Name_______________________________ **Student ID #** ________________________

Instructor___________________________ **Lab Period** _____ **Date Due** ___________

Lab 6
The Tangent

Objectives

1. To visualize the concept of the tangent.
2. To define the slope of the tangent line.
3. To develop a definition of the tangent line to a curve at a point.

The goal of this lab is to develop a mathematically accurate definition of the *tangent line to a curve at a point.* The first part of this lab will take you through a visual exploration that will help you develop an intuition about the concept of the tangent line. Based on this intuition, you will be asked to give an accurate definition of the tangent line. In the second part of the lab, you will test the accuracy of your "tangent line" definition on other examples and, if necessary, you will have the opportunity to revise your definition based on the information obtained from these examples.

Exploration 1 Secant Lines and the Tangent Line

A *secant line* is a line that passes through two different points on a curve.

In this exploration, you want to use a particular set of secant lines to help you formulate a definition of the tangent line to a curve at a point.

☞ Files: This lab requires that you set a specific domain and range, select a particular pen mode, and enter some functions and constants into *TEMATH*. If you have been given the *TEMATH* file *Lab 6 The Tangent*, skip the following set-up section and open this file by doing the following:

- If you have entered some objects into the Work window and it is not empty, select **Close Work ...** from the **File** menu.
- Select **Open Work ...** from the **File** menu.
- Select the file **"Lab 6 The Tangent"** and click the **Open** button.

Note that:

$f(x) =$ the example function under consideration
$x_0 =$ the x-value of the point of interest on the curve
$h =$ the horizontal distance between the two points on the secant line
$m(h) =$ the slope function of the secant line
$s(x) =$ the function representing the secant line

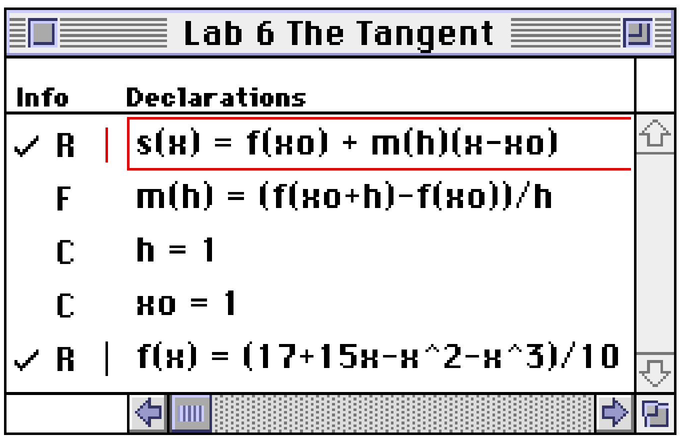
> If you have not been given the *TEMATH* file *Lab 6 The Tangent*, then follow the instructions given below in the set-up section. Remember, you can always save your work in a file on your own disk and open it at a later time.

Set-up Section

First, you need to set up *TEMATH* by doing the following:

- Click *off* autoscaling, set the domain to $-5 \le x \le 5$ and set the range to $-5 \le y \le 5$.
- Select **Pen...** from the **Options** menu. Click the box containing the × to the left of **Highlight Selected Plot** to remove the ×. Click the **OK** button. All graphs will now be plotted with a thin line.

You will also need to enter some functions and constants into *TEMATH* to aid you in your exploration. Instructions for doing this are given below and the goal is to have a *TEMATH* Work window that looks like the following:



You will use the cubic polynomial function $f(x) = (17 + 15x - x^2 - x^3)/10$ to generate a curve for this exploration. To enter this Function into *TEMATH*, do the following:

- Select **New Function** from the **Work** menu.
- Use the **Delete** key to delete the default name of the function in the **Work** window and enter **f(x) = (17+15x–x^2–x^3)/10**. Press the **Enter** or **Return** key.
- Select **Plot** from the **Graph** menu.

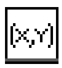
> ☞ After you've enter a function or constant into *TEMATH*'s Work window, you can easily change the definition of the function or constant by using any of the Macintosh's editing features and then pressing the **Enter** key to enter the new definition. Additionally, you can change the name a function or constant as long as it has not been used in the definition of another function or constant.

