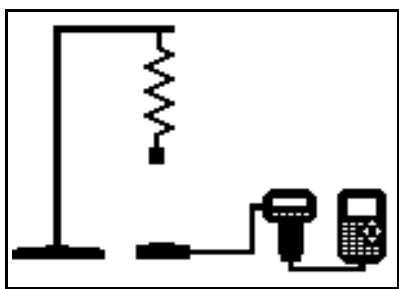


USING DIFFERENTIAL EQUATIONS TO MODEL REAL-WORLD DATA¹

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With the increasing availability of easy to use interactive differential equation software, we convinced ourselves to completely redesign the way we teach differential equations. Rather than have our students passively study a cookbook collection of special techniques for solving a few types of differential equations, we now have them actively involved in designing models and testing them with real-world data. Real-world data provides an extremely rich environment for developing, learning, and applying differential equations. Government sources, laboratory experiments, and research studies present a wealth of data that can be modeled by differential equations. Additionally, with the availability of sensors that attach to a computer or a graphing calculator, it has become an easy task to gather data from many different types of physical experiments. Thus, it has become common practice for our students to not only develop and test their own models, but, to test the validity and accuracy of the many differential equation models that are presented in a standard first year course. By seeing for themselves how well differential equations model physical phenomenon, they build up confidence in using differential equations in their studies, research, and future careers. In our courses, we emphasize the fact that not only do we want a model that fits the data well, but we want one that also makes sense from a theoretical and realistic point of view. Many models fit data well over a short time period (the data looks linear), but over longer time periods and for purposes of making projections into the future, one model may be much more appropriate than the others. We now present a few examples that use differential equations to model real-world data.

The Simple Harmonic Oscillator — Using Hooke's Law



With springs, masses, and supports borrowed from the Physics Department, we were able to set-up an harmonic oscillator experiment. We placed a motion sensor under an oscillating spring, attached the motion sensor to a Texas Instruments CBL[®], and gathered the data from the CBL with a TI-83 graphing calculator. Using the Texas Instruments Graph-Link software package, we downloaded the data from the TI-83 to a Macintosh computer. Next, we read the data into the TEMATH software package. Once we had access to the data, we found the central divided differences of the data and then found the central divided differences of the central divided differences. This gave us an estimate for the second derivative. We then plotted the second order central divided differences versus the y -values and used TEMATH's Least Squares fitting tool to fit these divided differences. Using this fit, we then obtained an estimate for the spring constant. The second order differential equation model that we obtained from the data was

$$\frac{d^2y}{dt^2} = -\frac{k_s}{m}y = -\frac{12.5}{0.22}y.$$

¹This article appeared in "Proceedings of the Tenth Annual International Conference on Technology in Collegiate Mathematics", Addison-Wesley Publishing Co., 1999, p. 246-250.

The experimental data and the solution to the differential equation are plotted in Figure 1.

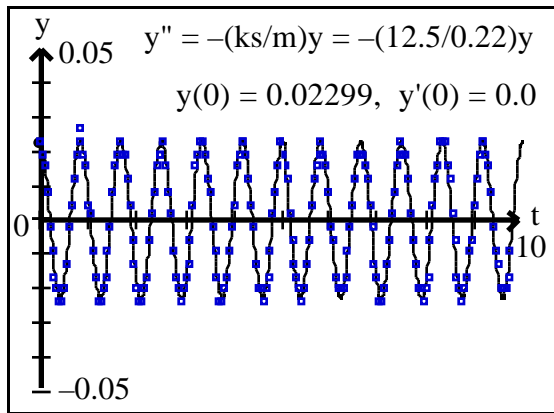


Figure 1 Spring-Mass Data and Fit

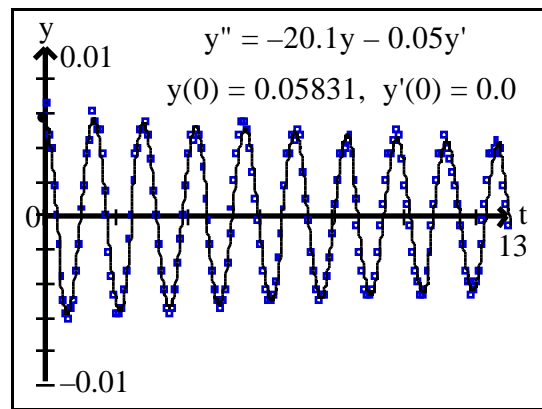


Figure 2 Damped Spring-Mass

As a second experiment, we attached the mass to a large paper plate to try to induce some damping. The data and the second order differential equation model that we obtained are shown in Figure 2. Note the excellent agreement between the data and the theory.

