Name________________________________ **Student ID #** _________________________
Instructor___________________________ **Lab Period** _____ **Date Due** ___________

Lab 11
Approximate Integration

Objectives

1. To become familiar with the right endpoint rule, the trapezoidal rule, and Simpson's rule.
2. To compare and contrast the properties of these three methods of approximate integration.

Exploration 1 *Right Endpoint Rule*

The value of the definite integral $\int_a^b f(x)dx$ can be approximated using the *right endpoint rule*

$$\int_a^b f(x)dx \approx R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \Delta x \sum_{i=1}^{n} f(x_i)$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$.

In this exploration, we will use the right endpoint rule to approximate the value of the known definite integral, $\int_0^{\pi/2} \sin(x)dx = 1$. But first, we must set up *TEMATH* by doing the following:

- Select **New Constant** from the **Work** menu. Delete the default constant name and enter **a = 0**.
- Select **New Constant** from the **Work** menu. Delete the default constant name and enter **b = π/2** (press the **Option p** key for π).
- Select **New Function** from the **Work** menu. Delete the default function name and enter **f(x) = sin(x)**.
- Select **New Function** from the **Work** menu. Delete the default function name and enter the right endpoint rule (press the **Option w** key for ∑)

R(n) = (b–a)/n ∑(i, 1, n, f(a+i(b–a)/n)).

Note that $R(n)$ is a function of n, where n is the number of right endpoints (or the number of rectangles).

To gain some insight into the properties of the right endpoint rule, let's evaluate $R(n)$ for increasingly larger values of n and observe how well the values of $R(n)$ approximate the definite integral. We can use *TEMATH*'s Expression Calculator to evaluate $R(n)$ by following these instructions:

- Select **Accuracy…** from the **Options** menu.
- Enter **15** for the number of significant digits and click the **OK** button. The results of all calculations performed in the Expression Calculator window will now be displayed with fifteen significant digits.
- Select **Calculators — Expression Calculator…** from the **Work** menu.
- Enter **R(10)**. Press the **Enter** key. Be sure that the flashing cursor remains on the same line as R(10). The right endpoint rule will be evaluated using ten rectangles and its value will be written on the next line in the Expression Calculator window.

> ☞ An approximation is said to have *d correct decimal digits* of accuracy if
>
> $$|\text{exact value} - \text{approximate value}| < 0.5 \times 10^{-d}.$$
>
> For our particular example, this becomes $|1 - \text{approximate value}| < 0.5 \times 10^{-d}$.

1. a) What is the value of the approximation $R(10)$? ……………………………………………

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - R(10)\right|$ of the approximation? ………………

   c) How many correct decimal digits are there in the approximation? ……………………

To gain some insight into the relationship between the absolute error and the value of n, let's write the value of the absolute error in the form $\frac{c}{n}$, where c is a constant and n is the number of rectangles. For example, if the absolute error is $0.076 \approx 0.08$ and $n = 10$, then we can solve the equation $0.08 = c/10$ for c to get $c = 0.8$ and write the absolute error in the form $\left|\int_0^{\pi/2} \sin(x)dx - R(10)\right| \approx \frac{0.8}{10}$. We will use this form of the absolute error in the following examples.

Next, let's change $R(10)$ to **R(100)** and press the **Enter** key. The right endpoint rule is now evaluated using one hundred rectangles and its value is written on the next line in the Expression Calculator window.

2. a) What is the value of the approximation $R(100)$?……………………………………………

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - R(100)\right|$ of the approximation?……………

   c) How many correct decimal digits are there in the approximation? ……………………

d) Write the value of the absolute error in the form $\frac{c}{n}$ ....

e) How many more correct decimal digits are there in this approximation than the previous one with $n = 10$? ....

Change $R(100)$ to **R(1000)** and press the **Enter** key. The right endpoint rule is evaluated using one thousand rectangles and its value is written on the next line.

3. a) What is the value of the approximation $R(1000)$? ....

b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - R(1000)\right|$ of the approximation? ....

c) How many correct decimal digits are there in the approximation? ....

d) Write the value of the absolute error in the form $\frac{c}{n}$ ....

e) How many more correct decimal digits are there in this approximation than the previous one with $n = 100$? ....

> ☞ If there is a constant k such that the absolute error can be written as
>
> $$|\text{exact value} - \text{approximate value}| < \frac{k}{n}$$
>
> for all $n > 0$, then we say that the approximations *converge linearly* to the exact value as n increases.

