Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6\ s_7, s_8\}$ and $T = \{t_1, t_2, t_3, t_4, t_5\}$.

Example 1: How to construct a function $f : S \to T$ from a partition P of S.

Solution: Let

$$S_1 = \{s_1, s_2, s_3\},\ S_2 = \{s_4, s_5\},\ S_3 = \{s_6\},\ S_4 = \{s_7, s_8\} \text{ and } P = \{S_1, S_2, S_3, S_4\}.$$

Then P is a partition of S since

$$S_1, S_2, S_3, S_4 \subset S\,,$$
$$S_i \cap S_j = \phi \text{ if } i \neq j,$$
$$S = S_1 \cup S_2 \cup S_3 \cup S_4$$

Now define $f : S \to T$ by

$$f(s_1) = f(s_2) = f(s_3) = t_1,$$
$$f(s_4) = f(s_5) = t_2,$$
$$f(s_6) = t_3,$$
$$f(s_7) = f(s_8) = t_4.$$

Then $f(S) \subset T$ and $f : S \to f(S)$ is a surjection. Next, let $s, s' \in S$. If define a relation on S by $s \sim_f s'$ $f(s) = f(s')$, then $\sim_f$ is an *equivalence relation* on S. Let $s \in S$ and $f(s) = t \in T$, then the equivalence class of s is

$$[s]_f = \{s' \in S \,|\, f(s') = f(s)\} = \{s' \in S \,|\, f(s') = t\} = f^{-1}(t) \subset S.$$

Thus, the equivalence classes of $\sim_f$ are

$$f^{-1}(t_1) = S_1,\ f^{-1}(t_2) = S_2,\ f^{-1}(t_3) = S_3,\ f^{-1}(t_4) = S_4\,.$$

Note that $f^{-1}(t_5) = \{\}$ and is ignored since $t_5 \notin f(S)$. Therefore,

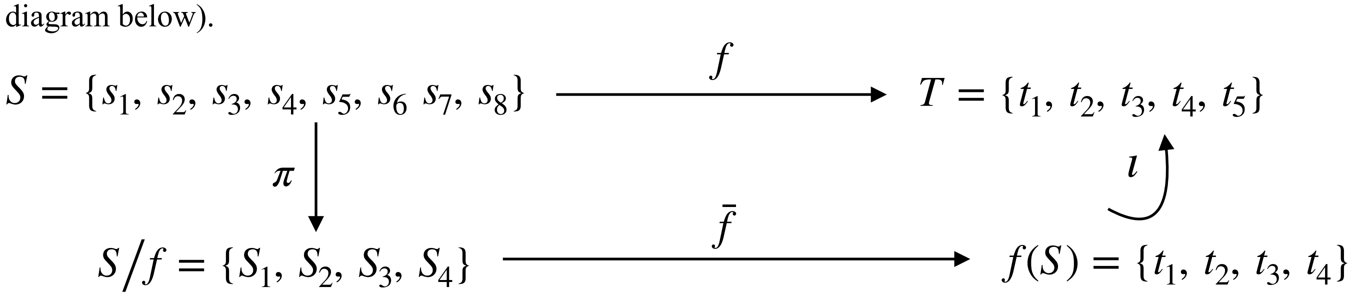
$$P = \{S_1, S_2, S_3, S_4\} = S/f\,,$$

$$\pi : S \to S/f \text{ is defined by } \pi(s) = S_i \iff s \in S_i.$$

and

$$\bar{f} : S/f \to f(S) \text{ is defined by } \bar{f}(S_i) = f(s) \text{ for } any\ s \in S_i$$

Then by its definition, $\bar{f} : S/f \to f(S)$ is a *bijection* and $f(s) = (\iota \circ \bar{f} \circ \pi)(s)$ for all $s \in S$ (see the diagram below).



The following examples shows that if the domain S and the range T are *groups* and the function $h : S \to T$ respects the group operation, then the equivalence classes all have the same size and are related!