Glucose Tolerance Test

We solicited the assistance of our colleagues in the Medical Laboratory Sciences Department to obtain data for a Glucose Tolerance Test. This test is given to patients to determine how well their bodies produce insulin when needed to reduce glucose levels in their blood. We obtained data for a normal patient, a Type I diabetic (insulin controlled), and a Type II diabetic (diet controlled). A typical glucose tolerance test is described below:

For two days prior to the test, the patient is told to eat a high carbohydrate diet in order to have maximum synthesis of insulin. Otherwise, the numbers may reflect a sluggish pancreas rather than an actual disease state. Then the patient must fast for twelve hours prior to the test. On the morning of the test, a blood specimen is drawn while the patient is still fasting — this specimen provides a baseline glucose level. Next, the patient is given a drink that contains 75 grams of glucose in approximately 250 mL of water. The patient must drink the glucose solution within five minutes. The remainder of the specimens are drawn at the time intervals 1/2, 1, 2, 3, and 4 hours later.

It should be noted that glucose is absorbed quickly into the blood stream and that insulin production is approximately 30-45 minutes “behind” the glucose values and it should be produced in proportion to the glucose level in the blood stream.

A reasonable differential equation model for the amount of glucose in the blood stream is

$$\frac{dg}{dt} = \text{GlucoseAbsorptionRate}(t) - k(g - \text{BaseLineValue})$$

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where $g(t)$ = amount of glucose in the blood stream at time t . A possible model for the rate of glucose absorption by the blood stream is the delayed surge function

$$GlucoseAbsorptionRate(t) = \begin{cases} 0 & 0 \leq t \leq \delta \\ a(t - \delta)e^{-b(t - \delta)} & t > \delta \end{cases}$$

A graph of the delayed surge function used to model the rate of glucose absorption is shown in Figure 3. The area under this curve is equal to the 75 grams of glucose contained in the glucose drink and this determines the value of the parameter a in the model. The delay δ was set to 0.4 hours — this takes into account the time to drink the glucose and the time for the stomach to begin the absorption process. Glucose data obtained for a normal patient is presented in Table 1.

t (Hours)	0	0.5	1	2	3	4
Glucose Level (mg/dL)	80	120	205	92	75	82

Table 1 Glucose Tolerance Test Data for a Normal Patient

The delayed surge function that we used in the differential equation to fit the data was

$$GlucoseAbsorptionRate(t) = \begin{cases} 0 & 0 \leq t \leq 0.4 \\ 675000(t - 0.4)e^{-3(t - 0.4)} & t > 0.4 \end{cases}$$

Figure 4 shows how well we were able to model the glucose test results.