4. Based on the above calculations, guess a value for k. ....

5. If the approximations *converge linearly* to the exact value, how many more correct decimal digits are there in the approximation when n is increased by a factor of 10?

....

Exploration 2 *Trapezoidal Rule*

The value of the definite integral $\int_a^b f(x)dx$ can be approximated using the *trapezoidal rule*

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right]$$

where $\Delta x = (b - a)/n$ and $x_i = a + i\Delta x$.

Let's use the trapezoidal rule to approximate the definite integral $\int_0^{\pi/2} \sin(x)dx = 1$ given in the previous exploration. If you've already done the previous exploration (entered *a*, *b*, and *f*(*x*)), you only need to enter the trapezoidal rule by doing the following:

- Select **New Function** from the **Work** menu. Delete the default function name and enter the trapezoidal rule (press the **Option w** key for ∑)

T(n) = (b–a)/(2n) (f(a)+2∑(i, 1, n–1, f(a+i(b–a)/n))+f(b)).

Note that $T(n)$ is a function of n, where n is the number of trapezoids used in approximating the integral on the interval $[a, b]$.

Use the Expression Calculator (as described in the previous exploration) to evaluate $T(10)$, $T(100)$, and $T(1000)$.

1. a) What is the value of the approximation $T(10)$? ....

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - T(10)\right|$ of the approximation? ....

   c) How many correct decimal digits are there in the approximation? ....

   d) Write the value of the absolute error in the form $\frac{c}{n^2}$ ....

2. a) What is the value of the approximation $T(100)$?...

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - T(100)\right|$ of the approximation?................

   c) How many correct decimal digits are there in the approximation?.......................

   d) Write the value of the absolute error in the form $\frac{c}{n^2}$ ....

   e) How many more correct decimal digits are there in this approximation than the previous one with $n = 10$? ....

3. a) What is the value of the approximation $T(1000)$?..

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - T(1000)\right|$ of the approximation? ....

   c) How many correct decimal digits are there in the approximation? ....

   d) Write the value of the absolute error in the form $\frac{c}{n^2}$ ....

e) How many more correct decimal digits are there in this approximation than the previous one with $n = 100$?……………………………………………………………

> ☞ If there is a constant k such that the absolute error can be written as
>
> $$|\text{ exact value} - \text{approximate value }| < \frac{k}{n^2}$$
>
> for all $n > 0$, then we say that the convergence of the approximations to the exact value is *quadratic*.

4. Based on the above calculations, guess a value for k.………………………………………

5. If the convergence is *quadratic*, how many more correct decimal digits are there in the approximation when n is increased by a factor of 10? ……………………………………

Exploration 3 *Simpson's Rule*

The value of the definite integral $\int_a^b f(x)dx$ can be approximated using Simpson's rule

$$\int_a^b f(x)dx \approx S_n = \frac{\Delta x}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(b)\right]$$

where $\Delta x = (b - a)/n$, $x_i = a + i\Delta x$, and n is an even positive integer.

Let's use Simpson's Rule to approximate the same definite integral $\int_0^{\pi/2} \sin(x)dx = 1$ given in the previous explorations. If you've already done the previous exploration (entered a, b, and $f(x)$), you only need to enter Simpson's rule by doing the following

- Select **New Function** from the **Work** menu. Delete the default function name and enter Simpson's rule (press the **Option w** key for ∑)

S(n) = (b–a)/(3n) (f(a)+2∑(i,1,n/2–1,f(a+2i(b–a)/n))+ 4∑(i,1,n/2,f(a+(2i–1)(b–a)/n))+f(b)).

Use the expression calculator (as described in the previous exploration) to evaluate $S(10)$, $S(100)$, and $S(1000)$.

