MTH 441 Basic Consequences of the Field Axioms V5 Fall 2019

Proposition 1: The additive and the multiplicative identity elements of a field (or ring) are *unique*.

Proof. Suppose F is a field (or ring) that contains two *additive* identity elements, say 0 and $0'$. Then by the field (or ring) axiom for additive identity elements (4.4.1v)

(1) $$0 + a = a, a + 0 = a \text{ for all } a \in F$$

and

(2) $$0' + a = a, a + 0' = a \text{ for all } a \in F.$$

Thus, by substitution of $0' \in F$ into the first equation of (1) for a, we have

(3) $$0 + 0' = 0'.$$

Also, by substitution of $0 \in F$ into the second equation of (2) for a, we have

(4) $$0 + 0' = 0.$$

Hence, by combining (3) and (4), we see that $0 = 0 + 0' = 0'$.

A similar proof shows that the multiplicative identity element is also unique. You should write this proof!
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Proposition 2: Let $0 \in F$ be the additive identity element of the field (or ring) F. Then

(5) $$0 \cdot b = 0 = b \cdot 0 \text{ for all } b \in F.$$

Proof: Observe that by the field (or ring) axiom for Additive Identity Elements (4.4.1v),

(6) $$0 + a = a, a + 0 = a \text{ for all } a \in F.$$

Thus, if 0 is substituted for a in (1), we see that

(7) $$0 + 0 = 0$$

This means that we can substitute the expression $0 + 0$ for 0 in the expression $0 \cdot b$ and obtain

(7a) $$0 \cdot b = (0 + 0) \cdot b.$$

Note: This can be thought of as multiplying both sides of (7) by b. Next, by the second Distributive Law for multiplication over addition (4.41iv) applied to the right-hand side of (7a), we see that

(8) $$0 \cdot b = 0 \cdot b + 0 \cdot b.$$

Now $0 \cdot b \in F$ because a field (or ring) is *closed* under multiplication. But by the Inverses Axiom 4.4.1vi (for a field (or ring), there is an element denoted by $(-0 \cdot b)$ such that

(9) $$(-0 \cdot b) + 0 \cdot b = 0 \text{ and } 0 \cdot b + (-0 \cdot b) = 0.$$

This, by adding $(-0 \cdot b)$ to both sides of (8), we find

(10) $$0 = (-0 \cdot b) + 0 \cdot b = (-0 \cdot b) + (0 \cdot b + 0 \cdot b).$$

Next by applying the Associative Law for Addition (4.4.1iii) to the right-hand side of equation (10), we see that

(11) $$0 = (-0 \cdot b) + 0 \cdot b = ((-0 \cdot b) + 0 \cdot b) + 0 \cdot b.$$

But by, applying the first equation of (9) , we find that

(12) $$0 = ((-0 \cdot b) + 0 \cdot b) + 0 \cdot b = 0 + 0 \cdot b.$$

Finally, by applying the first equation of (6) with $a = 0 \cdot b$, we see that

$$0 = 0 + 0 \cdot b = 0 \cdot b$$

and by the commutativity of multiplication, we see that $0 = b \cdot 0$. ∎

Proposition 3: Let $1 \in F$ be the multiplicative identity element of the field (or ring) F and let (-1) be its additive inverse. Then

(13) $$(-1)(-1) = 1.$$

Proof. First,

(14) $$(-1)(-1) = 0 + (-1)(-1)$$

where $0 \in F$ is the additive identity element. But

(15) $$0 = 1 + (-1)$$

by definition of (-1). Thus, substituting (15) into the right-hand side of (14) for 0, we find

(16) $$(-1)(-1) = \big(1 + (-1)\big) + (-1)(-1).$$

Now by the Associative Law for Addition applied to the right-hand side of (16),

(17) $$\big(1 + (-1)\big) + (-1)(-1) = 1 + \big[(-1) + (-1)(-1)\big].$$

Hence, by substitution of (17) into the right-hand side of (16), we see that

(18) $$(-1)(-1) = 1 + \big[(-1) + (-1)(-1)\big].$$

But by the definition of 1,

(19) $$\big(-1\big) = 1 \cdot \big(-1\big).$$

Thus, by substituting (19) for the underscored term on the right-hand side of (18), we see that

(20) $$(-1)(-1) = 1 + \big[\underline{1 \cdot (-1)} + (-1)(-1)\big].$$

Next by applying the second Distributive Law to the right-hand side of (20), we find

(21) $$(-1)(-1) = 1 + \underline{\big[1 + (-1)\big]}(-1).$$

But by substituting (15) into the right-hand side of (21), we see that

(22) $$(-1)(-1) = 1 + \underline{0} \cdot (-1).$$

Now, by applying **Proposition** 2 to the second term of (22), we find

(23) $$(-1)(-1) = 1 + 0.$$

So finally, by the definition of 0,

$$(-1) \cdot (-1) = 1\,.$$

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