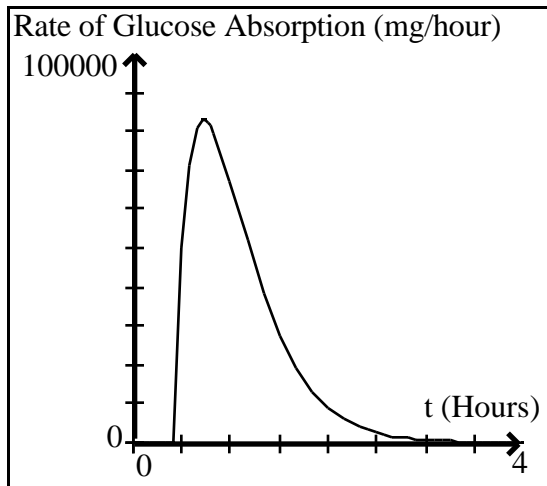


Figure 3 Glucose Absorption

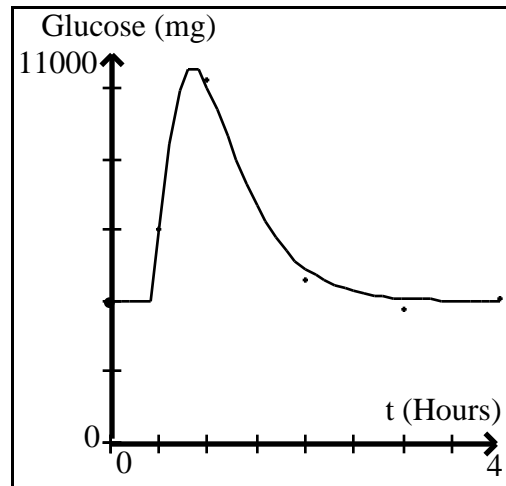


Figure 4 Fitting Glucose Data

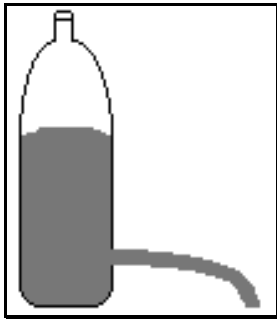
Torricelli's Law

The following experiment can be used to test Torricelli's law:

- Drill a small hole into the side (near the bottom) of a two liter clear plastic bottle.
- Attach a ruler to the side of the bottle with zero marking the position of the hole.

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Fill the bottle with water while covering the hole with a finger. Remove the finger and record the time and height of the water as the water is drained from the bottle. The height $h(t)$ of the water at time t is governed by Torricelli's law which states that $dh/dt = a\sqrt{h}$.



We performed Torricelli's experiment with an 1/8 inch hole in the two liter bottle. To estimate the value of the parameter a in Torricelli's law, we calculated the central differences of the data obtained from the experiment, plotted the central differences versus the measured heights of the water, and used TEMATH's Linear Least Square's Tool to find the linear fit for $dh/dt = a\sqrt{h}$ (see Figure 5). Once the value of a was estimated, we plotted the numerical solution of Torricelli's differential equation along with the data. Notice how well the solution models the experimental data as is shown in Figure 6.

