Name________________________________ **Student ID #** _________________________
Instructor___________________________ **Lab Period** _____ **Date Due** ___________

Lab 7
The Mean Value Theorem

Objectives

1. To graphically demonstrate the Mean Value Theorem.
2. To develop an understanding of the hypotheses of the Mean Value Theorem.

Many times when we use a theorem in solving a problem, we take for granted that the hypotheses given in the theorem are satisfied and we never check to see if that is in fact true. We simply assume that the result of the theorem will be true. In this lab, we will thoroughly investigate the importance of the hypotheses of a theorem and try to convince ourselves that before we solve a problem by applying a theorem, we should verify that the hypotheses given in the theorem are satisfied. The mean value theorem will be used as the basis for all our explorations in this lab.

The Mean Value Theorem

Let *f* be a function that satisfies the following hypotheses:

1. *f* is continuous on the closed interval $[a, b]$.
2. *f* is differentiable on the open interval (a, b).

Then there is a number *c* in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The mean value theorem states that under the specified hypotheses, there is a point in the interval of interest such that the slope of the tangent line at that point is equal to the slope of the secant line connecting the two endpoints of the graph of the function. In other words, the average rate of change of the function *f* on the interval $[a,b]$ is equal to the instantaneous rate of change of the function *f* at some point *c* in (a,b).

Exploration 1 Approximating the Value of *c*

In this exploration, we will graphically find an approximation to the value of *c* for which the mean value theorem is satisfied. Let $f(x) = 2x - x^2$ be the function in the mean value theorem and let $[a, b] = [0,3]$ be the closed interval in the mean value theorem. To find an approximation to the value of *c*, let's enter and plot the function

$\boldsymbol{f(x) = 2x - x^2}$ on the interval $\boldsymbol{0 \le x \le 3}$. Next, we need to draw the secant line and find its slope. To do this, first click the **Line Segment** tool, then, move the mouse to the left endpoint of the curve, click and hold down the mouse button, drag the mouse to the right endpoint of the curve, and release the mouse button. The secant line will be drawn and the slope of the secant line will be written into the Report window.

1. What is the slope of the secant line?…………………………………………………………

To find the value of *c* given in the Mean Value Theorem, we need to find a tangent line to the curve that has the same slope as the secant line. Examine the curve in the Graph window and try to visualize a point where the tangent to the curve will be parallel (same slope) to the secant line. When you have visualized this point, draw the tangent line at that point by clicking the **Tangent Line** tool to make it active and then clicking the point on the curve where you want the tangent line drawn. Note that the slope of the tangent line is written into the Report window. If the slope of the tangent line is not approximately equal to the slope of the secant line, click at other points on the curve to draw additional tangent lines until you find one with approximately the same slope as the secant line. If the graph gets messy with too many tangent lines, select **Delete All Tangents** from the **Edit** menu to erase all the tangent lines from the Graph window.

Note: You can also enter the *x*-value of the point where you want the tangent line drawn by entering its value into the ***x*-cell** of the **Domain & Range** window and by clicking the **Enter** button or pressing the **Enter** key. In this way, you can enter the *x*-value more accurately.

2. a) What is an approximate value of *c*? ……………………………………………………

   b) Find the exact value of *c* by using some algebra. Show all your work..………………

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Note: In the following explorations, make sure that **Auto Clear** is checked in the **Options** menu.

Exploration 2 Differentiability and the Mean Value Theorem

Are both the continuity and differentiability hypotheses really necessary for the result of the mean value theorem to be always true? What happens if we violate one of these hypotheses, for example, what if we pick a function that is not *differentiable* on the open interval (a, b). To find out what happens, let's examine the function $f(x) = \sqrt[3]{(x-1)^2}$ on the closed interval $[0,3]$.

1. Why is $f(x)$ not differentiable on $(0, 3)$? …………………………………………………

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Enter and plot the function $f(x) = \sqrt[3]{(x-1)^2}$ on the interval $[0,3]$. In *TEMATH*, the cube root function is entered as **rad(3, (x – 1)^2)**. Note that rad(n, g(x)) is *TEMATH*'s predefined function for the nth root of $g(x)$. Next, draw the secant line connecting the endpoints of the graph. Try drawing a tangent line to the curve that has the same slope as the secant line.

