Name ______________________________ **Student ID #** _______________________
Instructor __________________________ **Lab Period**_____ **Date Due** __________

Lab 4
The Limit of a Function

Objectives

1. To learn how to numerically estimate limits.
2. To develop an intuitive understanding of the limit of a function.
3. To become familiar with left-hand and right-hand limits.

Exploration 1   Estimating Limits

As the values of x become close to 1 but not equal to 1, what happens to the values of $f(x) = 3 - x^2$? To answer this question, let's do the following:

- Plot $f(x) = 3 - x^2$ on the interval $0 \le x \le 2$.
- Click the **Rectangular Tracker Tool** to make it active.
- Place the cursor on the upward pointing arrow ▲ at the *left end* of the x-axis.
- Press and hold down the mouse button and drag the arrow toward 1 along the x-axis. Note that you can read the values of the coordinates of a point on the curve with the tracker in the Graph window or you can read the values in the bottom two cells of the Domain & Range window. The values in the Domain & Range window are displayed with additional significant digits.

1. What happens to the values of $f(x)$ as x approaches 1 from the left? ....

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- Move the cursor to the *right end* of the x-axis.
- Press and hold down the mouse button and drag to the left toward 1 along the x-axis.

2. What happens to the values of $f(x)$ as x approaches 1 from the right? ....

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The Rectangular Tracker Tool provides informal evidence that the values of $f(x)$ approach 2 as x approaches 1. But in formal quantitative terms, can we make $f(x)$ as close to 2 as we like by taking x sufficiently close to 1? For example, suppose we want $f(x)$ to be within 0.0001 units of 2, that is, suppose we want $2 - 0.0001 < f(x) < 2 + 0.0001$ (or $1.9999 < f(x) < 2.0001$). How close does x have to be to 1 for this inequality to be true? To answer this question, let's generate a table of values for x and $f(x)$.

- Select the **Accuracy** item from the **Options** menu and enter **15** in the **upper cell**.
- Click the **OK** button or press the **Return** key. The results of all calculations will now be displayed with fifteen significant digits.
- Select **New Data Table — Keyboard Entry** from the **Work** menu.
- Double-click the **Resize** command in the data table command list. Enter **9** in the command area and press the **Enter** key. This places nine empty rows in the data table.
- To fill the x-column with values of x that get closer to 1 but are less than 1; select the ***x*-column**, double-click the **Fill** command, enter the fill function **x(i) = 1 – 10^(–i)**, and press the **Enter** key. Note that the variable **i** represents the row number in the data table.
- To generate the $f(x) = 3 - x^2$ values; select the ***x*-column**, double-click the **Generate** command, enter the generating function **y(x) = 3 – x^2**, and press the **Enter** key. The y-column is filled with the values of $f(x)$ corresponding to the x values in the first column.

We say that a is *within d* units of L if *a's* distance from L is less than d units as is shown in the open interval diagrams below.

This condition is expressed formally by the inequality

$$L - d < a < L + d$$

which can be simplified to

$$|a - L| < d.$$

3. Is there an entry in the x-column for which $f(x)$ is within 0.0001 of 2 for this entry and

   for *all larger* values in the x-column? ....

   If so, what is it? ....

   Next, let's see what happens when we approach 1 from the right.

- Fill the x-column using the fill function **x(i) = 1 + 10^(–i)** and generate the y-column for these x values.

4. Is there an entry in the x-column for which $f(x)$ is within 0.0001 of 2 for this entry and for *all smaller* values in the x-column? .... If so, what is it? ....

5. Based on your answers to questions 3 and 4 above, is there a number δ such that if x is within δ of 1 (that is, if $1-\delta < x < 1+\delta$), then $f(x)$ is within 0.0001 of 2 (that is, $2-0.0001 < f(x) < 2+0.0001$)? .... If so, what is δ? ....

Let's represent the number 0.0001 by the symbol ε , that is, let $\varepsilon = 0.0001$. The exploration above shows that if $1-\delta < x < 1+\delta$, then $2-\varepsilon < f(x) < 2+\varepsilon$. If we can find a δ for any ε no matter how small, then we will say that the limit of $f(x) = 3 - x^2$ as x approaches 1 is equal to 2 and we will write

$$\lim_{x \to 1}(3 - x^2) = 2.$$

Exploration 2 The Limit of a Rational Function

Using the procedure given in exploration 1 (use the fill functions **1 – 10^(–i)** and **1 + 10^(–i)**), let's estimate the value of

$$\lim_{x \to 1} \frac{\left(x^4 - x^3 + x^2 - 1\right)}{\left(x^3 - 2x^2 + 1\right)}.$$

1. Can you evaluate $\dfrac{\left(x^4 - x^3 + x^2 - 1\right)}{\left(x^3 - 2x^2 + 1\right)}$ at $x = 1$? .... If not, why not?..................