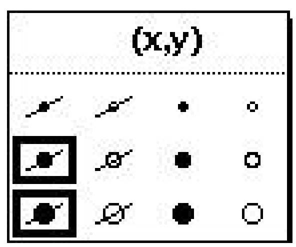
> ☞ You can remove a function or constant from the Work window by clicking on it to select it and then selecting **Remove** from the **Work** menu.

Later on in this lab you will need to overlay the plot of the secant line passing through the two points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ on top of the graph of $f(x)$. To set up *TEMATH* to do this for $x_0 = 1$ and $h = 1$, follow these instructions:

- Select **New Constant** from the **Work** menu.
- Delete the default name of the constant in the **Work** window and enter **xo = 1**.
- Press the **Enter** or **Return** key.
- Select **New Constant** from the **Work** menu.
- Delete the default name of the constant in the **Work** window and enter **h = 1**.
- Press the **Enter** or **Return** key.

To help visualize the point of interest on the curve, draw a dot on the curve at $x_0 = 1$ by doing the following:

- Click in the **Graph** window to make it the active window.
- Click the **Coordinate** tool  in the **Graph** window.
- Position the cursor over the top portion of the **Coordinate** tool icon . Press and hold down the mouse button. Drag the cursor to the medium sized solid dot on the left side of the pop-up menu



- Release the mouse button.
- Click in the **Domain & Range** window. Enter **1** into the **"x ="** cell in the lower portion of the window. Press the **Enter** key. A dot should be drawn on the curve at the point (1, 3).

In this exploration, you will need to find the slope of many different secant lines so it will be beneficial to enter the slope equation $m = \dfrac{f(x_0 + h) - f(x_0)}{h}$ into *TEMATH* by following these instructions:

- Select **New Function** from the **Work** menu.
- Delete the default name of the function in the Work window and enter **m(h) = (f(xo+h)–f(xo))/h**. Note that we entered the slope as a function of *h*, where *h* is the horizontal distance between the two points on the curve through which we want to draw the secant line.
- Press the **Enter** or **Return** key.

The point-slope form of the equation of the secant line passing through the two points $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ is $y = f(x_0) + m(x - x_0)$. Enter this equation into *TEMATH* as the function $y = s(x)$ by following these instructions:

- Select **New Function** from the **Work** menu
- Delete the default name of the function in the Work window and enter **s(x) = f(xo) + m(h)(x–xo)**. Press the **Enter** or **Return** key.

You have now completed the set up portion of this lab.

The purpose of this exploration is to see what happens to the secant lines as the value of h gets smaller and smaller, that is, as the magnitude of h gets closer and closer to zero. To begin, overlay the plot of the secant line for $h = 1$ by doing the following:

- If $s(x)$ in not selected in the Work window, then click on it to select it.
- Select **Overlay Plot** from the **Graph** menu. The secant line passing through the points $(1, 3)$ and $(2, 3.5)$ will be plotted.

To make h smaller in magnitude and to overlay the plot of the new secant line, do the following:

- Change the value of h to **0.5** in the **Work** window, that is, change $h = 1$ to $h = 0.5$. Remember to press the **Enter** key after you make this change.
- Click in the cell in the **Work** window that contains the secant line function $s(x)$. This selects the function for plotting.
- Change the plot color/pattern of $s(x)$ by clicking on the vertical line immediately to the left of $s(x)$ in the Work window. This will help you to distinguish the different secant lines that will be plotted in the Graph window.
- Select **Overlay Plot** from the **Graph** menu.

Repeat the above sequence of instructions for $h = 0.1$, 0.01, and 0.001.

1. Describe what happens to the graphs of the secant lines as the value of h gets closer and closer to zero, that is, as the two points on the curve get closer and closer to each other?