2. a) Is it possible to draw a tangent line that is parallel to the secant line?…………………

   b) Can you find a number *c* that satisfies the mean value theorem? ……………………

   c) If the differentiability hypothesis is violated, can we guarantee that a *c* will exist that satisfies the result of the Mean Value Theorem? Explain. ……………………………

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Exploration 3 Continuity and the Mean Value Theorem

In this exploration, we will investigate to see what happens if we violate the *continuity* hypothesis for $f(x)$ on the closed interval $[0,3]$?

Let's begin by entering and plotting the function $f(x) = \begin{cases} x & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$ on the interval $[0,3]$. In *TEMATH*, the piecewise function can be entered as **x if x ≠ 3; 2** (Press the **Option** = key to get ≠).

1. Why is $f(x)$ not continuous on $[0,3]$?………………………………………………………

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Next, draw the secant line connecting the *endpoints* of the graph. *Be careful*.

2. a) Is it possible to draw a tangent line that is parallel to the secant line?…………………

   b) Can you find a number *c* that satisfies the mean value theorem? ……………………

c) If the continuity hypothesis is violated, can we guarantee that a *c* will exist that satisfies the result of the Mean Value Theorem? Explain. ……………………………

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Exploration 4 Mean Value Theorem Problems

Enter and plot the function $f(x) = \begin{cases} \sin(2x) & \text{if } x \leq 2 \\ x - 2 & \text{if } x > 2 \end{cases}$ on the interval $[0,3]$. In *TEMATH*, the piecewise function can be entered as **sin(2x) if x ≤ 2; x–2**. Be sure that the secant line connects the *endpoints* of the graph.

1. a) Is $f(x)$ continuous on $[a,b]$? Why or why not?………………………………………

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b) Is $f(x)$ differentiable on (a,b)? Why or why not? …………………………………

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c) Can you draw a tangent line that is parallel to the secant line?………………………

If yes, what is the approximate value of *c*?……………………………………………

d) Is it possible for one or both of the hypotheses of the mean value theorem *not* to be satisfied but the results of the mean value theorem still be true?........... Explain your answer. ……………………………………………………………………………

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Enter and plot the function $f(x) = \dfrac{x}{x-1}$ on the interval $[0,3]$.

2. a) Is $f(x)$ continuous on $[a,b]$? Why or why not?……………………………………

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b) Is $f(x)$ differentiable on (a,b)? Why or why not? …………………………………

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c) Can you draw a tangent line that is parallel to the secant line?………………………

If no, describe why a tangent can't be drawn with the same slope as the secant line. ..

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Enter and plot the function $f(x) = \cos(x)$ on the interval $[0,3]$.

3. Is there more than one value of c that satisfies the mean value theorem? …………………

If yes, find *all* the values of c in $[0,3]$……………………………………………………

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Exploration 5 Finding Example Problems

In this exploration you will be asked to give some example functions that satisfy a specified set of criteria. In all the examples given below, <u>attach to this lab a printed copy of the graph and the tangent line(s) (if there is one)</u>. Give example functions that are different from those found in explorations 1-4.

1. a) Find a function $f(x)$ and an interval $[a,b]$ for which all the hypotheses of the mean value theorem are satisfied.…………………………………………………………

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b) Find the value of c that satisfies the mean value theorem. ……………………………

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2. a) Find a function $f(x)$ and an interval $[a,b]$ for which at least one of the hypotheses of the mean value theorem is <u>not</u> satisfied but a number *c* can still be found, that is, a tangent line can be drawn parallel to the secant line. <u>Note</u>: If the hypotheses of the mean value theorem are satisfied, then the mean value theorem <u>guarantees</u> the existence of a number *c*, however, if the hypotheses of the mean value theorem are <u>not</u> satisfied, then a number *c* may or may not exist, that is, there is no longer a guaranteed existence.……………………………………………………………………

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b) Find the value of *c* that satisfies the mean value theorem. ……………………………

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3. a) Find a function $f(x)$ and an interval $[a,b]$ for which all the hypotheses of the mean value theorem are satisfied and there is more than one number *c* in $[a,b]$ that satisfies the conclusion of the mean value theorem. …………………………………

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b) Find *all* the values of *c* that satisfies the mean value theorem. ……………………….

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4. a) Find a function $f(x)$ and an interval $[a,b]$ for which at least one of the hypotheses of the mean value theorem is <u>not</u> satisfied and a number *c* can <u>not</u> be found, that is, there is no tangent line that can be drawn parallel to the secant line.…………………

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b) Is $f(x)$ continuous on $[a,b]$? Why or why not?……………………………………

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c) Is $f(x)$ differentiable on (a,b)? Why or why not? …………………………………

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