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2. What happens to the values of $\dfrac{\left(x^4 - x^3 + x^2 - 1\right)}{\left(x^3 - 2x^2 + 1\right)}$ as x gets close to 1 from the left?.........

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3. What happens to the values of $\dfrac{\left(x^4 - x^3 + x^2 - 1\right)}{\left(x^3 - 2x^2 + 1\right)}$ as x gets close to 1 from the right?.......

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4. What is your estimate for the value of the limit (if it exists)?...................................

Exploration 3 The Limit of an Algebraic Function

Using the procedure given in exploration 1 (use the fill functions **8 – 10^(–i)** and **8 + 10^(–i)**), let's estimate the value of

$$\lim_{x \to 8} \frac{\left(x^{1/3} - 2\right)}{(x - 8)}.$$

In *TEMATH*, $x^{1/3}$ is entered as **rad(3, x).**

1. Can you evaluate $\frac{\left(x^{1/3} - 2\right)}{(x - 8)}$ at $x = 8$?...................... If not, why not?..................

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2. What happens to the values of $\frac{\left(x^{1/3} - 2\right)}{(x - 8)}$ as x gets close to 8 from the left?...................

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3. What happens to the values of $\frac{\left(x^{1/3} - 2\right)}{(x - 8)}$ as x gets close to 8 from the right?.................

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4. What is your estimate for the value of the limit (if it exists)?...................................

Exploration 4 The Limit of a Special Transcendental Function

Using the procedure given in exploration 1 (use the fill functions **–10^(–i)** and **10^(–i)**), let's estimate the value of

$$L = \lim_{x \to 0} (1 + x)^{1/x}.$$

Note that $(1+x)^{1/x}$ is entered as **(1 + x)^(1/x)**. It is known that this limit exists and that L is a number between 2 and 3.

1. Can you evaluate $(1+x)^{1/x}$ at $x = 0$?...................... If not, why not?..................

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2. What happens to the values of $(1+x)^{1/x}$ as x gets close to 0 from the left?...................

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3. What happens to the values of $(1+x)^{1/x}$ as x gets close to 0 from the right? ....

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4. What is your best estimate for L (give an eight digit approximation)?.........................

Exploration 5 Computers Only Simulate Arithmetic

A computer only *simulates* real arithmetic and it is very easy to exceed the limitations of this simulation. Thus, you must always decide for yourself if results calculated by a computer are correct. To see a simple example of this, do the following:

- Use the fill function **x(i) = 1024^(–i)** to fill the x-column of a 9-row data table.
- Select the ***x*-column** of the data table and double-click the **Generate** command in the data table's command list. Enter the expression $\mathbf{(x+1)-1}$ as the generating function *exactly* as it is written. Don't simplify this expression!
- Press the **Return** or **Enter** key.

1. What are the last five y-column entries generated by $(x+1)-1$? ....

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2. What are the *correct* values for the last five y-column entries? ....

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The errors produced above are caused by the *finite* precision of the computer's arithmetic simulation. The arithmetic of most calculators and computers is correct to somewhere between 10 and 16 significant decimal digits. However, the Macintosh's arithmetic is correct to almost 19 significant decimal digits. This means that the Macintosh represents a real number in a form that is roughly equivalent to $\pm 0.d_1 d_2 d_3 \cdots d_{19} \times 10^m$. Thus, two real numbers that differ only beyond the 19-th significant digit have identical representations in the computer. For example, if $|x| < 10^{-20}$, then $x+1$ and 1 both have the computer representation $0.1000000000000000000 \times 10^1$. Thus, $x+1$ equals 1 in the computer's simulation of arithmetic.

With the above in mind, try to estimate the right-hand limit below using the fill function **x(i) = 1024^(–i)**.

$$\lim_{x \to 0^+} \frac{x}{((x+1)-1)}$$

For your generating function, you should enter the rational expression **x/((x+1) – 1)** *exactly* as it is written. Don't simplify this expression!

3. Based on the y-column entries in the data table, what happens to the values of $\frac{x}{((x+1)-1)}$ as x approaches zero from the right? ....

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4. What is the *correct* value for this right-hand limit?...

5. Why did the computer produce the incorrect results given in question 3.? ....

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