Example 2: Let $S = T = Z_{12}$ and let $h : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$ be defined by $h([x]_{12}) = [9x]_{12}$. Notice that

$$h([x \pm y]_{12}) = [9(x \pm y)]_{12} = [9x \pm 9y]_{12} = [9x]_{12} \pm [9y]_{12} = h([x]_{12}) \pm h([y]_{12})$$

and

$$h([0]_{12}) = [0]_{12}$$

This shows that h respects the group operation and is called a *homomorphism* of groups. Clearly,

$$h([x]_{12}) = [0]_{12} \iff [9x]_{12} = [0]_{12} \iff 9x \equiv 0 \ (\mathbf{mod}\ 12)$$

$$\implies 3x \equiv 0 \ (\mathbf{mod}\ 4) \implies x \equiv 0 \ (\mathbf{mod}\ 4) \implies x \equiv 0,\ 4,\ 8 \ (\mathbf{mod}\ 12)\ .$$

Thus,

$$h^{-1}([0]_{12}) = \{[0]_{12},\ [4]_{12}, [8]_{12}\}.$$

<u>*Claim*</u>: Let $x \in \mathbf{Z}_{12}$. Elements of $\mathbf{Z}_{12}$ obtained by adding an element $h^{-1}([0]_{12})$ to x have the same image under h. Therefore, $x + h^{-1}([0]_{12}) = \{x + [0]_{12},\ x + [4]_{12}, x + [8]_{12}\} \subseteq h^{-1}([x]_{12})$

Proof: Let $z \in h^{-1}([0]_{12})$, then $h(x + z) = h(x) + h(z) = h(x) + [0]_{12}$. ∎

Using the claim, we see that

$$[0]_{12} + h^{-1}([0]_{12}) = \{[0]_{12},\ [4]_{12}, [8]_{12}\} \subseteq h^{-1}([0]_{12})$$
$$[1]_{12} + h^{-1}([0]_{12}) = \{[1]_{12},\ [5]_{12}, [9]_{12}\} \subseteq h^{-1}([9]_{12})$$
$$[2]_{12} + h^{-1}([0]_{12}) = \{[2]_{12},\ [6]_{12}, [10]_{12}\} \subseteq h^{-1}([2]_{12})$$
$$[3]_{12} + h^{-1}([0]_{12}) = \{[3]_{12},\ [7]_{12}, [11]_{12}\} \subseteq h^{-1}([3]_{12})$$

Therefore, the equivalence classes of $\sim_h$ are

$$h^{-1}([0]_{12}) = \{[0]_{12},\ [4]_{12}, [8]_{12}\} = E_0,$$
$$h^{-1}([1]_{12}) = \{[1]_{12},\ [5]_{12}, [9]_{12}\} = E_1,$$
$$h^{-1}([2]_{12}) = \{[2]_{12},\ [6]_{12}, [10]_{12}\} = E_2 \text{ and}$$
$$h^{-1}([3]_{12}) = \{[3]_{12},\ [7]_{12}, [11]_{12}\} = E_3.$$

Clearly,

$$|E_0| = |E_1| = |E_2| = |E_3| = 3,$$

$$E_1 = [1]_{12} + E_0, E_2 = [2]_{12} + E_0, E_3 = [3]_{12} + E_0,$$

$$\mathbf{Z}_{12}/h = \{E_0,\ E_1,\ E_2,\ E_3\},$$

$$h(\mathbf{Z}_{12}) = \{[0]_{12},\ [9]_{12},\ [6]_{12},\ [3]_{12}\} \subset \mathbf{Z}_{12},$$

$$\bar{h} : \mathbf{Z}_{12}/h \to h(\mathbf{Z}_{12}) \text{ is defined by } \bar{h}(E_i) = h(e) \text{ for any } e \in E_i,$$

$$\pi : \mathbf{Z}_{12} \to \mathbf{Z}_{12}/h \text{ is defined by } \pi(x) = E_i \iff x \in E_i, \text{ and}$$

$$h(x) = (\iota \circ \bar{h} \circ \pi)(x) \text{ for all } x \in \mathbf{Z}_{12}.$$

$$\begin{array}{ccc}
\mathbf{Z}_{12} = \{[0]_{12}, [1]_{12}, \cdots, [11]_{12}\} & \xrightarrow{\quad h \quad} & \mathbf{Z}_{12} = \{[0]_{12}, [1]_{12}, \cdots, [11]_{12}\} \\
\downarrow \pi & & \uparrow \iota \\
\mathbf{Z}_{12}/h = \{E_0, E_1, E_2, E_3\} & \xrightarrow{\quad \bar{h} \quad} & h(\mathbf{Z}_{12}) = \{[0]_{12}, [9]_{12}, [6]_{12}, [3]_{12}\}
\end{array}$$

∎