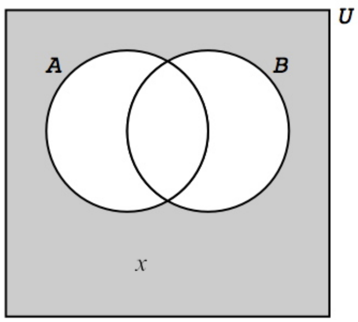
Proving $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ Using Elements V2

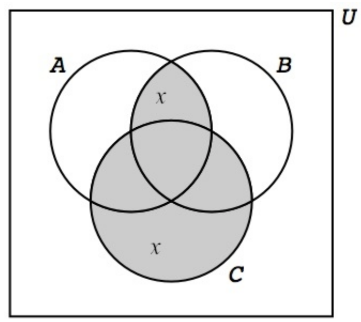
1. Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ by proving (a) $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and (b) $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

<u>Proof of (a)</u>: Let $x \in \overline{A \cup B}$ then $x \in U =$ the universal set and $x \notin A \cup B$. This means that $x \in U$ but $x \notin A$ and $x \notin B$ (see the figure on the left below). Thus, $x \in \overline{A}$ and $x \in \overline{B}$; hence, $x \in \overline{A} \cap \overline{B}$. Therefore, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

<u>Proof of (b)</u>: Let $x \in \overline{A} \cap \overline{B}$ then and $x \in \overline{A}$ and $x \in \overline{B}$. This means that $x \in U$ but $x \notin A$ and $x \notin B$. Thus, $x \in U$ and $x \notin A \cup B$; hence, $x \in \overline{A \cup B}$. Therefore, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$. □



$x \in \overline{A \cup B}$



$x \in (A \cap B) \cup C$

2. Show that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ by proving (a) $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$ and (b) $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

<u>Proof of (a)</u>: Let $x \in (A \cap B) \cup C$. Then either $x \in A \cap B$ or $x \in C$ (see the figure on the right above) since $A \cap B \subseteq (A \cap B) \cup C$ and $C \subseteq (A \cap B) \cup C$. Thus, there are two cases:

Case 1: $x \in C$. Then $x \in A \cup C$ and $x \in B \cup C$ because $C \subseteq A \cup C$ and $C \subseteq B \cup C$.
Case 2: $x \in A \cap B$. Then $x \in A$ and $x \in B$. But $A \subseteq A \cup C$ and $B \subseteq B \cup C$; hence, $x \in A \cup C$ and $x \in B \cup C$.

Thus, in both cases, $x \in A \cup C$ and $x \in B \cup C$ which means $x \in (A \cup C) \cap (B \cup C)$. Therefore, $(A \cap B) \cup C \subseteq (A \cup C) \cap (B \cup C)$.

<u>Proof of (b)</u>: Let $x \in (A \cup C) \cap (B \cup C)$, then $x \in A \cup C$ and $x \in B \cup C$. Again, there are two cases since $C \subseteq A \cup C$ and $C \subseteq B \cup C$:
Case 1: $x \in C$. Then $x \in (A \cap B) \cup C$ since $C \subseteq (A \cap B) \cup C$.
Case 1: $x \notin C$. Then $x \in A \cup C$ implies $x \in A$ since $A \cup C$ is the set of elements *either* in A *or* in C. Similarly, $x \in B \cup C$ implies that $x \in B$. But, $x \in A$ and $x \in B$ implies $x \in A \cap B$. Thus, $x \in (A \cap B) \cup C$ since $A \cap B \subseteq (A \cap B) \cup C$.
So in either case, $x \in (A \cap B) \cup C$. Therefore, $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$. □