Adaptive Radial Basis Function Methods with Residual Subsampling Technique for Interpolation and Collocation Problems

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Given data at nodes $\mathbf{x}_1, ..., \mathbf{x}_N$ in d dimensions, the basic form for an RBF approximation is

$$F(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j \phi(\epsilon_j ||\mathbf{x} - \mathbf{x}_j||),$$

where $|| \cdot ||$ denotes the Euclidean distance between two points and $\phi(r) = \sqrt{1 + r^2}$ is defined for $r \geq 0$.

$$f_i = f(\mathbf{x}_i)$$

$$\Longrightarrow$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$$

where $a_{ij} = \phi(\epsilon_j ||\mathbf{x}_i - \mathbf{x}_j||)$. Nonsingularity of A is guaranteed for many choices of ϕ with mild restrictions and constant shape parameters ϵ_j.

Advantages of RBF methods

- No need for a mesh / triangulation.
- Simple implementation and dimension independence.
- No staircasing / polygonization for boundaries.
- Depending on chosen RBFs, high-order/spectral convergence can be achieved.
- Easy to implement derivatives and boundary conditions.

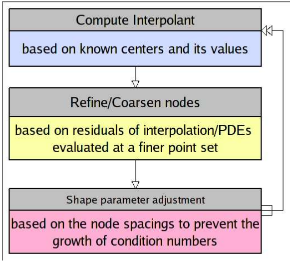
Challenges using RBF methods

- As the number of centers grows, the method needs to solve a relatively large algebraic system
- The matrix is full (except for compactly supported RBF).
- Choosing nodes and shape parameters.
- Ill-conditioning usually makes spectral convergence difficult to achieve.

Problems involve

- geometry
- steep gradients
- corners
- topological changes resulting from nonlinearity
- high degrees of localization in space and/or time

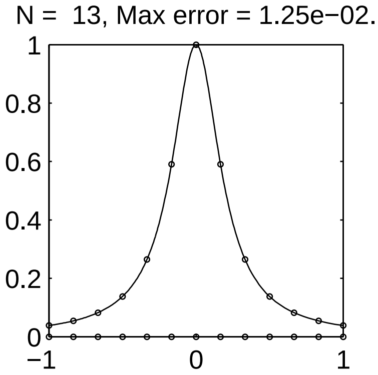
Residual Subsampling Scheme



Goal

Obtain an accurate solution using a minimal number of automatically chosen nodes.

Runge Function

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thetar = 2e-5; thetac = 1e-8; N = 13;
f = @(x) 1./(1+25*x.^2);
phi = @(r,epsilon) sqrt((epsilon*r).^2 + 1);
x = linspace(-1,1,N)';
ref = true;
while any(ref)
  N = length(x); dx = diff(x);
  epsilon = 0.75*min([Inf;1./dx],[1./dx;Inf]);
  y = x(1:N-1) + 0.5*dx;
  A = zeros(N); B = zeros(N-1,N);
  for j=1:N
    A(:,j) = phi(x-x(j),epsilon(j));
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  end
  lambda = A\f(x); resid = abs(B*lambda-f(y));
  ref = resid > thetar; x = sort([x;y(ref)]);
  coarsen = resid(1:N-2) < thetac & ...
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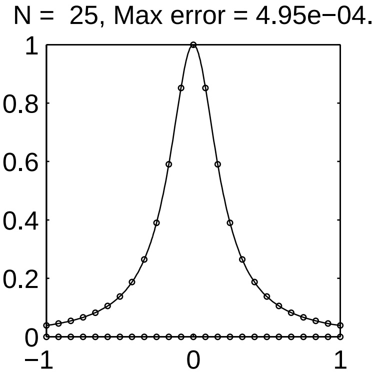


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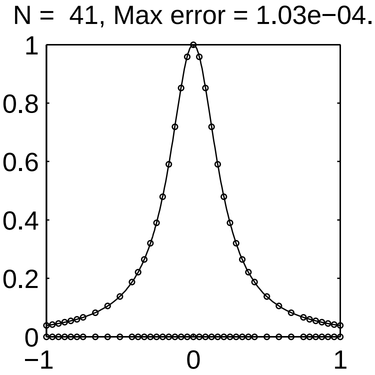


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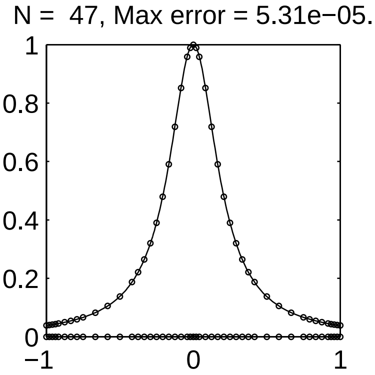


