Theorem: Let a and b be positive integers, then $\text{GCD}(a,b)\cdot\text{LCM}(a,b)=ab$.

Proof: Let $d=\text{GCD}(a,b)$. Then there exist positive integers a_1, b_1 and integers s and t such that

$$a=da_1, \qquad b=db_1 \qquad \text{and} \qquad d=as+bt .$$

Therefore, by using substitution,

$$d=as+bt=(da_1)s+(db_1)t=d(a_1s+b_1t).$$

Hence, since $d\neq 0$,

(1) $$1=a_1s+b_1t .$$

Now, let $L=da_1b_1\in\mathbf{Z}^+$. I will prove $L=\text{LCM}(a,b)$. To see this, first observe that, $L=(da_1)b_1=ab_1$ and $L=(db_1)a_1=ba_1$ which shows that $a\,|\,L$ and $b\,|\,L$. Therefore, by the definition of $\text{LCM}(a,b)$, we have

$$\text{LCM}(a,b)\,|\,L .$$

Next I will show that if m is a integer such that $a\,|\,m$ and $b\,|\,m$, then $L\,|\,m$: First, since a and b are divisors of m,

$$m=au \text{ and } m=bv$$

for some integers u and v. Consequently, by using (1) and these two different factorizations of m, we find

$$\begin{aligned} m=m\cdot 1 &= m(a_1s+b_1t)=m(a_1s)+m(b_1t)=(ma_1)s+(mb_1)t \\ &=(bva_1)s+(aub_1)t=(db_1\cdot va_1)s+(da_1\cdot ub_1)t \\ &=(db_1a_1)(vs+ut)=Lk \end{aligned}$$

where $k=vs+ut$ is an integer. Therefore, $L\,|\,m$. But $a\,|\,\text{LCM}(a,b)$ and $b\,|\,\text{LCM}(a,b)$; hence, $L\,|\,\text{LCM}(a,b)$, by the previous argument. So now we have shown

$$\text{LCM}(a,b)\,|\,L \text{ and } L\,|\,\text{LCM}(a,b)$$

where $\text{LCM}(a,b)$ and L are positive integers. Thus, $\text{LCM}(a,b)=L\cdot\alpha$ and $L=\text{LCM}(a,b)\cdot\beta$, for some positive integers α and β . Therefore,

$$L=\text{LCM}(a,b)\cdot\beta=L\cdot\alpha\cdot\beta$$

Hence, since $L\neq 0$,

$$1=\alpha\beta .$$

But the only positive integer solution of this equation is $\alpha=\beta=1$; thus, $\text{LCM}(a,b)=L$. So finally,

$$\text{GCD}(a,b)\cdot\text{LCM}(a,b)=d\cdot L=d\cdot(da_1b_1)=(da_1)(db_1)=ab .$$

□