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In everything that you have done so far, the second point on the secant line has always been to the right of the point $(x_0, f(x_0))$ (since h was positive). If you let h equal a negative number, then the point $(x_0 + h, f(x_0 + h))$ will be to the left of the point $(x_0, f(x_0))$. To see what happens to the secant lines for negative values of *h*, repeat the "moving secant line" process described above, but this time for negative values of *h*. However, before you do this, click in the Work window to make it active, select *f*(*x*), and plot *f*(*x*). This will clean up the Graph window by erasing the previously drawn secant lines. Next draw the dot at the point on the curve and draw the secant lines for $h = -1, -0.5, -0.1, -0.01,$ and -0.001.

2. Describe what happens to the graphs of the secant lines as h gets closer and closer to zero from the <u>negative</u> direction?……………………………………………………………

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To help you formulate an accurate definition of the tangent line, do the following:

- Plot *f*(*x*). This will clear all the secant lines from the Graph window.
- Overlay the plots of the secant lines for $h = 0.001$ and $h = -0.001$.

3. What do you observe about these two secant lines?………………………………………

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4. Do the secant lines approach the same line for positive values of h and for negative values of h?…………………………………………………………………………………

Let's call the line that all the secant lines are approaching as h gets closer and closer to zero the *tangent line*.

5. Using words, give an accurate definition for the *tangent line* to the curve $y = f(x)$ at the point $(x_0, f(x_0))$?………………………………………………………………………

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In order to find the equation of the tangent line, you need to know a point the tangent line passes through and the slope of the tangent line. Since all the secant lines pass through the point $(x_0, f(x_0)) = (1,3)$, you only need to find the slope. Since the secant lines are approaching the tangent line as h gets closer and closer to zero, the slope of these secant lines should also be getting closer and closer to the slope of the tangent line. To find this slope, you need to let h get closer and closer to zero and calculate the slopes of the corresponding secant lines. Remember, the Work window should already contain the slope function $m(h)$ for the secant lines. To calculate the slopes for different values of h, do the following:

- Select **Calculators — Expression Calculator** from the **Work** menu. The Expression Calculator window will open.
- Enter the expression **m(1)** and press the **Enter** key (make sure you press the Enter Key and not the Return Key). The slope of the secant line for $h = 1$ will be written on the next line. Record the value of this slope into the table given in question 6.
- Change m(1) to **m(0.1)**. Make sure the flashing cursor remains on the same line as m(0.1). Press the **Enter** key. Repeat this for **m(0.01)**, **m(0.001)**, **m(0.0001)**, **m(–1)**, **m(–0.1)**, **m(–0.01)**, **m(–0.001)**, and **m(–0.0001)**.

6. a) Enter the slopes of the secant lines into the following table:

h	*m(h)*	*h*	*m(h)*
1.0		–1.0	
0.1		–0.1	
0.01		–0.01	
0.001		–0.001	
0.0001		–0.0001	

b) As h gets closer and closer to zero from the positive direction, what value are the slopes of the secant lines getting close to? ……………………………………………

c) As h gets closer and closer to zero from the negative direction, what value are the slopes of the secant lines getting close to? …………………………………………

d) What value would you give to the slope of the tangent line?…………………………

7. What is the equation of the tangent line?…………………………………………………

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Exploration 2 Testing Your Definition of the Tangent

The goal of this exploration is to test your definition of the tangent line by using some other example functions. The first example consists of the function $f(x) = (x^3 - 3x^2 + 6x + 6)/10$ and the point on its curve at $x_0 = 1$. To enter and plot this function, do the following:

- Change the f(x) function in the Work window to **f(x) = (x^3–3x^2+6x+6)/10**.
- Select **Plot** from the **Graph** menu.

Using the process described in the first exploration, answer the following questions.

1. Describe what happens to the graphs of the secant lines as h gets closer and closer to zero?………………………………………………………………………………………

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To help you visualize the concept of the tangent line, do the following:

- Plot *f*(*x*).
- Overlay the plots of the secant lines for $h = 0.001$ and $h = -0.001$.

2. What do you observe about these two secant lines?………………………………………

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3. a) Enter the slopes of the secant lines into the following table:

h	*m(h)*	*h*	*m(h)*
1		–1	
0.1		–0.1	
0.01		–0.01	
0.001		–0.001	

b) As *h* gets closer and closer to zero from the positive direction (positive *h*), what value are the slopes of the secant lines getting close to? ………………………………

c) As *h* gets closer and closer to zero from the negative direction (negative *h*), what value are the slopes of the secant lines getting close to? ………………………………

d) Are the values in b) and c) the same? …………………………………………………

4. a) Does this curve have a tangent line at $x_0 = 1$? ……….. Explain why or why not.….