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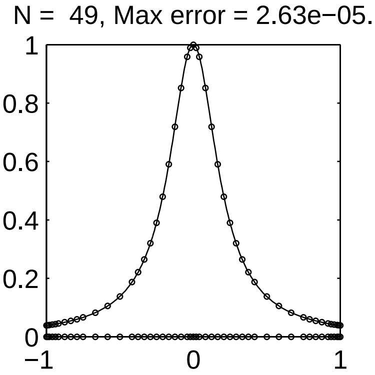


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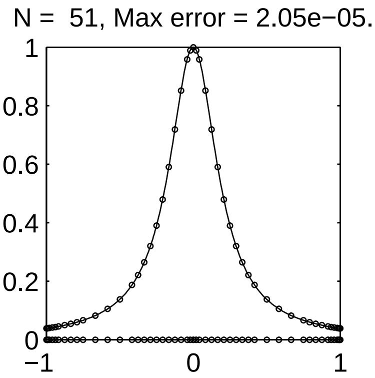


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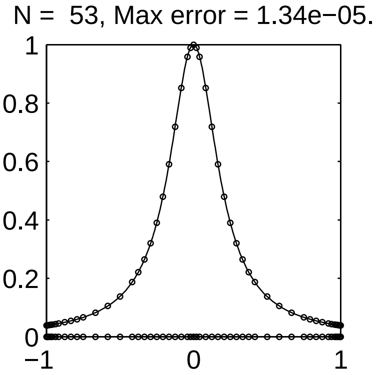


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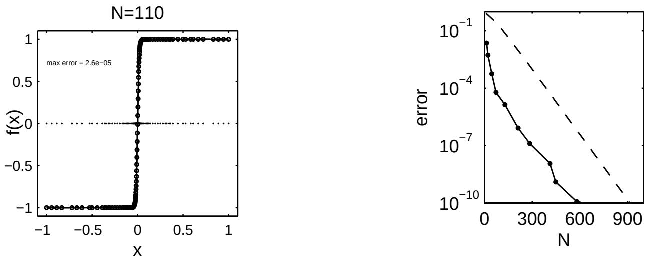
tanh(60x − .01)

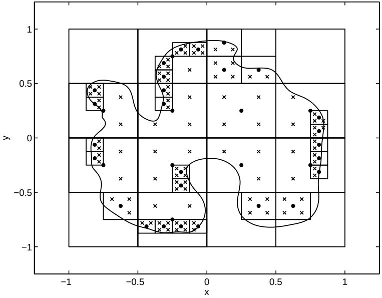


It	N	N_c	N_r	$\kappa(A)$	$\|\|.\|\|_\infty$
1	11	0	18	5.090e+02	7.9211e-01
2	29	0	34	2.632e+04	4.1501e-01
3	63	0	31	4.545e+05	1.2544e-01
4	94	3	30	4.330e+06	1.1129e-02
5	121	4	12	3.962e+07	2.6766e-04
6	129	2	2	3.180e+08	5.7980e-05
7	129	0	0	3.038e+08	2.5329e-05

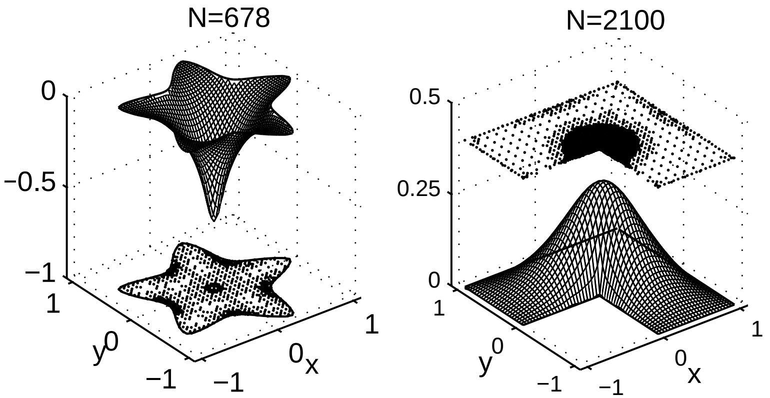
N_r ,N_c = Number of centers to be added/removed respectively.