1. a) What is the value of the approximation $S(10)$? ……………………………………………

   b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - S(10)\right|$ of the approximation? ………………

c) How many correct decimal digits are there in the approximation? ……………………

d) Write the value of the absolute error in the form $\dfrac{c}{n^4}$ ……………………………………

2. a) What is the value of the approximation $S(100)$?………………………………………

b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - S(100)\right|$ of the approximation?……………

c) How many correct decimal digits are there in the approximation? ……………………

d) Write the value of the absolute error in the form $\dfrac{c}{n^4}$ ……………………………………

e) How many more correct decimal digits are there in this approximation than the previous one with $n = 10$? ………………………………………………………………

3. a) What is the value of the approximation $S(1000)$? ……………………………………

b) What is the absolute error $\left|\int_0^{\pi/2} \sin(x)dx - S(1000)\right|$ of the approximation? ……………

c) How many correct decimal digits are there in the approximation? ……………………

d) Write the value of the absolute error in the form $\dfrac{c}{n^4}$ ……………………………………

e) How many more correct decimal digits are there in this approximation than the previous one with $n = 100$?……………………………………………………………

☞ If there is a constant k such that the absolute error can be written as

$$|\text{ exact value} - \text{approximate value }| < \frac{k}{n^p}$$

for all $n > 0$, then we say that the convergence of the approximations to the exact value is *of the order p*.

4. Based on the above calculations, guess a value for k.……………………………………

5. If the convergence is *of the order 4*, how many more correct decimal digits are there in the approximation when n is increased by a factor of 10? ………………………………

6. a) Which of the three approximate integration methods is the most *efficient*, that is, which method uses the fewest function evaluations to achieve a particular accuracy?

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b) Explain your answer to part a). ………………………………………………………

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c) Describe the procedure you would use to approximate the value of a definite integral (whose exact value you did not know) to ten correct decimal digits.…………………

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Exploration 4 *Error Bounds*

Are the patterns of error reduction found in the previous explorations the same for any definite integral? To investigate this question, let's approximate another known definite integral $\int_0^4 \sqrt{x}\,dx = 5.3333\ldots$ by using the methods described in the first three explorations. First, let's set up TEMATH by doing the following:

- Set **a = 0**, **b = 4**, and **f(x) = sqrt(x)** in the **Work** window.

1. Use the Expression Calculator to find the following:

a) $R(10)$…………………… $R(100)$………………… $R(1000)$……………………

b) $T(10)$…………………… $T(100)$………………… $T(1000)$……………………

c) $S(10)$…………………… $S(100)$………………… $S(1000)$……………………

2. As the value of n increased from 10 to 100 to 1000, was the pattern of the reduction of the approximation error the same as in the previous explorations?………………………

   Explain.…………………………………………………………………………………

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At this point, you should refer to the section in your calculus text book on approximate integration techniques and read it carefully.

It can be shown that if $|f'(x)| \le M_R$, $|f''(x)| \le M_T$, and $|f^{(4)}(x)| \le M_S$ for $a \le x \le b$ (that is, there exist a set of constants M_R, M_T, and M_S which bound the values of the derivatives of $f(x)$), then the absolute errors in the Right Endpoint Rule, the Trapezoidal Rule, and Simpson's Rule are bounded as follows:

$$|E_R| \le \frac{M_R(b-a)^2}{2n}, \quad |E_T| \le \frac{M_T(b-a)^3}{12n^2}, \quad \text{and} \quad |E_S| \le \frac{M_S(b-a)^5}{180n^4}$$

3. a) For $f(x) = \sin(x)$, find a bound M_R such that $|f'(x)| \le M_R$ for $0 \le x \le \pi/2$ ……

   ……………………………………………………………………………………………

   b) Find the theoretical value of k such that $|E_R| \le \frac{M_R(b-a)^2}{2n} = \frac{k}{n}$ ……………………

   c) Are the results of the first exploration consistent with the theoretical error bound calculated in part b)? Explain.…………………………………………………………

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4. Use the theoretical error bounds given above to explain the patterns of the approximation errors observed in the first three explorations for the Right Endpoint Rule, the Trapezoidal Rule, and Simpson's Rule. …………………………………………………

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5. Use the theoretical error bounds given above to explain why the pattern of approximation error reduction changed in exploration 4 with $f(x) = \sqrt{x}$. Give special attention to the derivatives of $f(x)$ on the interval (0, 4).…………………………………

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Exploration 5 *Evaluating Integrals*

The integral $\int_0^1 \sin(x^2)dx$ has no closed form solution. Use the Expression Calculator to find an approximation to this integral that has twelve correct decimal digits.

1. a) $\int_0^1 \sin(x^2)dx \approx$ ……………………………………………………………………

   b) Explain how you obtained the answer given in part a) and give reasons why the approximation is correct to twelve decimal digits. ……………………………………

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