}} v = n - 1$, it follows that $2r = n - 1$ and so $n = 2r + 1$ is odd. □

32. The graphs $G_1, G_2$ and $G_3$ in Figure 67 have the desired properties.

33. The graphs $H_1, H_2$ and $H_3$ in Figure 67 have the desired properties.

34. $H_1 \not\cong H_2$ since $H_1$ contains two adjacent vertices of degree 2 but $H_2$ does not. $H_2 \not\cong H_3$ since $H_2$ has order 6 and $H_3$ has order 5. $H_3 \not\cong H_4$ since $H_3$ has size 4 and $H_4$ has size 5. $H_4 \cong H_5$ since $H_5$ can be redrawn as shown in Figure 68, producing the isomorphism $f$ defined by $f(u_1) = u_5, f(w_1) = y_5, f(y_1) = v_5, f(x_1) = w_5, f(v_1) = x_5$.

35. An isomorphism $f$ from $V(G_1)$ to $V(G_2)$ can be defined by $f(u_1) = v_2, f(v_1) = w_2, f(w_1) = z_2, f(x_1) = u_2, f(y_1) = x_2, f(z_1) = y_2$. An isomorphism $g$ from $V(H_1)$ to $V(H_2)$ can be defined by $g(p_1) = r_2, g(q_1) = s_2, g(r_1) = p_2, g(s_1) = q_2, g(t_1) = t_2$.

36. (a) Yes. Any isomorphism from $V(G)$ to $V(H)$ must map adjacent vertices of degree 4 to adjacent vertices of degree 4.

(b) No. See Figure 69.

(c) Yes. Let $\alpha$ be an isomorphism from $G$ to $H$ and $\beta$ an isomorphism from $G$ to $F$. Then $\beta \circ \alpha^{-1}$ is an isomorphism from $H$ to $F$.

37. (a) The statement is true. **Proof.** Suppose that $A_1 = A_2$. Then $A_1$ and $A_2$ are both $n \times n$ matrices for some positive integer $n$. Hence the orders of $G_1$ and $G_2$ are $n$. Let
38. The edge set of $G$ is $E(G) = \{G_1G_4, G_1G_6, G_2G_4, G_4G_6\}$.

39. (a) The edge set of the positive slope graph $G$ of $S$ is $\{A_1A_3, A_3A_2, A_2A_4, A_4A_1\}$ and the vertex $A_5$ is not adjacent to any vertex in $G$.

(b) If $n$ is even, say $n = 2k$, then let
$$S = \{(i - 1, 1 - i) : 0 \leq i \leq k - 1\} \cup \{(i + 1, 2 - i) : 0 \leq i \leq k\}.$$ 
If $n$ is odd, say $n = 2k + 1$, then let
$$S = \{(i - 1, 1 - i) : 0 \leq i \leq k\} \cup \{(i + 1, 2 - i) : 0 \leq i \leq k\}.$$

(c) No.

Section 12.2. Connected Graphs

1. Let $G$ be the star $K_{1,3}$ with $V(G) = \{u, v, w, x\}$, where $\deg x = 3$. Then $W = (u, x, v, w, x, v)$ is a $u - v$ walk in which $wx$ is encountered once, $wv$ is encountered twice and $vx$ is encountered three times.

2. Let $P$ be a $u - v$ path and $P'$ a $v - w$ path. Then $P$ followed by $P'$ is a $u - w$ walk. Thus $G$ contains a $u - w$ path by Theorem 12.26.

3. (a) $(q, u, y, v, r, u, x)$; (b) $(t, x, u, y)$; (c) $(u, q, t, x, u, r, v, y, u)$; (d) $(s, v, z, w, s)$.

4. (a) $(v_2); (v_2, v_3); (v_2, v_3, v_7); (v_2, v_3, v_7, v_1); (v_2, v_3, v_7, v_1, v_5); (v_2, v_3, v_7, v_1, v_5, v_8); (v_2, v_3, v_7, v_1, v_5, v_8, v_3)$ are paths of lengths 0, 1, ..., 6.

(b) $(v_2, v_3, v_7, v_2)$ and $(v_1, v_3, v_8, v_7, v_1)$ are cycles of length 3 and 4.

5. See Figure 71.