Scheme

1. Initial coarse collection of nonoverlapping regular boxes in R^d that cover the domain Ω of interest.
2. Geometric adaptation.
3. Refi ning/Coarsening steps



Poisson Equation with Dirichlet condition



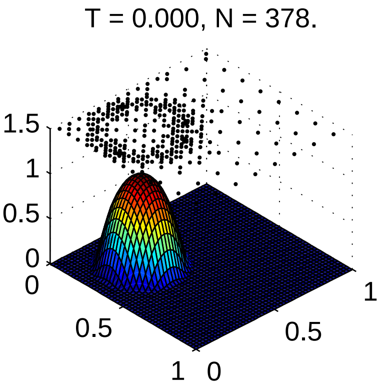
Burgers' Equation

$$\nu \Delta u - \nabla f(u) \cdot \vec{n} = \frac{\partial u}{\partial t}$$

$$u = 0 \text{ on } \partial\Omega$$

$$f(u) = \tfrac{1}{2}u^2$$

$$\nu = 2.10^{-3}$$



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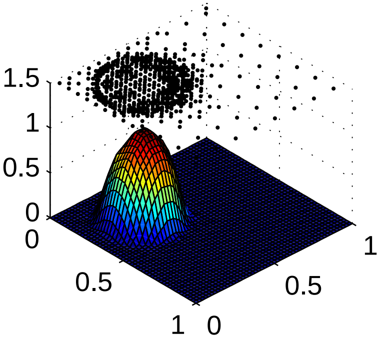
T = 0.010, N = 764.



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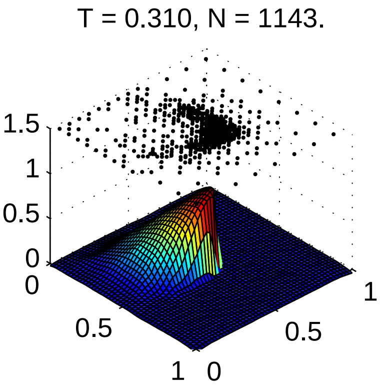

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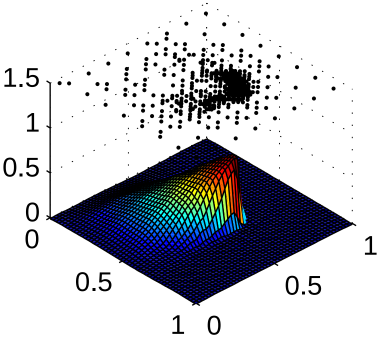
T = 0.510, N = 710.



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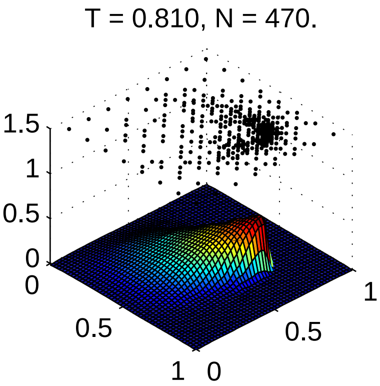

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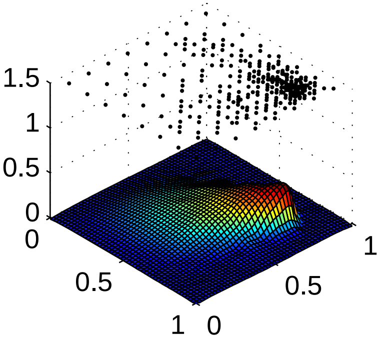
T = 1.190, N = 356.



Buckley-Leverett

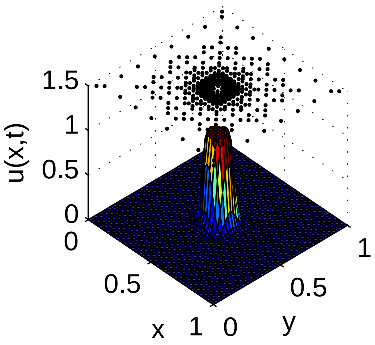
$$\nu\Delta u - \nabla f(u)\cdot\vec{n} = \frac{\partial u}{\partial t}$$

$$u = 0 \text{ on } \partial\Omega$$

$$f(u) = \frac{u^2}{u^2 + \mu(1-u)^2}$$

$$\nu = 10^{-3}\mu = \frac{1}{2}$$

T = 0.000, N = 484.



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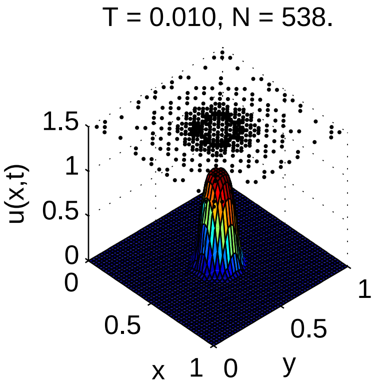

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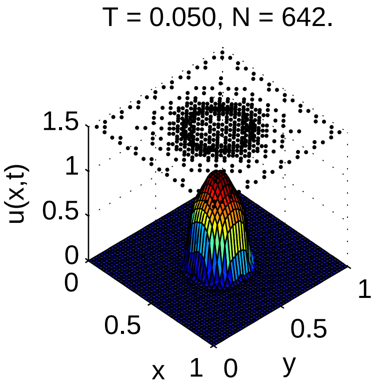