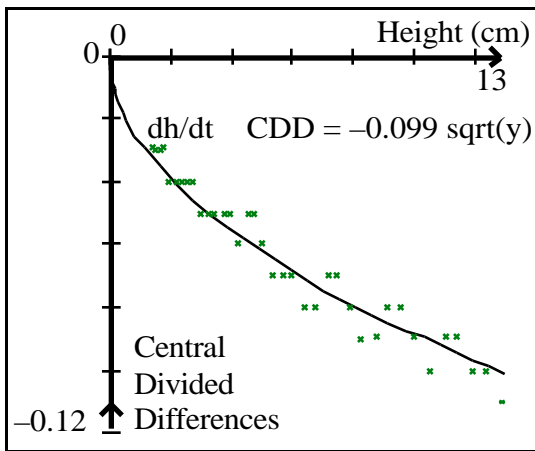


Figure 5 Fitting the Central Differences

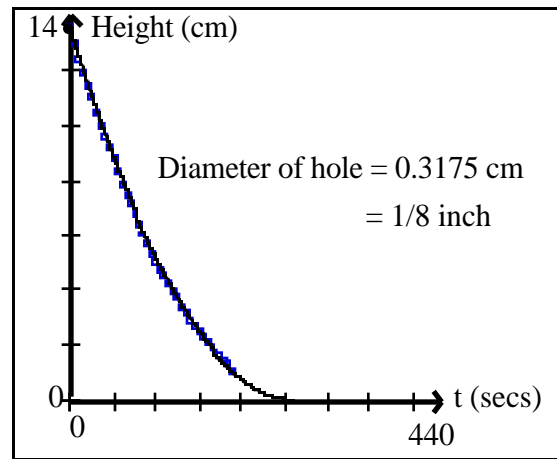


Figure 6 Modeling Torricelli's Law

The Standard Series RC Circuit Experiment

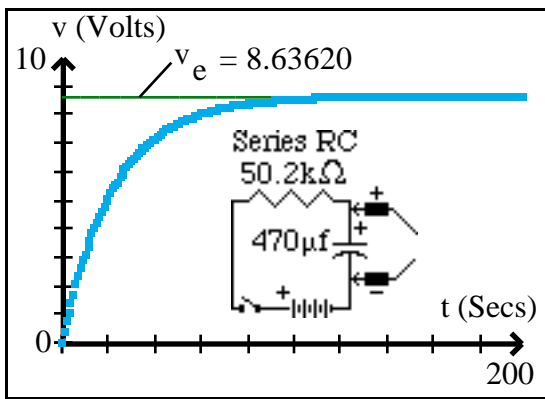


Figure 7 Charging Capacitor

In the standard series RC circuit experiment, the capacitor is allowed to charge by closing the switch. In theory, the differential equation that models the behavior of the voltage after the switch has been closed is $\frac{dv}{dt} = \frac{1}{RC} (v_{source} - v)$ where R is the resistance in ohms and C is the capacitance in farads and v_{source} is the source voltage. We used the CBL with a voltage probe to test this model with $R = 50.2k \pm 5\%$ and $C = 470 \mu f = 470 \cdot 10^{-6} f \pm 20\%$. A plot of the data we gathered and imported into TEMATH is shown in Figure 7.

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Unexpectedly, the measured equilibrium voltage was $v_e = 8.63620$ instead of the source voltage ($= 9.035$ volts). We used TEMATH to calculate the central divided differences as an approximation to dv/dt . A plot of these values versus v together with a fit of this data by a least squares line are shown in Figure 8.

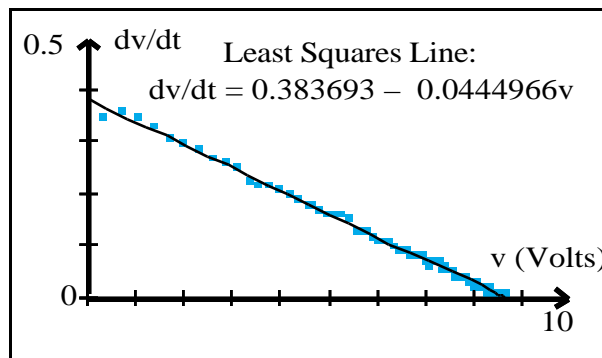


Figure 8 Least Squares Fit of (v, v') -data

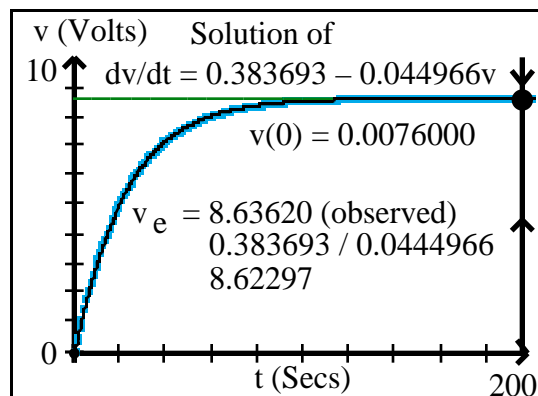


Figure 9 Solution and Data

Based on this fit, we decided to change the model to $\frac{dv}{dt} = 0.383693 - 0.0444966v$. We then used TEMATH to generate a numerical solution to this differential equation using $v(0) = 0.00760000$ as the initial condition. A plot of this solution which appears to fit the data very well is shown in Figure 9. We concluded that the CBL with the voltage probe attached has an internal resistance that prevents the capacitor from charging to the source voltage. We verified this by connecting a voltmeter across the capacitor and simultaneously disconnecting the voltage probe. The capacitor then resumed charging and continued to charge until the source voltage was reached as expected. If the circuit is viewed as two resistors in series, the internal resistance of the CBL-voltage probe unit can be found using the formula $R_{CBL} = \frac{v_e}{v_{source} - v_e} R = \frac{8.63620}{9.035 - 8.63620} 50200 \cdot 10^6$. Finally, the experimentally determined value of $\frac{1}{RC} = 0.0444966$ is in good agreement with the theoretical value of $\frac{1}{RC} = \frac{1}{470 \cdot 10^{-6} \cdot 50200} = 0.0423837$.

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