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b) If this curve has a tangent line at $x_0 = 1$, what is its equation? ………………………

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c) If this curve has a tangent line at $x_0 = 1$, is there anything that is different about it as compared to the tangent line found in exploration 1?…………………………………

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5. Does your original definition of the tangent line hold for this example?…………………

If not, explain why not and give a new definition..………………………………………

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Let's try another example. This time, let $f(x) = \left|1 - x^2\right|$ and $x_0 = 1$.

- Change the f(x) function in the Work window to **f(x) = abs(1–x^2)**.
- Select **Plot** from the **Graph** menu.

Using the process described in the first exploration (but only use h = 0.1, 0.01, 0.001, –0.1, –0.01, and –0.001), answer the following questions.

6. Describe what happens to the graphs of the secant lines as h gets closer and closer to zero?………………………………………………………………………………

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To help you visualize the concept of the tangent line, do the following:

- Plot $f(x)$.
- Overlay the plots of the secant lines for $h = 0.001$ and $h = -0.001$.

7. What do you observe about these two secant lines?…………………………………………

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8. a) Enter the slopes of the secant lines into the following table:

h	*m(h)*	*h*	*m(h)*
0.1		–0.1	
0.01		–0.01	
0.001		–0.001	
0.0001		–0.0001	

b) As h gets closer and closer to zero from the <u>positive</u> direction (positive h), what value are the slopes of the secant lines getting close to? ………………………………

c) As h gets closer and closer to zero from the <u>negative</u> direction (negative h), what value are the slopes of the secant lines getting close to? ………………………………

d) Are the values in b) and c) the same? ……………………………………………………

9. a) Does this curve have a tangent line at $x_0 = 1$?………. Explain why or why not.….

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b) If this curve has a tangent line at $x_0 = 1$, what is its equation? ………………………

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10. Does your definition of the tangent line hold for this example? …………………………

If not, explain why not and give a new definition.………………………………………

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Let's try one more example. This time, let $f(x) = 2(x-1)^{1/3} + 1$ and $x_0 = 1$.

- Change the f(x) function in the Work window to **f(x) = 2 rad(3, x–1) + 1**. Note that in *TEMATH*, the n^{th} root of a function $(g(x))^{1/n}$ is written as rad(n, g(x)).
- Select **Plot** from the **Graph** menu.

Using the process described in the first exploration (but only use *h* = 0.1, 0.01, 0.001, –0.1, –0.01, and –0.001), answer the following questions.

11. Describe what happens to the graphs of the secant lines as h gets closer and closer to zero?………………………………………………………………………………

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To help you visualize the concept of the tangent line, do the following:

- Plot $f(x)$.
- Overlay the plots of the secant lines for $h = 0.001$ and $h = -0.001$.

12. What do you observe about these two secant lines?……………………………………………

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13. a) Enter the slopes of the secant lines into the following table:

h	*m(h)*	*h*	*m(h)*
0.1		–0.1	
0.01		–0.01	
0.001		–0.001	
0.0001		–0.0001	

b) As h gets closer and closer to zero from the <u>positive</u> direction (positive h), what happens to the slopes of the secant lines? ……………………………………………

c) As h gets closer and closer to zero from the <u>negative</u> direction (negative h), what happens to the slopes of the secant lines? ……………………………………………

d) Are there any similarities between the slopes in b) and c)?……………………………

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14. a) Does this curve have a tangent line at $x_0 = 1$?………… Explain why or why not.….

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b) If this curve has a tangent line at $x_0 = 1$, what is its equation? ………………………

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c) If this curve has a tangent line at $x_0 = 1$, is there anything that is different about it as compared to the other tangent lines found previously?………………………………

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15. Does your definition of the tangent line hold for this example? …………………………

If not, explain why not and give a new definition..……………………………………………

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