Step and Adapt / Method of Lines

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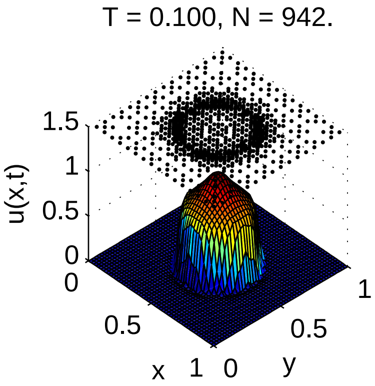

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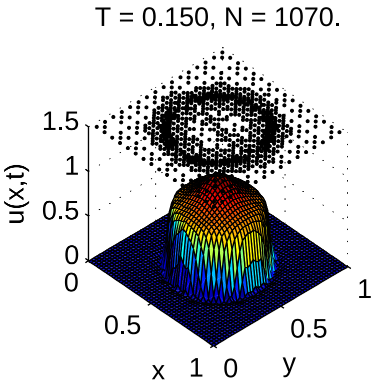

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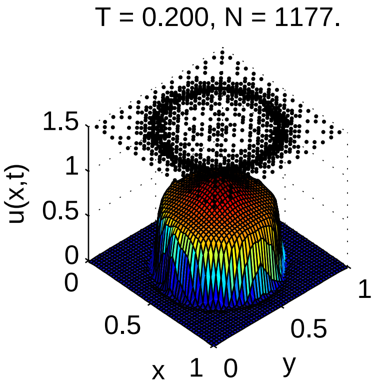
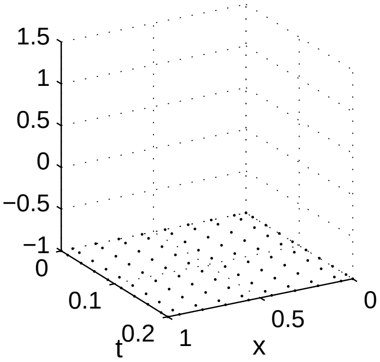


1-D Burgers' Equation

$$\nu u_{xx} - u u_x = u_t, \qquad 0 < x < 1$$
$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin(2\pi x) + \tfrac{1}{2}\sin(\pi x).$$

where, $\nu = 10^{-3}$



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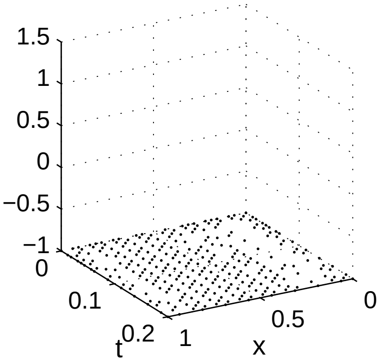

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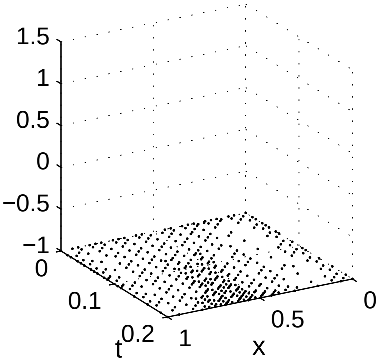

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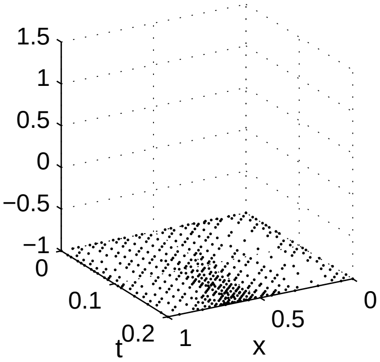

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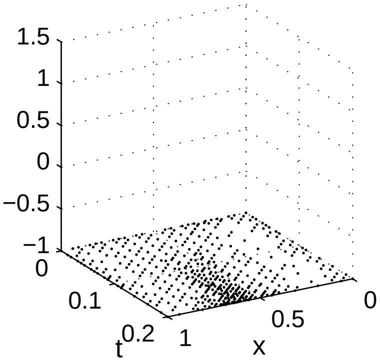

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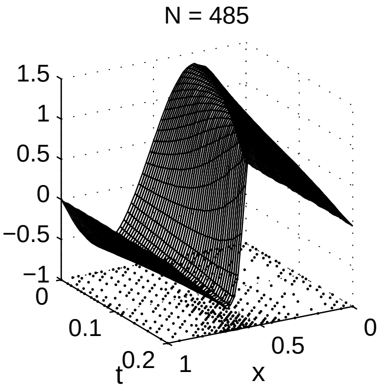

Things to be done

- Theory and model problems.
- Stability and Accuracy.
- Finding the best way to choose shape parameters.
- Applications (e.g. Lubrication theory in human eye).